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Reliabilities for (n, f, k) systems ☆

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Abstract

The (n, f, k) system consists of n components ordered in a line or a cycle, while the system fails if, and only if, there exist at least f failed components or at least k consecutive failed components. For the linear (n, f, k) system with equal component reliabilities, the system reliability formula was given by Sun and Liao (1990). In this paper, we obtain the system reliability formulas for the linear and the circular systems with different component reliabilities by means of a Markov chain method.

Keywords: Consecutive system; Markov chain; (n, f, k) system; Operations; Reliability

1. Introduction

As the systems in real world become more and more complicated, the notion of multiple failure criteria for systems is more important. The (n, f, k) system is such an example. The (n, f, k) system consists of n components ordered in a line or a cycle, while the system fails if, and only if, there exist at least f failed components or at least k consecutive failed components. The concept of an (n, f, k) system was first raised by Tung (1982) in a slightly different way for an application to a complex system such as the infrared (IR) detecting and signal processing portion of a system. The IR system consists of 112 detector channels and 28 MUX cards. The failure criteria are the occurrence of any of the following conditions:

1. more than five dead or noisy channels,
2. three or more dead or noisy channels adjacent to one another,
3. one or more dead or noisy channels in the central 10% of the array.

Sun and Liao (1990) generalized Tung's failure model, with criterion (3) removed. They called it the (n, f, k) system (note that their definition of f is slightly different from ours). The (n, f, k) system becomes popular

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as it models many practical problems, such as automatic payment systems in banks (Sun and Liao, 1990), evaluation of reliabilities for furnace systems (Zuo and Wu, 1996).

The system reliability formula for the linear (n,f,k) system with equal component reliabilities was given by Sun and Liao (1990). The purpose of this paper is to present the system reliability formulas for the linear and the circular (n,f,k) system with different components reliabilities. We employ a Markov chain method for the solution. Numerical examples are illustrated.

2. Markov chain representation for (n,f,k) systems

As the (n,f,k) system becomes the well-known f-out-of-n:F system for the case of $f \leqslant k$, in this paper we only consider the case of $f > k$.

We first give the system reliability formula for the linear (n,f,k) system in which component i has a working probability p_i.

The Markov chain method was first employed by Fu (1986), Fu and Hu (1987), and subsequently by Chao and Fu (1989,1991) in the study of system reliabilities. (For historical interest, the term "finite Markov chain imbedding" was formally introduced by Fu and Koutras 1994.) They showed that many important systems, such as series system, standby systems, k-out-of-n systems, consecutive k-out-of-n:F systems, deterioration systems, and repair systems, can be embedded into a Markov chain $\{Y(t)\}$ defined on the state space $S=\{1,2,\ldots,N\}$ and the discrete index space $T=\{1,2,\ldots,n\}$ while the system fails if there exists t_0 (with $1 \leqslant t_0 \leqslant n$) such that $Y(t)=N$ for all $t_0 \leqslant t \leqslant n$.

For the (n,f,k) system with $f>k$, we define the state space for process $Y(t)$ as

$$S=\{(i,j)\colon 0 \leqslant i \leqslant k-1 \text{ and } i \leqslant j \leqslant f-1\} \cup \{s_N\},$$

where (i,j) indicates a working state in which the system $(1,2,\ldots,t)$ has failed last i components but the $(t-i)$th component working and the system $(1,2,\ldots,t)$ has j failed components, and s_N indicates the state in which the system fails. We may view s_N as a join state of failed sub-states (i,j) while either $k \leqslant i$ or $f \leqslant j$, there are

$$N=|S|=(2f-k+1)k/2+1$$

states.

For convenience, we re-label state (i,j), with $0 \leqslant i \leqslant k-1$ and $i \leqslant j \leqslant f-1$, as state $s_{(2f-i-1)i/2+j+1}$. In other words, we regard

- state $(0,0)$ as state s_1, state $(0,1)$ as state $s_2,\ldots,$ state $(0,f-1)$ as state s_f,
- state $(1,1)$ as state s_{f+1}, state $(1,2)$ as state $s_{f+2},\ldots,$ state $(1,f-1)$ as state s_{2f-1},
- state $(2,2)$ as state s_{2f}, state $(2,3)$ as state $s_{2f+1},\ldots,$ state $(2,f-1)$ state s_{3f-2},

$\vdots$

- state $(k-1,k-1)$ as state $s_{N-f+k-1}$, state $(k-1,k)$ as state $s_{N-f+k},\ldots,$ state $(k-1,f-1)$ as state s_{N-1}.

We say that $\{Y(t)\}$ is a Markov chain with transition matrix

$$\Lambda_t(n)=\begin{pmatrix} A^{(1)}_{f\times f} & B^{(1)}_{f\times(f-1)} & \mathbf{0} & \mathbf{0} & C^{(1)}_{f\times 1} \\ A^{(2)}_{(f-1)\times f} & \mathbf{0} & B^{(2)}_{(f-1)\times(f-2)} & \mathbf{0} & C^{(2)}_{(f-1)\times 1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A^{(k)}_{(f-k+1)\times f} & \mathbf{0} & \mathbf{0} & B^{(k)}_{(f-k+1)\times(f-k)} & C^{(k)}_{(f-k+1)\times 1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}_{N\times N},$$

where

$$A^{(i)}_{(f-i+1)\times f} = \begin{pmatrix} & & & i & & \\ 0 & \dots & 0 & p_t & & \\ & & & & \ddots & \\ & & & & & p_t \end{pmatrix} \quad \text{for } i = 1, 2, \dots, k,$$

$$B^{(i)}_{(f-i+1)\times(f-i)} = \begin{pmatrix} p_t & & \\ & \ddots & \\ & & p_t \\ & & 0 \end{pmatrix} \quad \text{for } i = 1, 2, \dots, k-1,$$

$$B^{(k)}_{(f-i+1)\times(f-k)} = \mathbf{0},$$

$$C^{(i)}_{(f-i+1)\times 1} = (\, 0 \ \dots \ 0 \ q_t \,)^{\mathrm{T}} \quad \text{for } i = 1, 2, \dots, k-1,$$

$$C^{(k)}_{(f-k+1)\times 1} = (\, q_t \ \dots \ q_t \,)^{\mathrm{T}}.$$

It is clear that $\{Y(t)\}$ is a Markov chain in which self-transitions for the states $s_1, s_2, \dots, s_f$ form the sub-matrix $A^{(1)}_{f\times f}$, the transitions $s_{f+1} \to s_2,\ s_{f+2} \to s_3, \dots, s_{2f-1} \to s_f$ form the submatrix $A^{(2)}_{(f-1)\times f}, \dots,$ the transitions $s_{N-f+k-1} \to s_k,\ s_{N-f+k} \to s_{k+1}, \dots, s_{N-1} \to s_f$ form the submatrix $A^{(k)}_{(f-k+1)\times f}$; the transitions $s_1 \to s_{f+1},\ s_2 \to s_{f+2}, \dots, s_{f-1} \to s_{2f-1}$ form the submatrix $B^{(1)}_{f\times(f-1)}$, the transitions $s_{f+1} \to s_{2f},\ s_{f+2} \to s_{2f+1}, \dots, s_{2f-2} \to s_{3f-3}$ form the submatrix $B^{(2)}_{(f-1)\times(f-2)}, \dots,$ the transitions $s_{N-2f+2k-3} \to s_{N-f+k-1},\ s_{N-2f+2k-2} \to s_{N-f+k}, \dots, s_{N-f+k-3} \to s_{N-2}$ form the submatrix $B^{(k-1)}_{(f-k)\times(f-k-1)}$; the transition $s_f \to s_N$ forms the submatrix $C^{(1)}_{f\times 1}$, the transition $s_{2f-1} \to s_N$ forms the submatrix $C^{(2)}_{(f-1)\times 1}, \dots,$ the transition $s_{N-f+k-2} \to s_N$ forms the submatrix $C^{(k-1)}_{(f-k)\times 1}$, the transitions $s_{N-f+k-1} \to s_N,\ s_{N-f+k} \to s_N, \dots, s_{N-1} \to s_N$ form the submatrix $C^{(k)}_{(f-k+1)\times 1}$; the transition $s_N \to s_N$ forms the submatrix 1.

We summarize the transition rules as follows.

1. Each i $(1 \leqslant i \leqslant f)$ has a self-transition and $\min\{i-1, k-1\} + 1$ inputs (including the self-transition).
2. Each j (for all j except the down state) has 2 outputs, since every component has two states – “working state” and “failed state”.

Thus, if we assume that the initial probabilities are $\pi_0 = (1, 0, \dots, 0)$, then the system reliability is

$$R_{\mathrm{L}}(n, f, k) = \pi_0 \prod_{t=1}^{n} \Lambda_t(n) U_0^{\mathrm{T}},$$

where $U_0 = (1, \dots, 1, 0)_{1\times N}$.

It takes N^2 multiplications and $(N-1)^2$ additions to calculate $\pi_0 \Lambda_1(n)$. If we treat both multiplication and addition as unit operations, then computing $\pi_0 \Lambda_1(n)$ costs $\mathrm{O}(N^2)$ operations. Thus it costs $\mathrm{O}(nN^2)$ operations to compute $\pi_0 \prod_{t=1}^{n} \Lambda_t(n) U_0^{\mathrm{T}}$.

Next, we consider the circular (n, f, k) system. For the system to work, the necessary condition is that the line must end with exactly i failed components for some $0 \leqslant i \leqslant k-1$. We treat each such case separately.

For example, consider the case of exactly i failed components, we will break the cycle between components $n-i$ and $n-i-1$, and treat the first $i+1$ components with fixed states as the initial state of a line with $n-i-1$ components. The initial state is the state (i,i), or $s_{if-i(i-1)/2+1}$, and the initial probability π_i is a vector with $p_{n-i}\prod_{m=0}^{i-1} q_{n-m}$ at position $if-i(i-1)/2+1$ and 0 elsewhere. Finally, we add up the reliabilities from various initial states to obtain

$$R_{\mathrm{C}}(n,f,k)=\sum_{i=0}^{k-1} p_{n-i}\prod_{m=0}^{i-1} q_{n-m}\pi_i \prod_{t=1}^{n-i-1} \Lambda_t(n-i-1)U_0^{\mathrm{T}}.$$

The computing of $p_{n-i}\prod_{m=0}^{i-1} q_{n-m}\pi_i\prod_{t=1}^{n-i-1}\Lambda_t(n-i-1)U_0^{\mathrm{T}}$ needs $\mathrm{O}(nN^2)$ operations. Hence, it needs $\mathrm{O}(knN^2)$ operations to compute $R_{\mathrm{C}}(n,f,k;\,p_j)$.

We conclude this section by noting that for the i.i.d. case, Hwang (1986) obtained

$$R_{\mathrm{C}}(n,f,k)=\sum_{j=0}^{f-1} N_{\mathrm{C}}(j,n,k)p^{n-j}q^{j},$$

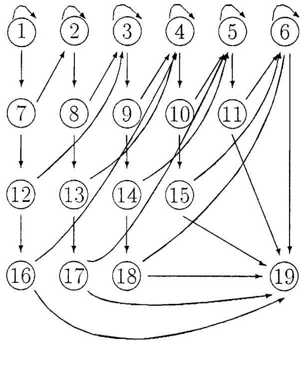
where

$$N_{\mathrm{C}}(j,n,k)=\frac{n}{n-j}\sum_{i\geqslant 0}(-1)^i\binom{n-j+1}{i}\binom{n-kj}{n-j}$$

is the number of ways of arranging j failed components and $n-j$ working components into a cycle that contains no k consecutive failed components.

3. Numerical examples

For a linear $(n,f,k)=(20,6,4)$ system, with component reliabilities $p_i=0.9+0.01i$ $(1\leqslant i\leqslant 10)$ and $p_j=0.55+0.02j$ $(11\leqslant j\leqslant 20)$, we have $N=19$ and the Markov chain transition graph is as follows:



The Markov transition matrix is

$$\Lambda_t(20) = \left(\begin{array}{cccccc|ccccc|cccc|ccc|c}
p_t &&&&&& q_t &&&&&&&&&&&& \\
& p_t &&&&&& q_t &&&&&&&&&&& \\
&& p_t &&&&&& q_t &&&&&&&&&& \\
&&& p_t &&&&&& q_t &&&&&&&&& \\
&&&& p_t &&&&&& q_t &&&&&&&& \\
&&&&& p_t &&&&& 0 &&&&&&&& q_t \\ \hline
& p_t &&&&&&&&&& q_t &&&&&&& \\
&& p_t &&&&&&&&&& q_t &&&&&& \\
&&& p_t &&&&&&&&&& q_t &&&&& \\
&&&& p_t &&&&&&&&&& q_t &&&& \\
&&&&& p_t &&&&&&&&& 0 &&&& q_t \\ \hline
&& p_t &&&&&&&&&&&&& q_t &&& \\
&&& p_t &&&&&&&&&&&&& q_t && \\
&&&& p_t &&&&&&&&&&&&& q_t & \\
&&&&& p_t &&&&&&&&&&&& 0 & q_t \\ \hline
&&& p_t &&&&&&&&&&&&&&& q_t \\
&&&& p_t &&&&&&&&&&&&&& q_t \\
&&&&& p_t &&&&&&&&&&&&& q_t \\ \hline
&&&&&&&&&&&&&&&&&& 1
\end{array}\right)$$

Using the Mathematica software, we get

$$R_{\mathrm{L}}(20,6,4) = \pi_0 \prod_{t=1}^{20} \Lambda_t(20) U_0^{\mathrm{T}} = 0.989292.$$

For a circular $(n, f, k) = (20, 6, 4)$ system with the same component reliabilities as the linear system, we get

$$R_{\mathrm{C}}(20,6,4;s_1) = 0.940946,$$

$$R_{\mathrm{C}}(20,6,4;s_7) = 0.0451895,$$

$$R_{\mathrm{C}}(20,6,4;s_{12}) = 0.00290479,$$

$$R_{\mathrm{C}}(20,6,4;s_{16}) = 0.000205044.$$

So the system reliability is

$$R_{\mathrm{C}}(20,6,4) = \sum_{i=0}^{3} R_{\mathrm{C}}(20,6,4;s_{6i-i(i-1)/2+1}) = 0.989245.$$

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