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On the generation of higher order numerical integration methods using lower order Adams–Bashforth and Adams–Moulton methods

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Abstract

In this paper, a new explicit numerical integration method is proposed. The proposed method is based on the relationship that m-step Adams–Moulton method is the linear convex combination of $(m-1)$-step Adams–Moulton and m-step Adams–Bashforth methods with a fixed weighting coefficient. By examining the order of accuracy and stability regions, we conclude that the present method is superior to the traditional Adams–Bashforth–Moulton predictor–corrector method. A simple harmonic oscillator problem is used to demonstrate the efficiency of the proposed method. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Adams–Moulton and Adams–Bashforth numerical integrator; Accuracy and stability analysis

1. Introduction

Differential equations are often used to model complex problems in science and engineering. In most practical problems, these differential equations are highly nonlinear and are too complicated to solve analytically. Hence, with given initial conditions, these differential equations are frequently solved approximately using appropriate explicit/implicit numerical integration methods. Note that the numerical integration methods using the approximation with only one of the previous mesh points are called one-step methods. On the other hand, numerical methods using the approximation with more than one previous mesh point to determine the approximation of the next point are called multi-step methods. In general, to solve nonlinear differential equations, Adams–Bashforth–Moulton predictor–corrector method is the most popular and commonly used multi-step integration method for the reasons that it is simple to implement and is 'strongly stable' [1].

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In this paper, we show that the m-step Adams–Moulton method is the linear convex combination of the $(m-1)$-step Adams–Moulton and m-step Adams–Bashforth methods with a fixed weighting coefficient. Based on this relationship, a modified Adams–Bashforth–Moulton predictor–corrector (MABMPC) method is proposed. By examining the order of accuracy and stability domain of the proposed method, we conclude that the proposed method is superior to the traditional Adams–Bashforth–Moulton predictor–corrector (ABMPC) method. A simple harmonic oscillator problem is used to demonstrate the efficiency of the proposed method.

2. Preliminary

To begin the derivation of the multi-step methods, if we integrate the initial-value problem over the interval $[t_i, t_{i+1}]$, then the following property exists:

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\,\mathrm{d}t, \tag{1}$$

where $f(t, y(t))$ is the first derivative of $y(t)$. To derive an Adams–Bashforth explicit m-step (AB-m) method, Newton backward difference formula with a set of equal spacing points, $t_{i+1-m}, \ldots, t_{i-1}, t_i$, is used to approximate the integral $\int_{t_i}^{t_{i+1}} f(t, y(t))\,\mathrm{d}t$ which is

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\,\mathrm{d}t = h \cdot \sum_{k=0}^{m-1} \left[\nabla^k f(t_i, y(t_i)) \cdot (-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s \right] + h^{m+1} f^{(m)}(\mu_i, y(\mu_i))(-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s, \tag{2}$$

where

$$C_k^{-s} = \frac{-s(-s-1)\cdots(-s-k+1)}{k!} \tag{3}$$

and $\nabla f(x_i)$ represents the back difference operator that is defined by

$$\nabla f(x_i) = f(x_i) - f(x_{i-1}). \tag{4}$$

Higher orders of back difference operator are defined recursively by

$$\nabla^k f(x_i) = \nabla(\nabla^{k-1} f(x_i)). \tag{5}$$

On the other hand, the Adams–Moulton implicit $(m-1)$ step (AM_$(m-1)$) method are derived by using the set of equal spacing points, $t_{i+2-m}, \ldots, t_i, t_{i+1}$, and the integral $\int_{t_i}^{t_{i+1}} f(t, y(t))\,\mathrm{d}t$ is approximated by

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\,\mathrm{d}t = h \cdot \sum_{k=0}^{m-1} \left[\nabla^k f(t_{i+1}, y(t_{i+1})) \cdot (-1)^k \int_0^1 C_k^{-s+1}\,\mathrm{d}s \right] + h^{m+1} f^{(m)}(\xi_i, y(\xi_i))(-1)^m \int_0^1 C_m^{-s+1}\,\mathrm{d}s, \tag{6}$$

where

$$C_k^{-s+1} = \frac{(-s+1)(-s)(-s-1)\cdots(-s-k+2)}{k!}. \tag{7}$$

To simplify the notation, we define $f(t_i) \equiv f(t_i, y(t_i))$. By using Eqs. (2) and (6), Adams–Bashforth m step and Adams–Moulton $m-1$ step methods can be expressed as follows:
Adams–Bashforth m step method:

$$y^{\mathrm{AB}\_m}(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\,\mathrm{d}t = y(t_i) + h \cdot \sum_{k=0}^{m-1} \left[\nabla^k f(t_i) \cdot (-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s \right]. \tag{8}$$

Adams–Moulton $m-1$ step method:

$$y^{\mathrm{AM}\_(m-1)}(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\,\mathrm{d}t = y(t_i) + h \cdot \sum_{k=0}^{m-1} \left[\nabla^k f(t_{i+1}) \cdot (-1)^k \int_0^1 C_k^{-s+1}\,\mathrm{d}s \right]. \tag{9}$$

Both integrals $(-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s$ and $(-1)^k \int_0^1 C_k^{-s+1}\,\mathrm{d}s$ for various values of k are easily evaluated and are listed in Table 1.

Table 1
The values of $(-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s$ and $(-1)^k \int_0^1 C_k^{-s+1}\,\mathrm{d}s$ for different k

k	0	1	2	3	4	5
$(-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s$	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{251}{720}$	$\frac{95}{288}$
$(-1)^k \int_0^1 C_k^{-s+1}\,\mathrm{d}s$	1	$-\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{19}{720}$	$-\frac{3}{160}$

Note that the local truncation error for both Adams–Bashforth m step method and Adams–Moulton $m-1$ step method are m-th order of the integration step size h.

3. Method of generating higher order Adams–Moulton integrators

In order to generate the high order Adams–Moulton Integrators, the following propositions are given.

Proposition 1.

$$\nabla^k f(t_{i+1}) = f(t_{i+1}) - \sum_{r=0}^{k-1} \nabla^r f(t_i). \tag{10}$$

Proof. By induction:

(i) $k = 1$ hold

(ii) Suppose $k = n$, hold, i.e.,

$$\nabla^n f(t_{i+1}) = f(t_{i+1}) - \sum_{r=0}^{n-1} \nabla^r f(t_i) \tag{11}$$

when $k = n + 1$, then

$$\begin{aligned}\nabla^{n+1} f(t_{i+1}) &= \nabla(\nabla^n f(t_{i+1})) = \nabla\left(f(t_{i+1}) - \sum_{r=0}^{n-1} \nabla^r f(t_i)\right)\\ &= \nabla f(t_{i+1}) - \sum_{r=0}^{n-1} \nabla^r f(t_i) = f(t_{i+1}) - f(t_i) - \sum_{r=1}^{n} \nabla^r f(t_i)\\ &= f(t_{i+1}) - \sum_{r=0}^{n} \nabla^r f(t_i). \qquad \Box\end{aligned}$$

Proposition 2.

$$(-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s - (-1)^k \int_0^1 C_k^{-s+1}\,\mathrm{d}s = (-1)^{k-1} \int_0^1 C_{k-1}^{-s}\,\mathrm{d}s. \tag{12}$$

Proof.

$$\begin{aligned}(-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s &= (-1)^k \int_0^1 \frac{-s(-s-1)(-s-2)\cdots(-s-k+1)}{k!}\,\mathrm{d}s\\ &= \int_0^1 \frac{s(s+1)(s+2)\cdots(s+k-1)}{k!}\,\mathrm{d}s,\end{aligned} \tag{13}$$

$$\begin{aligned}(-1)^k \int_0^1 C_k^{-s+1}\,\mathrm{d}s &= (-1)^k \int_0^1 \frac{(-s+1)(-s)(-s-1)\cdots(-s-k+2)}{k!}\,\mathrm{d}s\\ &= \int_0^1 \frac{(s-1)s(s+1)\cdots(s+k-2)}{k!}\,\mathrm{d}s,\end{aligned} \tag{14}$$

$$\begin{aligned}&(-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s - (-1)^k \int_0^1 C_k^{-s+1}\,\mathrm{d}s(-1)^{k-1}\\ &= \int_0^1 \frac{s(s+1)(s+2)\cdots(s+k-2)(s+k-1)}{k!}\,\mathrm{d}s\\ &\quad - \int_0^1 \frac{(s-1)s(s+1)(s+2)\cdots(s+k-2)}{k!}\,\mathrm{d}s\\ &= \int_0^1 \frac{[(s+k-1)-(s-1)]s(s+1)(s+2)\cdots(s+k-2)}{k!}\,\mathrm{d}s\end{aligned}$$

$$= \int_0^1 \frac{s(s+1)(s+2)\cdots(s+k-2)}{(k-1)!}\,\mathrm{d}s$$

$$= (-1)^{k-1}\int_0^1 C_{k-1}^{-s}\,\mathrm{d}s. \qquad \square$$

Proposition 3.

$$\sum_{r=0}^{k}(-1)^r\int_0^1 C_l^{-s+1}\,\mathrm{d}s = (-1)^k\int_0^1 C_k^{-s}\,\mathrm{d}s. \tag{15}$$

Proof. By induction:

(i) $k = 0$ arbitary.

(ii) when $k = 1$

$$1 + (-1)\int_0^1 C_1^{-s+1}\,\mathrm{d}s = \frac{1}{2} = (-1)\int_0^1 C_1^{-s}\,\mathrm{d}s. \tag{16}$$

(iii) Suppose $k = n$ hold, i.e.,

$$\sum_{r=0}^{n}(-1)^r\int_0^1 C_r^{-s+1}\,\mathrm{d}s = (-1)\int_0^1 C_n^{-s}\,\mathrm{d}s$$

when $k = n+1$

$$\begin{aligned}\sum_{r=0}^{n+1}(-1)^r\int_0^1 C_l^{-s+1}\,\mathrm{d}s &= \sum_{r=0}^{n}(-1)^r\int_0^1 C_r^{-s+1}\,\mathrm{d}s + (-1)^{n+1}\int_0^1 C_{n+1}^{-s+1}\,\mathrm{d}s\\ &= (-1)^n\int_0^1 C_n^{-s}\,\mathrm{d}s + (-1)^{n+1}\int_0^1 C_{n+1}^{-s+1}\,\mathrm{d}s\\ &= (-1)^{n+1}\int_0^1 C_{n+1}^{-s}\,\mathrm{d}s. \qquad \square\end{aligned}$$

Note that, Adams–Bashforth m step method is given by

$$y^{\mathrm{AB\_}m}(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\,\mathrm{d}t = y(t_i) + h\cdot\sum_{k=0}^{m-1}\left[\nabla^k f(t_i)\cdot(-1)^k\int_0^1 C_k^{-s}\,\mathrm{d}s\right] \tag{17}$$

whereas Adams–Moulton $(m-1)$ step method is given as

$$\begin{aligned}y^{\mathrm{AM\_}(m-1)}(t_{i+1}) &= y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\,\mathrm{d}t\\ &= y(t_i) + h\cdot\sum_{k=0}^{m-1}\left[\nabla^k f(t_{i+1})\cdot(-1)^k\int_0^1 C_k^{-s+1}\,\mathrm{d}s\right],\end{aligned} \tag{18}$$

$$y^{\mathrm{AM\_}(m-1)}(t_{i+1}) = y(t_i) + hf(t_{i+1})(-1)^{m-1}\int_0^1 C_{m-1}^{-s}\,\mathrm{d}s + h\cdot\sum_{k=0}^{m-1}\left\{\nabla^k f(t_i)\cdot\left[-(-1)^{m-1}\int_0^1 C_{m-1}^{-s}\,\mathrm{d}s + (-1)^k\int_0^1 C_k^{-s}\,\mathrm{d}s\right]\right\}. \tag{19}$$

To establish the relationship between Adams–Bashforth m step method and Adams–Moulton $(m-1)$ step method, the following theorem is proposed and proved.

Theorem 1.

$$\frac{W_1}{W_1+W_2}y^{\mathrm{AB\_}m}(t_{i+1}) + \frac{W_2}{W_1+W_2}y^{\mathrm{AM\_}(m-1)}(t_{i+1}) = y^{\mathrm{AM-}m}(t_{i+1}), \tag{20}$$

where

$$\begin{aligned} W_1 &= -(-1)^m\int_0^1 C_m^{-s+1}\,\mathrm{d}s,\\ W_2 &= (-1)^m\int_0^1 C_m^{-s}\,\mathrm{d}s,\\ W_1+W_2 &= (-1)^{m-1}\int_0^1 C_{m-1}^{-s}\,\mathrm{d}s. \end{aligned} \tag{21}$$

Proof. Define

$$\theta(t_{i+1}) \equiv \frac{W_1}{W_1+W_2}y^{\mathrm{AB\_}m}(t_{i+1}) + \frac{W_2}{W_1+W_2}y^{\mathrm{AM\_}(m-1)}(t_{i+1}). \tag{22}$$

Evaluating the coefficient of $y(t_i), h\cdot f(t_{i+1})$, and $h\cdot\nabla^k f(t_i)$ in $\theta(t_{i+1})$

$$y(t_i):\quad \frac{W_1}{W_1+W_2}\cdot 1 + \frac{W_2}{W_1+W_2}\cdot 1 = 1; \tag{23}$$

$$h\cdot f(t_{i+1}):\quad \frac{W_2}{W_1+W_2}\cdot(-1)^{m-1}\int_0^1 C_{m-1}^{-s}\,\mathrm{d}s = (-1)^m\int_0^1 C_m^{-s}\,\mathrm{d}s; \tag{24}$$

$$\begin{aligned} h\cdot\nabla^k f(t_i):\quad &\frac{W_1}{W_1+W_2}\cdot(-1)^k\int_0^1 C_k^{-s}\,\mathrm{d}s + \frac{W_2}{W_1+W_2}\\ &\cdot\left[-(-1)^{m-1}\int_0^1 C_{m-1}^{-s}\,\mathrm{d}s + (-1)^k\int_0^1 C_k^{-s}\,\mathrm{d}s\right]\\ &= -(-1)^m\int_0^1 C_m^{-s}\,\mathrm{d}s + (-1)^k\int_0^1 C_k^{-s}\,\mathrm{d}s; \end{aligned} \tag{25}$$

$$\theta(t_{i+1}) = y(t_i) + hf(t_{i+1})(-1)^m\int_0^1 C_m^{-s}\,\mathrm{d}s + h\cdot\sum_{k=0}^{m-1}\left\{\nabla^k f(t_i)\cdot\left[-(-1)^m\int_0^1 C_m^{-s}\,\mathrm{d}s + (-1)^k\int_0^1 C_k^{-s}\,\mathrm{d}s\right]\right\}$$

$$= y(t_i) + hf(t_{i+1})(-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s + h$$

$$\cdot \sum_{k=0}^{m} \left\{ \nabla^k f(t_i) \cdot \left[-(-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s + (-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s \right] \right\} \tag{26}$$

($\because\ \nabla^m f(t_i) \cdot [-(-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s + (-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s] = 0$).

$\theta(t_{i+1})$ is the formulation of Adams–Moulton m step method. □

4. Modified Adams–Bashforth–Moulton predictor–corrector method

Adams–Bashforth–Moulton m-step predictor–corrector method is given as follows:

$$y^{\mathrm{p}}(t_{i+1}) = y(t_i) + h \cdot \sum_{k=0}^{m-1} \left[\nabla^k f(t_i) \cdot (-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s \right], \tag{27a}$$

$$y^{\mathrm{c}}(t_{i+1}) = y(t_i) + hf(t_{i+1}, y^{\mathrm{p}}(t_{i+1}))(-1)^{m-1} \int_0^1 C_{m-1}^{-s}\,\mathrm{d}s$$

$$+h \cdot \sum_{k=0}^{m-1} \left\{ \nabla^k f(t_i) \cdot \left[-(-1)^{m-1} \int_0^1 C_{m-1}^{-s}\,\mathrm{d}s + (-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s \right] \right\} \tag{27b}$$

and Modified Adams–Bashforth–Moulton m-step predictor–corrector method is given by

$$y^{\mathrm{p}}(t_{i+1}) = y(t_i) + h \cdot \sum_{k=0}^{m-1} \left[\nabla^k f(t_i) \cdot (-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s \right], \tag{28a}$$

$$y^{\mathrm{c}}(t_{i+1}) = y(t_i) + hf(t_{i+1}, y^{\mathrm{p}}(t_{i+1}))(-1)^{m-1} \int_0^1 C_{m-1}^{-s}\,\mathrm{d}s$$

$$+h \cdot \sum_{k=0}^{m-1} \left\{ \nabla^k f(t_i) \cdot \left[-(-1)^{m-1} \int_0^1 C_{m-1}^{-s}\,\mathrm{d}s + (-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s \right] \right\}, \tag{28b}$$

$$y^{\mathrm{mpc}}(t_{i+1}) = \frac{W_1}{W_1 + W_2} \cdot y^{\mathrm{p}}(t_{i+1}) + \frac{W_2}{W_1 + W_2} \cdot y^{\mathrm{c}}(t_{i+1}). \tag{28c}$$

To examine the robustness and efficiency of the proposed method, the stability and accuracy analysis are given as follows:

4.1. Stability analysis

Calculate for the linear test problem

$$\dot{z} = \lambda z. \tag{29}$$

Figs. 1–3 show that domain of stability of the characteristic equations (28) connecting the points $h\lambda$ for which the roots of these equations have a modulus less than unity. From Figs. 1–3, we conclude that the stability domains of MABMPC methods are larger than ABMPC methods.

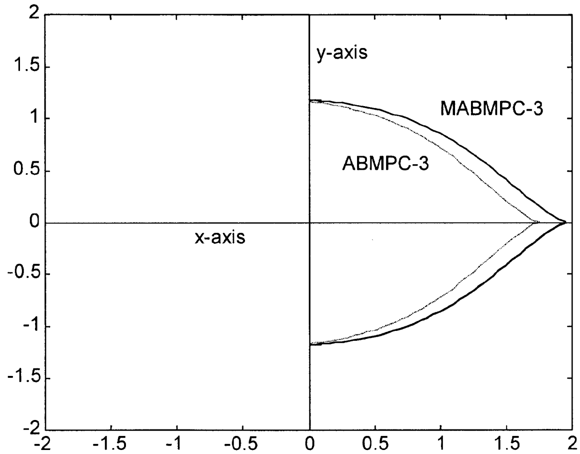

Fig. 1. The stability region of MAMPC-3 and AMPC-3.

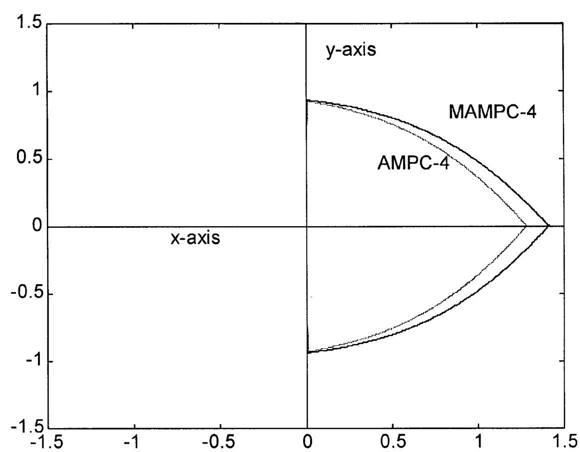

Fig. 2. The stability region of MAMPC-4 and AMPC-4.

4.2. Accuracy analysis

Since $y^{\mathrm{mpc}-m}$ and $y^{\mathrm{AM}-m}$ can be approximated by

$$y^{\mathrm{mpc}-m}(t_{i+1}) = y(t_i) + hf(t_{i+1}, y^{\mathrm{p}}(t_{i+1}))(-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s$$

$$+h \cdot \sum_{k=0}^{m} \left\{ \nabla^k f(t_i) \cdot \left[-(-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s + (-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s \right] \right\}, \tag{30}$$

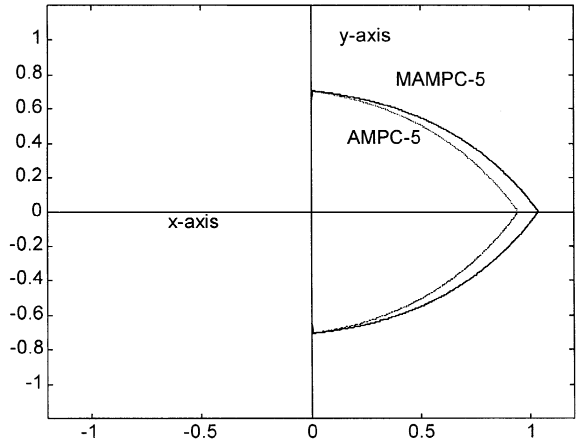

Fig. 3. The stability region of MAMPC-5 and AMPC-5.

$$y^{\mathrm{AM}\_m}(t_{i+1}) = y(t_i) + hf(t_{i+1}, y(t_{i+1}))(-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s + h \cdot \sum_{k=0}^{m} \left\{ \nabla^k f(t_i) \cdot \left[-(-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s + (-1)^k \int_0^1 C_k^{-s}\,\mathrm{d}s \right] \right\}. \tag{31}$$

To proceed further, we make the assumption that $y(t_i)$ and $\nabla^k f(t_i)$ in Eqs. (30) and (31) are equal at first. With this assumption, we have

$$\begin{aligned} y^{\mathrm{AM}\_m}(t_{i+1}) - y^{\mathrm{mpc}\_m}(t_{i+1}) &= h(-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s \cdot [f(t_{i+1}, y(t_{i+1})) - f(t_{i+1}, y^{\mathrm{p}}(t_{i+1}))] \\ &= h(-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s \cdot hf'(\vartheta_i) \cdot (y(t_{i+1}) - y^{\mathrm{p}}(t_{i+1})) \\ &\approx h^{m+2} \left[(-1)^m \int_0^1 C_m^{-s}\,\mathrm{d}s \right]^2 f'(\vartheta_i) f^{(m)}(\mu_i, y(\mu_i)). \end{aligned} \tag{32}$$

Hence, the dominant truncation error of y^{mpc_m} and y^{AM_m} have the order of same h^{m+1}. In Eq. (32), we have shown that y^{mpc_m} and y^{AM_m} have the same order of accuracy. Thus, we conclude that MABMPC methods are more accurate than ABMPC methods (with the same step size) about one order of magnitude.

5. Numerical examples

In the present numerical experiment, we use the following example to demonstrate the accuracy of the present method in comparison with Adams–Moulton predictor–corrector methods.

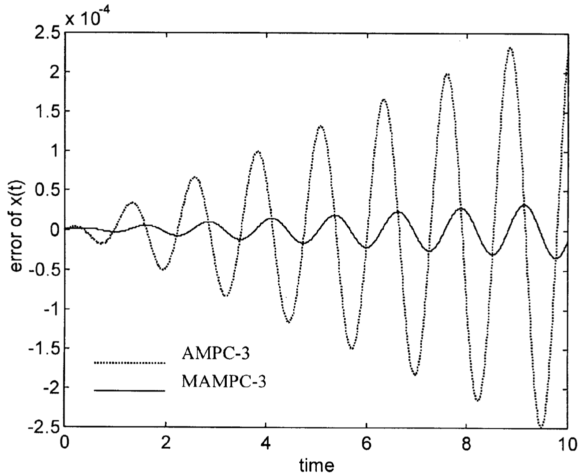

Fig. 4. Error of position $x(t)$, $h = 0.01$ s.

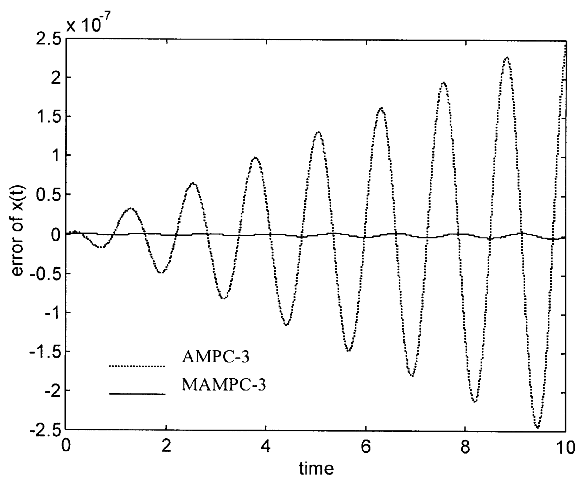

Fig. 5. Error of position $x(t)$, $h = 0.001$ s.

Example (A harmonic oscillator problem)**.** The equations of motion of a harmonic oscillator are given as

$$\dot{x} = v, \qquad \dot{v} = -25x, \tag{33}$$

and its initial conditions are

$$x(0) = 1, \quad v(0) = 0. \tag{34}$$

Note that the exact solution of Eq. (33) is $x(t) = \cos(5t)$.

The Adams–Moulton third order predictor–corrector method (AMPC_3) and the presented MAMPC_3 method are used in this numerical example with two different integration step sizes

$h=0.01$ and 0.001 s. The starting procedure of these two methods is calculated by using Runge–Kutta–Fehlberg method with truncation error equals to 10^{-12}. The error of state variable x is, $e(t)=\hat{x}(t)-x(t)$, which is shown in Fig. 4 ($h=0.01$ s) and Fig. 5 ($h=0.001$ s). Here, Fig. 4 shows that the error of state variable x is reduced to 14% if one replaces AMPC_3 by MAMPC_3 with $h=0.01$ s. Fig. 5 shows that the error of state variable x is reduced to 1.3% when $h=0.001$ s.

6. Conclusion

A modified predictor–corrector method has been presented in this paper. By making a small modification, the proposed method has increased the accuracy about one order of magnitude in comparison with Adams–Moulton predictor–corrector methods. Numerical example of nonlinear differential equations has been used here to show the superiority of the proposed integration method.

Reference

[1] R.L. Burden, J.D. Faires, Numerical Analysis, PWS-KENT Publishing Company, Boston, MA, 1993.