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Circular chromatic numbers of Mycielski's graphs

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Abstract

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski developed a graph transformation that transforms a graph G into a new graph $\mu(G)$, we now call the Mycielskian of G, which has the same clique number as G and whose chromatic number equals $\chi(G)+1$. Let $\mu^n(G)=\mu(\mu^{n-1}(G))$ for $n\geqslant 2$. This paper investigates the circular chromatic numbers of Mycielski's graphs. In particular, the following results are proved in this paper: (1) for any graph G of chromatic number n, $\chi_c(\mu^{n-1}(G))\leqslant\chi(\mu^{n-1}(G))-\frac{1}{2}$; (2) if a graph G satisfies $\chi_c(G)\leqslant\chi(G)-\frac{1}{d}$ with $d=2$ or 3, then $\chi_c(\mu^2(G))\leqslant\chi(\mu^2(G))-\frac{1}{d}$; (3) for any graph G of chromatic number 3, $\chi_c(\mu(G))=\chi(\mu(G))=4$; (4) $\chi_c(\mu(K_n))=\chi(\mu(K_n))=n+1$ for $n\geqslant 3$ and $\chi_c(\mu^2(K_n))=\chi(\mu^2(K_n))=n+2$ for $n\geqslant 4$.

Keywords: Circular chromatic number; Mycielski's graphs; Girth; Homomorphism; Connectivity; Critical graph

1. Introduction

All graphs in this paper are simple, i.e., finite, undirected, loopless, and without multiple edges.

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski [15] developed an interesting graph transformation as follows. For a graph G with vertex set $V(G)=V$ and edge set $E(G)=E$, the *Mycielskian* of G is the graph $\mu(G)$ with vertex set $V\cup V'\cup\{u\}$, where $V'=\{x' \colon x\in V\}$, and edge set $E\cup\{xy' \colon xy\in E\}\cup\{y'u \colon y'\in V'\}$. The vertex x' is called the *twin* of the vertex x (and x is

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also called the twin of x'); and the vertex u is called the *root* of $\mu(G)$. If there is no ambiguity we shall always use u as the root of $\mu(G)$. For $n \geqslant 2$, let $\mu^n(G) = \mu(\mu^{n-1}(G))$.

Mycielski [15] showed that $\chi(\mu(G)) = \chi(G) + 1$ for any graph G and $\omega(\mu(G)) = \omega(G)$ for any graph G with at least one edge. Hence $\mu^n(K_2)$ is a triangle-free graph of chromatic number $n + 2$. Besides such interesting properties involving clique numbers and chromatic numbers, Mycielski's graphs also have some other parameters that behave in a predictable way. For example, it was shown by Larsen et al. [14] that $\chi_f(\mu(G)) = \chi_f(G) + \frac{1}{\chi_f(G)}$ for any graph G, where $\chi_f(G)$ is the fractional chromatic number of G. Mycielski's graphs were also used by Fisher [6] as examples of optimal fractional colorings that have large denominators.

The purpose of this paper is to investigate the circular chromatic numbers of Mycielski's graphs. The circular chromatic number $\chi_c(G)$ of a graph G is a variation of the chromatic number of G, introduced by Vince [17] in 1988, as the 'star chromatic number' of a graph. Let k and d be integers such that $0 < d \leqslant k$. A (k,d)-*coloring* of G is a coloring c of vertices of G with k colors $\{0, 1, \ldots, k-1\}$ such that for any edge xy, $d \leqslant |c(x) - c(y)| \leqslant k - d$. The *circular chromatic number* $\chi_c(G)$ of G is the minimum ratio $\frac{k}{d}$ for which there exists a (k,d)-coloring of G. (To be precise, the minimum in the definition should be infimum. However, it was shown in [17] that the infimum is attained.) Observe that a $(k,1)$-coloring of a graph G is just an ordinary k-coloring of G. It follows that $\chi_c(G) \leqslant \chi(G)$. On the other hand, it is also not difficult to see [3,17,18] that $\chi(G) - 1 < \chi_c(G)$. Therefore, $\chi(G) = \lceil \chi_c(G) \rceil$. In some sense the circular chromatic number is a refinement of the chromatic number of a graph, and it contains more information about the graph. Readers are referred to [1–5,7–13,16–23] for more information on circular chromatic numbers of graphs.

In this paper, we show that circular chromatic numbers of Mycielski's graphs exhibit interesting patterns. The problem of determining if $\chi_c(G) = \chi(G)$ or $\chi_c(G)$ is 'close to' $\chi(G) - 1$ is hard and has been extensively studied for general graphs. This paper reports some progress for Mycielski's graphs in this direction. In Section 3, we prove that $\chi_c(\mu^{n-1}(G)) \leqslant \chi(\mu^{n-1}(G)) - \frac{1}{2}$ for any graph G of chromatic number n, and $\chi_c(\mu^2(G)) \leqslant \chi(\mu^2(G)) - \frac{1}{d}$ for any graph G with $\chi_c(G) \leqslant \chi(G) - \frac{1}{d}$, $d = 2$ or 3. Section 4 establishes that $\chi_c(\mu(G)) = \chi(\mu(G)) = 4$ for any graph G of chromatic number 3, $\chi_c(\mu(K_n)) = \chi(\mu(K_n)) = n + 1$ for $n \geqslant 3$, and $\chi_c(\mu^2(K_n)) = \chi(\mu^2(K_n)) = n + 2$ for $n \geqslant 4$.

These results yield many graphs with special properties having particular circular chromatic numbers. For example, it follows that there are triangle-free 4-critical graphs whose circular chromatic numbers are 4. This disproves a conjecture in [16]. It also follows from these results that there are triangle-free and color-critical graphs G of high connectivity for which $\chi_c(G) \leqslant \chi(G) - \frac{1}{2}$.

Along the way to proving these results, we also refine some tools used by others in the study of the relationship between the circular chromatic number and the chromatic number of a graph. We believe that the results obtained here are just a fraction of a family of interesting properties concerning the circular chromatic numbers of Mycielski's graphs. In Section 5, a few questions are raised.

2. Preliminary results

The *connectivity* $\kappa(G)$ of a graph G is the minimum non-negative integer k such that $G\backslash S$ is disconnected or trivial for some vertex set S of size k. A graph G is *k-critical* if $\chi(H) < \chi(G) = k$ for any proper subgraph H of G; or equivalently, G is connected and $\chi(G\backslash e) < \chi(G)=k$ for any edge e in G. The following lemma is surely folkloric:

Lemma 1. *If G has no isolated vertices, then $\kappa(\mu(G))\geqslant\kappa(G)+1$. If G is k-critical, then $\mu(G)$ is $(k+1)$-critical.*

Proof. Suppose $V(G)=V$ and $V(\mu(G))=V\cup V'\cup\{u\}$. Let S be a subset of $V(\mu(G))$ of size $\kappa(G)$. If $|S\cap V| < \kappa(G)$, then $G\backslash(S\cap V)$ is connected. Also, for any vertex $x\in V$, x' is adjacent to at least $\kappa(G)$ vertices of V in $\mu(G)$. So, any such vertex x' of $\mu(G)\backslash S$ is adjacent to at least one vertex in $G\backslash(S\cap V)$. And u is adjacent to all such vertices x' of $\mu(G)\backslash S$. Thus, $\mu(G)\backslash S$ is connected. If $|S\cap V|=\kappa(G)$, then $S\subseteq V$. Since G has no isolated vertices, any vertex $x\in V\backslash S$ is adjacent to some vertex y' in V', which is in turn adjacent to u. Thus, $\mu(G)\backslash S$ is also connected. Therefore, $\kappa(\mu(G))\geqslant\kappa(G)+1$.

For the proof of the second half of this lemma, assume that G is k-critical. Since G is connected, so is $\mu(G)$. Let e be any edge of $\mu(G)$. We consider the following three cases.

Suppose $e=ab$ for some $a\in V$ and $b\in V$. Let c be a proper $(k-1)$-coloring of $G\backslash e$. Then the following c' is a proper k-coloring of $\mu(G)\backslash e$: $c'(x)=c(x)$ for all $x\in V$, $c'(x')=k-1$ for all $x'\in V'$, and $c'(u)=0$.

Suppose $e=ab'$ for some $a\in V$ and $b'\in V'$. Let c be a proper $(k-1)$-coloring of $G\backslash ab$. Then the following c' is a proper k-coloring of $\mu(G)\backslash e$: $c'(b)=k-1$, $c'(x)=c(x)$ for all $x\in V\backslash\{b\}$, $c'(x')=c(x)$ for all $x'\in V'$, and $c'(u)=k-1$.

Suppose $e=a'u$ for some $a'\in V'$. Suppose c is a proper $(k-1)$-coloring of $G\backslash a$. Then the following c' is a proper k-coloring of $\mu(G)\backslash e$: $c'(x)=c'(x')=c(x)$ for all $x\in V\backslash\{a\}$ and $c'(a)=c'(a')=c'(u)=k-1$. $\square$

For an n-coloring $c: V(G)\mapsto\{0,1,\ldots,n-1\}$ of G, we denote by $D_c(G)$ the directed graph with vertex set $V(G)$ in which there is an arc from x to y if and only if $xy\in E(G)$ and $c(x)+1\equiv c(y) \pmod n$. It was shown in [10], in the corollary of Theorem 1, that an n-chromatic graph G satisfies $\chi_c(G)<n$ if and only if G has an n-coloring c for which $D_c(G)$ is acyclic. For our purposes in this paper, we refine this result in two respects.

Lemma 2. *If x_0 is a vertex of an n-chromatic graph G for which $\chi_c(G)<n$, then there is an n-coloring c of G such that $D_c(G)$ is acyclic, $c(x_0)=1$, and $c(x)\notin\{0,1\}$ for all vertices x adjacent to x_0.*

Proof. Suppose $\chi_c(G) = \frac{k}{d} < n$ and $d > 1$. Then G has a (k,d)-coloring h with $h(x_0) = d-1$. Define c: $V(G) \mapsto \{0,1,\ldots,n-1\}$ by $c(v) = \lfloor \frac{h(v)+1}{d} \rfloor$ for each $v \in V(G)$. It is straightforward to check that c is a proper coloring, $D_c(G)$ is acyclic, $c(x_0) = 1$, and $c(x) \notin \{0,1\}$ for all vertices x adjacent to x_0. □

Corollary 3. *If $\mu(G)$ with root u satisfies $\chi_c(\mu(G)) < \chi(\mu(G)) = n$, then there is an n-coloring c of $\mu(G)$ such that $D_c(\mu(G))$ is acyclic, $c(u) = 1$, and $c(x') \notin \{0,1\}$ for all $x' \in V'$. Moreover, for any such coloring c, there is an edge $ab \in E(G)$ such that $c(a) = 0$, $c(b) = 1$, and $c(a') = c(b')$.*

Proof. Applying Lemma 2 to $\mu(G)$ with $x_0 = u$, we obtain an n-coloring c such that $D_c(\mu(G))$ is acyclic, $c(u)=1$, and $c(x') \notin \{0,1\}$ for all $x' \in V'$. To prove the 'moreover' part, we assume to the contrary that $c(a') \neq c(b')$ for all edges $ab \in E(G)$ with $c(a)=0$ and $c(b)=1$. Let c' be the coloring defined by $c'(x)=c(x)$ if $c(x) \notin \{0,1\}$ and $c'(x) = c(x')$ if $c(x) \in \{0,1\}$. It is straightforward to verify that c' is an $(n-2)$-coloring of G, contrary to the assumption that $\chi(\mu(G)) = n$. □

Lemma 4. *Suppose G is an n-chromatic graph and that there is an n-coloring c: $V(G) \mapsto \{0,1,\ldots,n-1\}$ of G such that $D_c(G)$ is acyclic. Let $\mathscr{P}$ be the set of all directed paths of $D_c(G)$. For any $P \in \mathscr{P}$, let $z(P)$ be the number of vertices of P which are colored 0 and let $d = \max\{z(P)+1\colon P \in \mathscr{P}\}$. If $n \geqslant 3$, then $\chi_c(G) \leqslant n - \frac{1}{d}$.*

Proof. For each vertex x of G, let $\mathscr{P}_x$ be the set of all directed paths of $D_c(G)$ that end at x and let $\ell(x) = \max\{z(P){:}P \in \mathscr{P}_x\}$. Define an $(nd-1,d)$-coloring h of G by $h(x) = (c(x)d + \ell(x)) \bmod (nd-1)$. Since $0 \leqslant c(x) \leqslant n-1$ and $\ell(x) \leqslant d-1$, it follows that $0 \leqslant c(x)d + \ell(x) \leqslant nd-1$ and then $h(x) = c(x)d + \ell(x)$, except $h(x) = 0$ for $c(x) = n-1$ and $\ell(x) = d-1$. We show that h is indeed an $(nd-1,d)$-coloring of G.

Suppose xy is an edge of G. Assume that $c(x) < c(y)$. First consider the case that $2 \leqslant c(y) - c(x) \leqslant n-2$. If $c(y) \leqslant n-2$ or $c(y) = n-1$ but $\ell(y) < d-1$, then $c(y)d \leqslant h(y) \leqslant c(y)d + d - 1$ and $c(x)d \leqslant h(x) \leqslant c(x)d + d - 1$. Hence,

$$d \leqslant c(y)d - (c(x)d + d - 1) \leqslant h(y) - h(x) \leqslant c(y)d + d - 1 - c(x)d \leqslant nd - 1 - d.$$

If $c(y) = n-1$ and $\ell(y) = d-1$, then $h(y) = 0$. Since $1 \leqslant c(x) \leqslant n-3$, it follows that $d \leqslant h(x) \leqslant (n-2)d - 1$. Hence, $d \leqslant h(x) - h(y) \leqslant nd - 1 - d$.

Next, we assume that $c(y) - c(x) = 1$. In this case, xy is an arc of $D_c(G)$. Therefore, $\ell(y) \geqslant \ell(x)$. If $c(y) \leqslant n-2$ or $c(y) = n-1$ but $\ell(y) < d-1$, then $h(y) = c(y)d + \ell(y)$ and $h(x) = c(x)d + \ell(x)$. Hence, $d \leqslant h(y) - h(x) \leqslant 2d - 1 \leqslant nd - 1 - d$. If $c(y) = n-1$ and $\ell(y) = d-1$, then $h(y) = 0$. Since $c(x) = n-2$, it follows that $(n-2)d \leqslant h(x) \leqslant (n-2)d + d - 1$. Hence, $d \leqslant h(x) - h(y) \leqslant nd - 1 - d$.

Finally, we assume that $c(y) = n-1$ and $c(x) = 0$. In this case, yx is an arc of $D_c(G)$ and $\ell(x) \geqslant \ell(y) + 1$. Therefore, $\ell(y) < \ell(x) \leqslant d - 1$. Hence, $h(y) = c(y)d + \ell(y)$ and

$h(x) = \ell(x) \geqslant \ell(y) + 1$. It follows that $d \leqslant h(y) - h(x) \leqslant nd - 1 - d$. This completes the proof of the lemma. □

Corollary 5. *Suppose $n \geqslant 3$ and that G is an n-chromatic graph having an n-coloring c such that $D_c(G)$ is acyclic. Let $\mathscr{P}$ be the set of all directed paths of $D_c(G)$. For each $P \in \mathscr{P}$, let $s(P)$ be the number of arcs in P and let $s = \max\{s(P){:}\ P \in \mathscr{P}\}$. If $d = \lfloor \frac{s}{n} \rfloor + 2$, then $\chi_c(G) \leqslant n - \frac{1}{d}$.*

Proof. Since each directed path of $D_c(G)$ has at most s arcs, it follows that the path contains at most $\lfloor \frac{s}{n} \rfloor + 1$ vertices with color 0. The result then follows from Lemma 4. □

3. Graphs G with $\chi_c(\mu(G)) < \chi(\mu(G))$

Note that $\mu(K_2)$ is a pentagon that has circular chromatic number $\frac{5}{2}$. Indeed, it is not difficult to see that for any bipartite graph G, $\mu(G)$ has circular chromatic number $\frac{5}{2}$. The purpose of this section is to study G and m for which $\chi_c(\mu^m(G)) \leqslant \chi(\mu^m(G)) - \frac{1}{d}$ for some d.

To work with such graphs, we need to take special care with the names of the vertices. We now introduce a system for naming the vertices of $\mu^m(G)$. It turns out that using this naming system provides an easy method for determining the adjacency of vertices and for telling which vertex is the twin of another vertex at a certain level.

For any two non-negative integers i and j, let $i\&j$ denote the integer whose binary representation is the logical '*and*' of the binary representations of i and j. For instance, $14\&25 = 01110_2\&11001_2 = 01000_2 = 8$ and $10\&17 = 01010_2\&10001_2 = 00000_2 = 0$. If $i > 0$, let $f(i)$ denote the maximum factor of i that is a power of 2. For instance, $f(1) = f(3) = 1$, $f(6) = f(18) = 2$ and $f(12) = 4$. For any graph G and any non-negative integer m, let G_m be the graph whose

$$\text{vertex set } V(G_m) = \{x^i{:}\ x \in V(G) \text{ and } 0 \leqslant i < 2^m\} \cup \{u^i{:}\ 1 \leqslant i < 2^m\},$$

$$\begin{aligned}\text{edge set } E(G_m) = {} & \{x^i y^j{:}\ xy \in E(G) \text{ and } i\&j = 0\} \cup \{x^i u^j{:}\ i\&j = f(j)\} \\ & \cup\ \{u^i u^j{:}\ i\&j = \max\{f(i), f(j)\} \text{and} f(i) \neq f(j)\}.\end{aligned}$$

Note that $G \cong G_0$.

Lemma 6. *For any graph G and any non-negative integer m, G_{m+1} is isomorphic to $\mu(G_m)$. Consequently, $\mu^m(G) \cong G_m$ for any $m \geqslant 0$.*

Proof. Consider the function $h{:}\ V(G_{m+1}) \mapsto V(\mu(G_m))$ defined by

$$h(x^i) = x^i \quad \text{and} \quad h(x^{i+2^m}) = (x^i)' \quad \text{for } x \in V(G) \text{ and } 0 \leqslant i < 2^m,$$

$$h(u^i) = u^i \quad \text{and} \quad h(u^{i+2^m}) = (u^i)' \quad \text{for } 1 \leqslant i < 2^m,$$

$$h(u^{2^m}) = u.$$

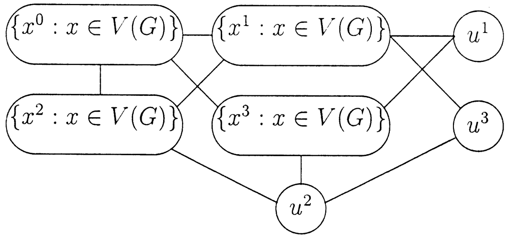

Fig. 1. $\mu^2(G)$.

It is straightforward to check that h is an isomorphism between G_{m+1} and $\mu(G_m)$ by using the following facts:

(i) If $0 \leqslant i,\ j < 2^m$, then $(i+2^m)\&j = i\&(j+2^m) = i\&j$ and $(i+2^m)\&\,(j+2^m) = (i\&j)+2^m$.

(ii) If $1 \leqslant i < 2^m$, then $f(2^m) = 2^m > f(i) = f(i+2^m)$.

An induction with the basis $G \cong G_0$ proves that $\mu^m(G) \cong G_m$ for any $m \geqslant 0$. $\square$

It follows from Lemma 6 that for any graph G, we may simply take the definition of G_m as a naming system for the vertices of $\mu^m(G)$. For the remainder of this paper, we use $V(G_m)$ and $E(G_m)$ to denote the vertex set and the edge set of $\mu^m(G)$, respectively. Fig. 1 shows $\mu^2(G)$. Note that a link between the two sets $\{x^i\colon x \in V(G)\}$ and $\{y^j\colon y \in V(G)\}$ means that $i\&j = 0$, i.e., $x^i y^j \in E(\mu^2(G))$ if and only if $xy \in E(G)$; and a link between $\{x^i\colon x \in V(G)\}$ and u^j means $i\&j = f(j)$, i.e., $x^i u^j \in E(\mu^2(G))$ for all $x \in V(G)$.

Theorem 7. *If G is a graph for which $\chi_c(G) \leqslant \chi(G) - \frac{1}{d}$ with $d = 2$ or 3, then $\chi_c(\mu^2(G)) \leqslant \chi(\mu^2(G)) - \frac{1}{d}$.*

Proof. Suppose $\chi(G) = k$ and that $\chi_c(G) \leqslant k - \frac{1}{d}$. Let $c : V(G) \mapsto \{0, 1, \ldots, dk-2\}$ be a $(dk-1, d)$-coloring of G. Define $c' : V(\mu^2(G)) \mapsto \{0, 1, \ldots, dk+2d-2\}$ as

$$c'(x^i) = \begin{cases} dk+d & \text{if } i = 0 \text{ and } c(x) = dk-d, \\ c(x)+dk-1 & \text{if } i = 2 \text{ and } c(x) \leqslant d-2, \\ dk-1 & \text{if } i = 3 \text{ and } c(x) \leqslant d-2, \\ c(x) & \text{otherwise,} \end{cases}$$

$$c'(u^i) = \begin{cases} dk+d-1 & \text{if } i = 1, \\ dk+2d-2 & \text{if } i = 2, \\ dk+d-2 & \text{if } i = 3. \end{cases}$$

It is straightforward to verify that $d \leqslant |c'(a) - c'(b)| \leqslant (dk+2d-1) - d$ for each $ab \in E(\mu^2(G))$ (see Fig. 1). Hence c' is a $(dk+2d-1, d)$-coloring of $\mu^2(G)$. $\square$

Corollary 8. *If G is a graph for which $\chi_c(G) \leqslant \chi(G) - \frac{1}{d}$, $d = 2$ or 3, and k is a non-negative integer, then $\chi_c(\mu^{2k}(G)) \leqslant \chi(\mu^{2k}(G)) - \frac{1}{d}$.*

For any integer $n \geqslant 4$, in order to find an n-chromatic graph G for which $\chi_c(\mu(G)) \leqslant \chi(\mu(G)) - \frac{1}{2}$, we may take any graph H such that $\chi(H) = n-1$ and $\chi_c(H) \leqslant n - \frac{3}{2}$ (see [17] for a proof of the existence of such graphs) and let $G = \mu(H)$. It follows from Theorem 7 that $\chi_c(\mu(G)) = \chi_c(\mu^2(H)) \leqslant \chi(\mu^2(H)) - \frac{1}{2} = \chi(\mu(G)) - \frac{1}{2}$. Therefore there are many graphs G whose Mycielskians have circular chromatic numbers strictly less than their chromatic numbers. Our next result concerns graphs obtained by repeatedly taking Mycielski transformations of a graph.

Theorem 9. *If G is a graph of chromatic number n, then $\chi_c(\mu^{n-1}(G)) \leqslant \chi(\mu^{n-1}(G)) - \frac{1}{2}$.*

Proof. First of all, we construct a $(2n-1)$-coloring c of $\mu^{n-1}(K_n)$ such that $D_c(\mu^{n-1}(K_n))$ is acyclic. For the sake of clarity, we first color the vertices of $\mu^{n-2}(K_n)$.

Let the vertices of K_n be $x_1, x_2, \ldots, x_n$. Then by our naming system, the vertex set of $\mu^{n-2}(K_n)$ is $\{x_j^i\colon 1 \leqslant j \leqslant n \text{ and } 0 \leqslant i < 2^{n-2}\} \cup \{u^i\colon 1 \leqslant i < 2^{n-2}\}$. Let $I = \{i\colon 0 \leqslant i < 2^{n-2}\}$. We partition I into subsets I_t for $1 \leqslant t \leqslant n-1$, where $I_t = \{i \in I\colon 2^{n-2} - 2^{t-1} \leqslant i < 2^{n-2} - 2^{t-2}\}$. Let c be the $(2n-1)$-coloring of $\mu^{n-2}(K_n)$ defined as follows:

$$c(x_j^i) = \begin{cases} 2j-1 & \text{if } 2 \leqslant j \leqslant n-1, \\ 0 & \text{if } j = n \text{ and } i \in I_1 = \{2^{n-2}-1\}, \\ 2 & \text{if } j = 1 \text{ and } i \in I_1 \cup I_{n-1}, \\ 2t & \text{if } j \in \{1,n\} \text{ and } i \in I_t \text{ for } 2 \leqslant t \leqslant n-2, \\ 2n-2 & \text{if } j = n \text{ and } i \in I_{n-1}, \end{cases}$$

$$c(u^i) = \begin{cases} 17 & \text{if } i = 2^{n-3}, \\ 2t+4 & \text{otherwise, where } f(i) = 2^t. \end{cases}$$

We first verify that c is a proper coloring of $\mu^{n-2}(K_n)$. The graph $\mu^{n-2}(K_n)$ has three types of edges: $x_j^i x_{j'}^{i'}$, $u^i u^j$, and $x_j^i u^k$. If $x_j^i x_{j'}^{i'} \in E(\mu^{n-2}(K_n))$, then $j \neq j'$ and $i \& i' = 0$. It follows that $c(x_j^i) \neq c(x_{j'}^{i'})$, since $i \& i' \neq 0$ when i and i' are both in I_t for some $2 \leqslant t \leqslant n-2$. If $u^i u^j \in E(\mu^{n-2}(K_n))$, then $f(i) \neq f(j)$, which implies that $c(u^i) \neq c(u^j)$. Suppose $x_j^i u^k \in E(\mu^{n-2}(K_n))$. Note that $u^{2^{n-3}}$ is the only vertex of color 1. Thus, we may assume that $k \neq 2^{n-3}$. Then $c(u^k)$ is an even integer and $4 \leqslant c(u^k) \leqslant 2n-4$. Suppose to the contrary that both end vertices of the edge $x_j^i u^k$ are colored with color $c(x_j^i) = c(u^k) = 2s+4$, where $2 \leqslant s+2 \leqslant n-2$. It then follows from the definition that $f(k) = 2^s$, $j \in \{1,n\}$, and $i \in I_{s+2} = \{2^{n-2} - 2^{s+1}, \ldots, 2^{n-2} - 2^s - 1\}$. However this implies that $i \& k \neq f(k)$ and hence, $x_j^i u^k \notin E(\mu^{n-2}(K_n))$. Therefore, c is indeed a proper coloring of $\mu^{n-2}(K_n)$.

Next we color the remaining vertices of $\mu^{n-1}(K_n) = \mu(\mu^{n-2}(K_n))$. All these vertices will be colored the same color as their twins in $\mu^{n-2}(K_n)$, except that $c(x_n^{2^{n-1}-1}) = c(u^{2^{n-2}+2^{n-3}}) = 2n-2$ and $c(u^{2^{n-2}}) = 1$.

To verify that c is a proper coloring of $\mu^{n-1}(K_n)$, it suffices to consider the three exceptionally colored vertices $x_n^{2^{n-1}-1}, u^{2^{n-2}+2^{n-3}}$, and $u^{2^{n-2}}$. The only other vertex with color 1 is $u^{2^{n-3}}$, which is not adjacent to $u^{2^{n-2}}$. The only other vertices of color $2n-2$ are those x_n^i with $0 \leqslant i < 2^{n-3}$, which are not adjacent to $x_n^{2^{n-1}-1}$ and $u^{2^{n-2}+2^{n-3}}$. Therefore, c is indeed a proper coloring of $\mu^{n-1}(K_n)$.

Next we show that $D_{\mathrm{c}}(\mu^{n-1}(K_n))$ is acyclic. Assume to the contrary that $D_{\mathrm{c}}(\mu^{n-1}(K_n))$ contains a directed cycle. Note that $x_n^{2^{n-2}-1}$ is the only vertex of color 0. We conclude that this cycle has a length of $2n-1$. It starts with $x_n^{2^{n-2}-1}$; and then $u^{2^{n-3}}$, which is the unique vertex of color 1 adjacent to $x_n^{2^{n-2}-1}$; and ends with $u^{2^{n-2}+2^{n-3}}$, which is the unique vertex of color $2n-2$ adjacent to $x_n^{2^{n-2}-1}$.

Let us call the vertices of this cycle $Y=(y_0, y_1, \ldots, y_{2n-2})$, where $c(y_i)=i$. Therefore, $y_0 = x_n^{2^{n-2}-1}, y_1 = u^{2^{n-3}}$, and $y_{2n-2} = u^{2^{n-2}+2^{n-3}}$.

Define $J_t^* = \{x_j^i\colon 0 \leqslant i < 2^{t-1}$ or $0 \leqslant i - 2^{n-2} < 2^{t-1}\}$ and $I_t^* = \{x_j^i : i \in I_t$ or $i - 2^{n-2} \in I_t\}$ for $1 \leqslant t \leqslant n-2$. It is easy to verify that for any $1 \leqslant t \leqslant n-2$ and $x_j^i x_{j'}^{i'} \in E(\mu^{n-1}(K_n))$, if $x_j^i \in I_t^*$, then $i \& i' = 0$ and so $x_{j'}^{i'} \in J_t^*$. Moreover, any vertex $x_j^i \in J_t^*$ is not adjacent to any vertex u^k colored by $2t+2$, since $0 \leqslant i$ (or $i - 2^{n-2}$) $< 2^{t-1}$ and $f(k) = 2^{t-1}$. Since $y_1 = u^{2^{n-3}}$, we may conclude that $y_2 \in I_1^*$. It then follows that $y_3 \in J_1^*, y_4 \in I_2^*, y_5 \in J_2^*, \ldots, y_{2n-4} \in I_{n-2}^*$, and $y_{2n-3} \in J_{n-2}^*$. Thus, y_{2n-3} is not adjacent to $y_{2n-2} = u^{2^{n-2}+2^{n-3}}$, contrary to the assumption that Y is a cycle. Therefore, $D_{\mathrm{c}}(\mu^{n-1}(K_n))$ is indeed acyclic. Since there is only one vertex colored with color 0, it follows from Lemma 4 that $\chi_{\mathrm{c}}(\mu^{n-1}(K_n)) \leqslant 2n-1-\frac{1}{2} = \chi(\mu^{n-1}(K_n)) - \frac{1}{2}$.

If G is an arbitrary n-chromatic graph, then there is a homomorphism from G to K_n. It follows that there is a homomorphism from $\mu^{n-1}(G)$ to $\mu^{n-1}(K_n)$. Therefore, $\chi_{\mathrm{c}}(\mu^{n-1}(G)) \leqslant \chi(\mu^{n-1}(K_n)) - \frac{1}{2} = \chi(\mu^{n-1}(G)) - \frac{1}{2}$. □

The following corollary follows easily from Theorems 7 and 9:

Corollary 10. *If G is an n-chromatic graph and t is a non-negative integer, then* $\chi_{\mathrm{c}}(\mu^{n-1+2t}(G)) \leqslant \chi(\mu^{n-1+2t}(G)) - \frac{1}{2}$.

By Lemma 1, if G is color-critical, then so is $\mu^m(G)$. It also follows from Lemma 1 that $\mu^m(G)$ has high connectivity. If G is triangle-free, then so is $\mu^m(G)$. Thus, it follows from Corollary 10 that there are triangle-free and color-critical graphs G of high connectivity for which $\chi_{\mathrm{c}}(G) \leqslant \chi(G) - \frac{1}{2}$ (for example, $\mu^{k-3}(C_5)$ is such a k-critical graph). This gives another proof of Theorem 4 in [2], which asserts that there exist k-critical $(k-1)$-connected triangle-free graphs G for which $\chi_{\mathrm{c}}(G) \leqslant k - \frac{1}{2}$.

4. Graphs G with $\chi_{\mathrm{c}}(\mu(G)) = \chi(\mu(G))$

This section investigates graphs G for which $\chi_{\mathrm{c}}(\mu(G)) = \chi(\mu(G))$. We first prove that the Mycielskian of any 3-chromatic graph has circular chromatic number 4.

Theorem 11. *If* $\chi(G) = 3$, *then* $\chi_c(\mu(G)) = \chi(\mu(G)) = 4$.

Proof. Suppose that, to the contrary, there is a 3-chromatic graph G for which $\chi_c(\mu(G)) < 4$. By Corollary 3, there is a 4-coloring c of $\mu(G)$ such that $D_c(\mu(G))$ is acyclic, $c(u) = 1$, $c(x') \notin \{0, 1\}$ for all $x' \in V'$; and there is an edge $xy \in E(G)$ such that $c(x) = 0, c(y) = 1$, and $c(x') = c(y')$. Assume that c is such a coloring with a least number of 0–1 edges (i.e., edges with two end vertices colored 0 and 1, respectively). Assume $c(x') = c(y') = 2$ (the case in which $c(x') = c(y') = 3$ is symmetric). Then $c(z) = 1$ for each $z \in N_G(x)$, otherwise $xyx'z$ is a directed 4-cycle in $D_c(\mu(G))$. For each $z \in N_G(x)$, if $c(z') = 3$, then $c(w) = 0$ for each $w \in N_G(z) \backslash \{x\}$, otherwise $xzwz'$ is a directed 4-cycle in $D_c(\mu(G))$. We re-color z' with color 2 for each $z \in N_G(x)$, and re-color x and x' with color 3. It is straightforward to verify that this new coloring c' is still a proper 4-coloring of $\mu(G)$ and that $D_{c'}(G)$ is acyclic. However c' has fewer 0–1 edges than c, contrary to the choice of c. $\square$

It was conjectured in [16] that triangle-free n-critical graphs have circular chromatic numbers strictly less than n. However, it follows from Lemma 1 and Theorem 11 that for $k \geqslant 2$, $\mu(C_{2k+1})$ is a triangle-free 4-critical graph that has the circular chromatic number 4. Therefore the conjecture fails for $n = 4$. We do not know whether the conjecture fails for any other integer n.

It was shown in [16] that n-critical graphs of 'large girth' have circular chromatic numbers 'close to' $n - 1$. However, it is unknown how large the girth of an n-critical graph G must be to guarantee that $\chi_c(G) < n$. For each integer n, let $g(n)$ be the minimum integer such that any n-critical graph of girth greater than $g(n)$ has $\chi_c(G) < n$. It follows from the corollary of Theorem 1 in [10] that $g(n) \leqslant n$. The above argument shows that $g(4) \geqslant 4$ and hence $g(4) = 4$. It is easy to show that $g(3) = 3$. The value of $g(n)$ is unknown for $n \geqslant 5$.

The next four results concern the Mycielskian of the complete graphs K_n. When $n=2$, $\mu(K_2)$ is the pentagon, and hence has circular chromatic number $\frac{5}{2}$. It follows from Theorem 11 that $\chi_c(\mu^2(K_2))=4$, and follows from Corollary 8 that $\chi_c(\mu^{2k+1}(K_2)) \leqslant 2k + 2 + \frac{1}{2}$. In the following we consider the case of $n \geqslant 3$.

Theorem 12. *If* $n \geqslant 3$, *then* $\chi_c(\mu(K_n)) = \chi(\mu(K_n)) = n + 1$.

Theorem 13. *If* $n \geqslant 4$, *then* $\chi_c(\mu^2(K_n)) = \chi(\mu^2(K_n)) = n + 2$.

To prove Theorems 12 and 13, it suffices to show that for any $(n + 1)$-coloring c of $\mu(K_n)$ and any $(n + 2)$-coloring c' of $\mu^2(K_n)$, the directed graphs $D_c(\mu(K_n))$ and $D_{c'}(\mu^2(K_n))$ contain directed cycles (see Lemma 2). However, we prove two stronger results that seem to be potentially useful for more general graphs.

First, we introduce notation. Suppose G is a graph and that $c : V(G) \mapsto \{0, 1, \ldots, k - 1\}$ is a proper coloring of G. Let $C = (x_1, x_2, \ldots, x_m)$ be a cycle of G. We say that C is *consistently colored* if there is an index i such that $c(x_i) < c(x_{i+1}) < \cdots < c(x_m) <$

$c(x_1) < \cdots < c(x_{i-1})$. We may view the colors as cyclically ordered, such that i precedes $i+1$ and $k-1$ precedes 0. Then a cycle C is consistently colored only if the colors of the vertices of C are in the same cyclic order as C. To be precise, for two colors i and j we let $[i,j]_k$ denote the set $\{i,i+1,i+2,\dots,j\}$, where addition is carried out modulo k. For example, $[2,5]_8=\{2,3,4,5\}$ and $[5,2]_8=\{5,6,7,0,1,2\}$. We let $(i,j)_k=[i,j]_k-\{i,j\}$ and $[i,j)_k=[i,j]_k-\{j\}$. Then a cycle $C=(x_1,x_2,\dots,x_m)$ is consistently colored if for any index $q\notin\{p,p+1\}$, $c(x_q)\notin[c(x_p),c(x_{p+1})]_k$.

It is trivial that for any proper coloring of K_n there is a consistently colored n-cycle. In the next two theorems, we show that for $n\geqslant 3$, every proper coloring of $\mu(K_n)$ has a consistently colored $(n+1)$-cycle; and for $n\geqslant 4$, every proper coloring of $\mu^2(K_n)$ has a consistently colored $(n+2)$-cycle.

Theorem 14. *If* $n\geqslant 3$ and $c:V(\mu(K_n))\mapsto\{0,1,\dots,k-1\}$ *is a proper coloring of* $\mu(K_n)$, *then there is a consistently colored* $(n+1)$-cycle.

Proof. The restriction of c to $V=V(K_n)$ has a consistently colored n-cycle which we assume is $C=(x_1^0,x_2^0,\dots,x_n^0)$. The colors $(c(x_1^0),c(x_2^0),\dots,c(x_n^0))$ form a cycle with respect to the cyclic order of the colors. For each j, we consider the color $c(x_j^1)$ (recall that x_j^1 is the twin of x_j^0). If $c(x_j^1)\in[c(x_i^0),c(x_{i+1}^0)]_k$ for some $i\notin\{j-1,j\}$, then we obtain an $(n+1)$-cycle $(x_1^0,x_2^0,\dots,x_i^0,x_j^1,x_{i+1}^0,\dots,x_n^0)$ which is consistently colored (note that x_j^1 is adjacent to both x_{i+1}^0 and x_i^0, hence $c(x_j^1)\in(c(x_i^0),c(x_{i+1}^0)))$.

Assume now that for each j, we have $c(x_j^1)\in(c(x_{j-1}^0),c(x_{j+1}^0))_k$. Let i be the index such that $c(u^1)\in[c(x_i^0),c(x_{i+1}^0))_k$. We now consider the relative positions of the colors $c(x_i^1)$, $c(x_{i+1}^1)$, and $c(u^1)$. If $c(u^1)\in(c(x_i^1),c(x_{i+1}^1))_k\subseteq(c(x_{i-1}^0),c(x_{i+2}^0))_k$, then $(x_1^0,x_2^0,\dots,x_{i-1}^0,x_i^1,u^1,x_{i+1}^1,x_{i+2}^0,\dots,x_n^0)$ is a consistently colored $(n+1)$-cycle. If $c(u^1)\in(c(x_{i+1}^1),c(x_{i+1}^0))_k\subseteq(c(x_{i+1}^1),c(x_{i+2}^1))_k$, then $(x_1^0,x_2^0,\dots,x_i^0,x_{i+1}^1,u^1,x_{i+2}^1,x_{i+3}^0,\dots,x_n^0)$ is a consistently colored $(n+1)$-cycle. Otherwise $c(u^1)\in[c(x_i^0),c(x_i^1))_k\subseteq(c(x_{i-1}^1),c(x_i^1))_k$, and in this case $(x_1^0,x_2^0,\dots,x_{i-2}^0,x_{i-1}^1,u^1,x_i^1,x_{i+1}^0,x_{i+2}^0,\dots,x_n^0)$ is a consistently colored $(n+1)$-cycle. $\square$

We note from the proof of Theorem 14 that there are two types of consistently colored $(n+1)$-cycles in $\mu(K_n)$ (see Fig. 2). Type I is an $(n+1)$-cycle $C_{\mathrm{I}}(i,j)=(x_1^0,\dots,x_i^0,x_j^1,x_{i+1}^0,\dots,x_n^0)$ obtained from the n-cycle $C=(x_1^0,x_2^0,\dots,x_n^0)$ in K_n by adding

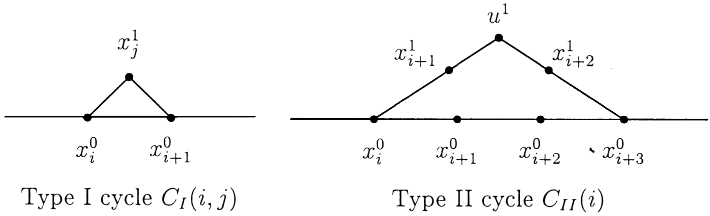

Fig. 2. Two types of $(n+1)$-cycles.

a vertex x_j^1 between x_i^0 and x_{i+1}^0. Type II is an $(n+1)$-cycle $C_{\mathrm{II}}(i)=(x_1^0,\ldots,x_i^0,x_{i+1}^1,u^1,x_{i+2}^1,x_{i+3}^0,\ldots,x_n^0)$ obtained from the n-cycle in K_n by replacing x_{i+1}^0,x_{i+2}^0 with x_{i+1}^1,u^1,x_{i+2}^1.

We call $\{x_i^0,x_{i+1}^0\}$ the *base* of a Type I cycle $C_{\mathrm{I}}(i,j)$ and $\{x_i^0,x_{i+1}^0,x_{i+2}^0,x_{i+3}^0\}$ the *base* of a Type II cycle $C_{\mathrm{II}}(i)$.

Theorem 15. *If $n\geqslant 4$ and $c : V(\mu^2(K_n)) \mapsto \{0,1,\ldots,k-1\}$ is a proper coloring of $\mu^2(K_n)$, then there is a consistently colored $(n+2)$-cycle.*

Proof. Recall that the vertex set of $\mu^2(K_n)$ is $(\bigcup_{i=0}^3 V_i)\cup\{u^1,u^2,u^3\}$, where $V_i=\{x^i\colon x\in V(K_n)\}$ for $0\leqslant i\leqslant 3$ (see the naming system introduced in the previous section and Fig. 1). Then we have five copies of $\mu(K_n)$ in $\mu^2(K_n)$ that are induced by the following vertex sets: $V_0\cup V_1\cup\{u^1\}$, $V_0\cup V_1\cup\{u^3\}$, $V_0\cup V_2\cup\{u^2\}$, $V_0\cup V_3\cup\{u^1\}$, $V_0\cup V_3\cup\{u^2\}$. By Theorem 14, the copy of $\mu(K_n)$ with vertex set $V_0\cup V_r\cup\{u^s\}$ has a consistently colored $(n+1)$-cycle $C_{r,s}$, which is either of Type I or of Type II.

Assume to the contrary that $\mu^2(K_n)$ has no consistently colored $(n+2)$-cycles. We consider the relative positions of the above five $(n+1)$-cycles $C_{r,s}$. First of all, the bases of any two $(n+1)$-cycles have at least two common vertices, otherwise their union would induce a consistently colored $(n+2)$-cycle. Therefore, for any two consistently colored $(n+1)$-cycles $C_{r,s}$ and $C_{r',s'}$, one of the following relative positions holds.

(i) $C_{r,s}=C_{\mathrm{I}}(i,j)$ and $C_{r',s'}=C_{\mathrm{I}}(i,j')$ (see Fig. 3).

(ii) $C_{r,s}=C_{\mathrm{II}}(i)$ and $C_{r',s'}=C_{\mathrm{I}}(i,j)$ or $C_{\mathrm{II}}(i)$ or $C_{\mathrm{I}}(i+2,j)$ or $C_{\mathrm{II}}(i+2)$ (see Fig. 4), or vice versa.

(iii) $C_{r,s}=C_{\mathrm{II}}(i)$ and $C_{r',s'}=C_{\mathrm{I}}(i+1,j)$ or $C_{\mathrm{II}}(i+1)$ (see Fig. 5), or vice versa.

Claim. *If u^s is adjacent to all vertices of $V_{r'}$ and $u^{s'}$ is adjacent to all vertices of V_r, then* (i) *or* (ii) *holds.*

Proof. Suppose to the contrary that (iii) holds. For the case in which $C_{r,s}=C_{\mathrm{II}}(i)$ and $C_{r',s'}=C_{\mathrm{I}}(i+1,j)$, since $x_j^{r'}$ is adjacent to u^s (i.e., the dashed line in Fig. 5 is an edge in $\mu^2(K_n)$), we have

$$(x_1^0,\ldots,x_{i+1}^0,x_j^{r'},u^s,x_{i+2}^r,x_{i+3}^0,\ldots,x_n^0) \quad \text{or} \quad (x_1^0,\ldots,x_i^0,x_{i+1}^r,u^s,x_j^{r'},x_{i+2}^0,\ldots,x_n^0)$$

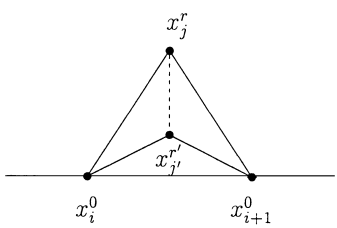

Fig. 3. Relative position (i).

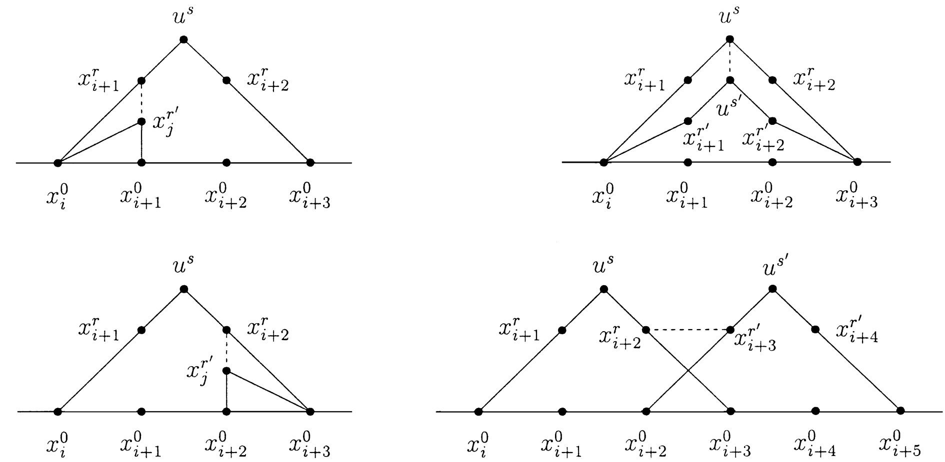

Fig. 4. Relative position (ii).

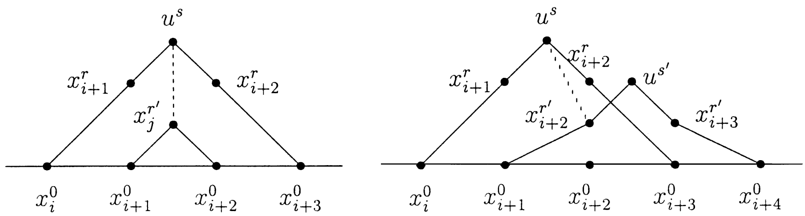

Fig. 5. Relative position (iii).

is a consistently colored $(n+2)$-cycle. Similarly, for the case in which $C_{r,s} = C_{\mathrm{II}}(i)$ and $C_{r',s'} = C_{\mathrm{II}}(i+1)$, $x^{r'}_{i+2}$ is adjacent to u^s and so,

$$(x^0_1,\dots,x^0_{i+1},x^{r'}_{i+2},u^s,x^r_{i+2},x^0_{i+3},\dots,x^0_n)$$
$$\text{or}\quad (x^0_1,\dots,x^0_i,xi+1^r,u^s,x^{r'}_{i+2},u^{s'},x^{r'}_{i+3},x^0_{i+4},\dots,x^0_n)$$

is a consistently ordered $(n+2)$-cycle. □

Note that if $C_{r,s}$ and $C_{r',s'}$ have relative position (i) or (ii), and that $C_{r',s'}$ and $C_{r'',s''}$ have relative position (i) or (ii), then $C_{r,s}$ and $C_{r'',s''}$ also have relative position (i) or (ii). In other words, two $(n+1)$-cycles having relative positions (i) or (ii) form an equivalence relation among the five $(n+1)$-cycles.

This fact along with an application of the above claim to the $(n+1)$-cycle pairs $(C_{2,2},C_{3,2})$, $(C_{3,2},C_{3,1})$, $(C_{3,1},C_{1,1})$, $(C_{1,1},C_{1,3})$ leads to the conclusion that $C_{2,2}$ and $C_{1,3}$ have relative positions (i) or (ii).

Suppose $C_{2,2}$ and $C_{1,3}$ have relative position (ii), say $C_{2,2}=C_{\mathrm{II}}(i)$ and $C_{1,3}=C_{\mathrm{I}}(i,j)$ or $C_{\mathrm{II}}(i)$ or $C_{\mathrm{I}}(i+2,j)$ or $C_{\mathrm{II}}(i+2)$. If $C_{2,2}=C_{\mathrm{II}}(i)$ and $C_{1,3}=C_{\mathrm{I}}(i,j)$, then x_j^1 is adjacent to x_{i+1}^2 and so

$$(x_1^0,\dots,x_i^0,x_j^1,x_{i+1}^2,u^2,x_{i+2}^2,x_{i+3}^0,\dots,x_n^0) \quad \text{or} \quad (x_1^0,\dots,x_i^0,x_{i+1}^2,x_j^1,x_{i+1}^0,\dots,x_n^0)$$

is a consistently colored $(n+2)$-cycle. If $C_{2,2}=C_{\mathrm{II}}(i)$ and $C_{1,3}=C_{\mathrm{II}}(i)$, then u^3 is adjacent to u^2 and so

$$(x_1^0,\dots,x_i^0,x_{i+1}^1,u^3,u^2,x_{i+2}^2,x_{i+3}^0,\dots,x_n^0)$$
$$\text{or} \quad (x_1^0,\dots,x_i^0,x_{i+1}^2,u^2,u^3,x_{i+2}^1,x_{i+3}^0,\dots,x_n^0)$$

is a consistently colored $(n+2)$-cycle. If $C_{2,2}=C_{\mathrm{II}}(i)$ and $C_{1,3}=C_{\mathrm{I}}(i+2,j)$, then x_j^1 is adjacent to x_{i+2}^2 and so

$$(x_1^0,\dots,x_{i+2}^0,x_j^1,x_{i+2}^2,x_{i+3}^0,\dots,x_n^0) \quad \text{or} \quad (x_1^0,\dots,x_i^0,x_{i+1}^2,u^2,x_{i+2}^2,x_j^1,x_{i+3}^0,\dots,x_n^0)$$

is a consistently colored $(n+2)$-cycle. If $C_{2,2}=C_{\mathrm{II}}(i)$ and $C_{1,3}=C_{\mathrm{II}}(i+2)$, then x_{i+3}^1 is adjacent to x_{i+2}^2 and so

$$(x_1^0,\dots,x_{i+2}^0,x_{i+3}^1,x_{i+2}^2,x_{i+3}^0,\dots,x_n^0)$$
$$\text{or} \quad (x_1^0,\dots,x_i^0,x_{i+1}^2,u^2,x_{i+2}^2,x_{i+3}^1,u^3,x_{i+4}^1,x_{i+5}^0,\dots,x_n^0)$$

is a consistently colored $(n+2)$-cycle.

Suppose $C_{2,2}$ and $C_{1,3}$ have relative position (i), say $C_{2,2}=C_{\mathrm{I}}(i,j)$ and $C_{1,3}=C_{\mathrm{I}}(i,j')$. Then $j=j'$, otherwise, x_j^2 is adjacent to $x_{j'}^1$ and so

$$(x_1^0,\dots,x_i^0,x_j^2,x_{j'}^1,x_{i+1}^0,\dots,x_n^0) \quad \text{or} \quad (x_1^0,\dots,x_i^0,x_{j'}^1,x_j^2,x_{i+1}^0,\dots,x_n^0)$$

is a consistently colored $(n+2)$-cycle. We may assume that both $V_0\cup V_2\cup\{u^2\}$ and $V_0\cup V_1\cup\{u^3\}$ have only one Type I $(n+1)$-cycle, for otherwise the union of two cycles with different bases is a consistently colored $(n+2)$-cycle, or we may choose j,j' so that $j\neq j'$, contrary to the conclusion above. Therefore $c(x_p^1)$, $c(x_p^2)\in(c(x_{p-1}^0),c(x_{p+1}^0))_k$ for each $p\neq j$. We may assume that none of $V_0\cup V_2\cup\{u^2\}$ and $V_0\cup V_1\cup\{u^3\}$ has Type II cycles, for otherwise we may choose $C_{2,2}$ and $C_{1,3}$ so that they have relative position (ii), which has been discussed in the previous paragraph. It follows that $c(u^2)\in(c(x_{j-1}^2),c(x_{j+1}^2))_k$ and $c(u^3)\in(c(x_{j-1}^1),c(x_{j+1}^1))_k$, (cf. the proof of Theorem 14). Note that $i\neq j-1,j$, without loss of generality, we may assume that $1\leqslant i\leqslant j-2<n$ and $c(u^2)\in(c(u^3),c(x_{j+2}^0))_k$. If $i<j-2$, then since u^3 is adjacent to u^2,

$$(x_1^0,\dots,x_i^0,x_j^1,x_{i+1}^0,\dots,x_{j-2}^0,x_{j-1}^1,u^3,u^2,x_{j+1}^2,x_{j+2}^0,\dots,x_n^0)$$

is a consistently colored $(n+2)$-cycle. If $i=j-2$, then either

$$(x_1^0,\dots,x_{j-2}^0,x_{j-1}^1,x_j^2,x_{j-1}^0,\dots,x_{j+2}^0,\dots,x_n^0)$$
$$\text{or} \quad (x_1^0,\dots,x_{j-2}^0,x_j^2,x_{j-1}^1,u^3,u^2,x_{j+1}^2,x_{j+2}^0,\dots,x_n^0)$$

is a consistently colored $(n+2)$-cycle.

We have omitted some details of the proof, which can be added easily. For example, we did not explicitly use the condition $n\geqslant 4$, which is a necessary condition. Indeed, when $n=4$, the fourth picture in Fig. 4 looks different. The vertices x^0_{i+4}, x^0_{i+5} are equal to x^0_i, x^0_{i+1}, respectively. A dashed line should be added between x^r_{i+1} and x^r_{i+4}. However, a consistently colored $(n+2)$-cycle can still be found in the corresponding cases. □

It follows from Theorems 14 and 15 that for any $(n+1)$-coloring c of $\mu(K_n)$ and any $(n+2)$-coloring c' of $\mu^2(K_n)$, the directed graphs $D_c(\mu(K_n))$ and $D_{c'}(\mu^2(K_n))$ contain directed cycles. Therefore, Theorems 12 and 13 have been proven.

We close this section with the following conjecture:

Conjecture 1. If $n \geqslant m+2$, then $\chi_c(\mu^m(K_n)) = \chi(\mu^m(K_n)) = n+m$.

5. Further research

We have established some results for circular chromatic numbers on Mycielski's graphs. However, many questions remain open. We list below some related questions.

Question 1. Given a graph G, what can we say about the sequence $(\chi(\mu^m(G)) - \chi_c(\mu^m(G))$: $m=1,2,\ldots)$? Does it approach a limit? What are the possible accumulating points of such a sequence?

It follows from Corollary 10 that there are infinitely many integers m for which $\chi(\mu^m(G)) - \chi_c(\mu^m(G)) \geqslant \frac{1}{2}$. We do not know whether there are infinitely many integers m for which $\chi(\mu^m(G)) - \chi_c(\mu^m(G)) < \frac{1}{2}$.

Question 2. What is $\chi_c(\mu^n(K_n))$?

We know that $\chi_c(\mu^2(K_2)) = \chi(\mu^2(K_2)) = 4$, but we do not know the value $\chi_c(\mu^n(K_n))$ for any other n.

Question 3. What determines whether $\chi_c(\mu(G)) = \chi(\mu(G))$?

We have many examples G for which $\chi_c(\mu(G)) = \chi(\mu(G))$, and also many examples G for which $\chi_c(\mu(G)) < \chi(\mu(G))$. However, it seems difficult to characterize those graphs G for which $\chi_c(\mu(G)) = \chi(\mu(G))$. For two integers k and d such that $k > 2d$, G^d_k is the graph with vertex set $\{0, 1, \ldots, k-1\}$ in which ij is an edge if and only if $d \leqslant |i-j| \leqslant k-d$. Vince [17] showed that $\chi_c(G^d_k) = \frac{k}{d}$. It is easy to prove (see [3]) that a graph G is (k,d)-colorable if and only if there exists a homomorphism from G to G^d_k. Therefore, in the study of circular chromatic numbers, graphs G^d_k play the role of complete graphs in the study of chromatic numbers. Theorem 12 says that for $n \geqslant 3$, $\chi_c(\mu(K_n)) = \chi(\mu(K_n))$. An interesting question is:

Question 4. Does $\chi_c(\mu(G^d_k)) = \chi(\mu(G^d_k))$?

Remark. Question 4 has now been answered in the affirmative in [13].

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