Characterizing Bit Permutation Networks

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Abstract: In recent years, many multistage interconnection networks using 2×2 switching elements have been proposed for parallel architectures. Typical examples are baseline networks, banyan networks, shuffle-exchange networks, and their inverses. As these networks are blocking, such networks with extra stages have also been studied extensively. These include Benes networks and $\Delta \oplus \Delta'$ networks. Recently, Hwang et al. studied k-extra-stage networks, which are a generalization of the above networks. They also investigated the equivalence issue among some of these networks. In this paper, we studied a more general class of networks, which we call $(m + 1)$-stage d-nary bit permutation networks. We characterize the equivalence of such networks by sequence of positive integers. Networks 33: 261–267, 1999

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1. INTRODUCTION

Consider a multistage interconnection network $\mathcal{V}$ with $N = d^{n+1}$ inputs and outputs and which has $m + 1$ stages of N/d crossbars of size $d \times d$. Let the jth crossbar in a stage be labeled by j in the d-nary number (with n bits). A *bit-i group* consists of those crossbars whose labels are identical except the ith bit. Such a group will be labeled by a d-nary number x with n bits which is identical to any member in the group except that bit i is replaced by the symbol x_0, which stands for the set $\{0, 1, \ldots, d - 1\}$. $\mathcal{V}$ will be called a *$(m + 1)$-stage d-nary bit permutation network* if the linking between stage k to stage $k + 1$ is always from a bit-i_k group G to a bit-j_k group G', where G' is a permutation of G, for $k = 0, 1, \ldots, m - 1$. For a detailed description and notation of bit permutation networks, see Section 2.

Note that $(n + 1)$-stage binary bit permutation networks include all self-routing networks like Omega, banyan, baseline, and their inverse networks. Binary bit permutation networks have been widely studied in the literature [1, 3, 6, 8, 11] for their topological equivalence. Bermond et al. [3] characterized the Omega-equivalent class by the $P(i, j)$ property. An $(n + 1)$-stage network satisfies the $P(i, j)$ property if the subnetwork from stage i to stage j has exactly 2^{n-j+i} components. Then, an $(n + 1)$-stage network with the unique path property is in the Omega-equivalent class if and only if it satisfies the $P(i, j)$ property for all $0 \le i \le j \le n$.

Another special class of bit permutation networks consists of $(n + 1)$-stage networks with extra stages. A k-extra-stage network is a cascade of a $(n + 1)$-stage network with k extra stages also satisfying the bit permutation linking pattern. Lea and Shyy [7, 9] proposed adding extra stages to a binary inverse banyan network while the

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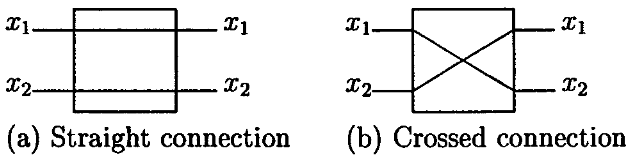

Fig. 1. A 2 × 2 switching element.

k extra stages are added by pattern F^{-1} (see below). Hwang et al. [5] generalized the study of equivalence by adding extra stages to a binary Omega-equivalent network with the following patterns for extra stages:

(i) F: They are identical to the first k stages of the network;
(ii) F^{-1}: Identical to the mirror image of the first k stages;
(iii) L: Identical to the last k stages;
(iv) L^{-1}: Identical to the mirror image of the last k stages.

In this paper, we determined the equivalence classes among all $(m + 1)$-stage d-nary bit permutation networks. We characterize such a network by an m-sequence over $\{1, 2, \ldots, n\}$, namely, every $(m + 1)$-stage d-nary bit permutation network is reduced to an m-sequence over $\{1, 2, \ldots, n\}$ and equivalence is determined by some easily computable sequence statistics. Note that the sequence is independent of d. For $m = n$, this characterization, of course, corresponds to the $P(i, j)$ characterization. But the sequence-graph correspondence is not in an obvious way. With the power and convenience of the sequence characterization, we easily give an explicit solution of the size of the s-stage bit permutation class. Recently, Hu et al. [4] gave an $O(N^4 \log N)$-time algorithm to check the equivalence of combined $(2n - 1)$-stage networks, which are obtained by cascading two Omega-equivalent networks. We give an (mn)-time algorithm for checking the equivalence of two $(m + 1)$-stage bit permutation networks. In particular, the running time is $O(\log^2 N)$ when the network has $2n - 1$ stages.

2. NETWORKS

We start the discussion of bit permutation networks by examining the following classical example: A typical Omega-equivalent network consists of N input terminals, N output terminals, and $\log_2 N$ columns (stages) of 2×2 switching elements in which each column has $N/2$ switching elements. Figure 1 shows a 2×2 switching element, and Figure 2 shows a baseline network with $N = 16$, in which a terminal i is represented by its binary number representation (x_3, x_2, x_1, x_0) and is adjacent to a switching element named by (x_3, x_2, x_1).

One can view the baseline network in Figure 2 as a graph whose vertices are those 32 switching elements named by $(x_3, x_2, x_1)_i$, where $0 \le i \le 3$ and $x_1, x_2, x_3 \in \{0, 1\}$, and there are links

from $(x_3, x_2, x_1)_0$ to $(x_0, x_3, x_2)_1$,
from $(x_3, x_2, x_1)_1$ to $(x_3, x_0, x_2)_2$, and
from $(x_3, x_2, x_1)_2$ to $(x_3, x_2, x_0)_3$,

where $x_0 \in \{0, 1\}$, meaning there are two links, one with $x_0 = 0$ and the other with $x_0 = 1$. A link from a switching element x at stage i to a switching element y at stage $i + 1$ exists if the bits of y can be obtained from the bits of x by a permutation depending only on i. For instance, we can represent the links from stage 0 to stage 1 by a permutation f_1 on $\{0, 1, 2, 3\}$ with

$$f_1(0) = 1, \quad f_1(1) = 2, \quad f_1(2) = 3, \quad f_1(3) = 0.$$

In this way, the links from stage 0 to stage 1 are those from $(x_3, x_2, x_1)_0$ to $(x_{f_1(3)}, x_{f_1(2)}, x_{f_1(1)})_1$. We can also say that for a link from $x = (x_3, x_2, x_1)_0$ to $y = (x_0, x_3, x_1)_1$ a coordinate x_j at the jth position of x moves to the $f_1^{-1}(j)$th position of y, where the coordinate x_0's moving from "outside" into y means there are two such links. Similarly, the following permutations f_2 and f_3 represent links from stage 1 to stage 2 and stage 2 to stage 3, respectively:

$$f_2(0) = 1, \quad f_2(1) = 2, \quad f_2(2) = 0, \quad f_2(3) = 3;$$
$$f_3(0) = 1, \quad f_3(1) = 0, \quad f_3(2) = 2, \quad f_3(3) = 3.$$

Throughout this paper, we shall use the *cycle* notation for

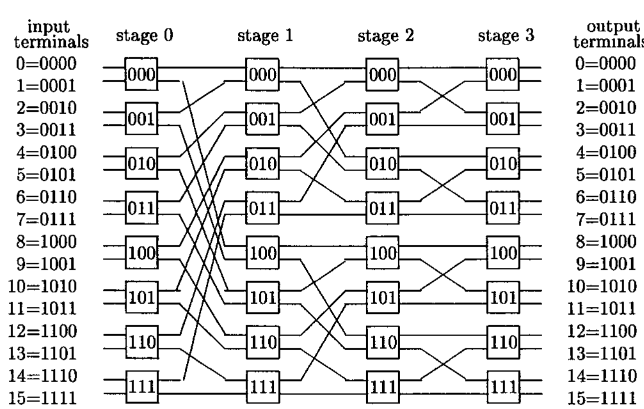

Fig. 2. A baseline network with $N = 16$.

permutations, that is, the cycle $(i_1, i_2, \ldots, i_n)$ represents the permutation f with

$$f(i_1) = i_2, \quad f(i_2) = i_3, \ldots, \quad f(i_{n-1}) = i_n, \quad f(i_n) = i_1, \quad \text{and } f(j) = j \text{ for all other } j.$$

Then, f_1 can be represented by $(0, 1, 2, 3)$, f_2 by $(0, 1, 2)$, and f_3 by $(0, 1)$.

Not only can the baseline network in Figure 2 be represented by the permutations f_i, but also all bit permutation networks can be represented in this way. We can also use general $d \times d$ switching elements instead of 2×2 switching elements. A general setting is as follows:

Definition. Suppose that n is a positive integer and $f_1, f_2, \ldots, f_m$ are $m \geq 0$ permutations on $\{0, 1, \ldots, n\}$ such that $f_i(0) \neq 0$ for $1 \leq i \leq m$. The $(m+1)$-*stage d-nary bit permutation network* $N_d(n, f_1, f_2, \ldots, f_m)$ is the network whose vertices are those $(x_n, x_{n-1}, \ldots, x_1)_i$ with $0 \leq i \leq m$ and $x_j \in \{0, 1, \ldots, d-1\}$ for $1 \leq j \leq n$, and each $(x_n, x_{n-1}, \ldots, x_1)_{i-1}$ is adjacent to $(x_{f_i(n)}, x_{f_i(n-1)}, \ldots, x_{f_i(1)})_i$, where $x_0 \in \{0, 1, \ldots, d-1\}$. In other words, there is an edge from $(x_n, x_{n-1}, \ldots, x_1)_{i-1}$ to $(y_n, y_{n-1}, \ldots, y_1)_i$ whenever $y_j = x_{f_i(j)}$ for $1 \leq j \leq n$.

The following examples show $(m+1)$-stage binary bit permutation networks with $N = 2^{n+1}$ input and output terminals. Examples 1–3 have been worked out by Wu and Feng [11]. We give the presentation in our format for easier use later.

Example 1. The *baseline network BL* is precisely $N_2(n, f_1, f_2, \ldots, f_n)$, with

$$f_i = (0, 1, \ldots, n-i+1) \quad \text{for } 1 \leq i \leq n.$$

The *inverse baseline network* BL^{-1} is $N_2(n, \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n)$, with

$$\bar{f}_i = f^{-1}_{n-i+1} = (i, i-1, \ldots, 1, 0) \quad \text{for } 1 \leq i \leq n.$$

Example 2. The *banyan network BY* or the *indirect binary n-cube network* is $N_2(n, f_1, f_2, \ldots, f_n)$, with

$$f_i = (0, i) \quad \text{for } 1 \leq i \leq n.$$

The *modified data manipulator* is $N_2(n, \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n)$, with

$$\bar{f}_i = f^{-1}_{n-i+1} = (0, n-i+1) \quad \text{for } 1 \leq i \leq n$$

and, hence, is the inverse banyan network BY^{-1}.

Example 3. The *Omega* (or *shuffle-exchange*) *network SE* is $N_2(n, f_1, f_2, \ldots, f_n)$, with

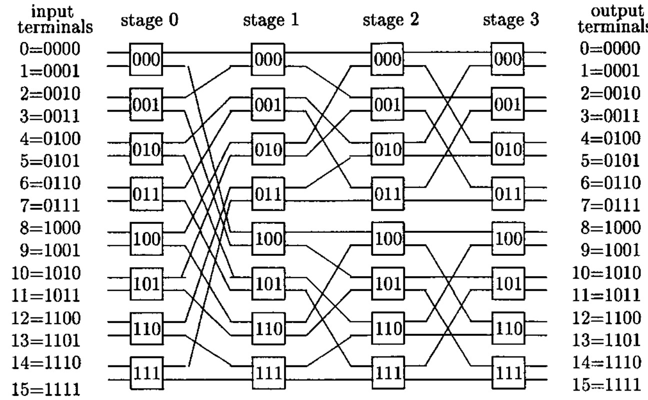

Fig. 3. The network in Example 4.

$$f_i = (n, n-1, \ldots, 1, 0) \quad \text{for } 1 \leq i \leq n.$$

The *flip* (or *inverse-shuffle-exchange*) *network* SE^{-1} is $N_2(n, \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n)$, with

$$\bar{f}_i = f_i^{-1} = (0, 1, \ldots, n) \quad \text{for } 1 \leq i \leq n.$$

Example 4. Interchange the role of the coordinates x_1 and x_2 of (x_3, x_2, x_1) in stages 2 and 3 of the baseline network in Figure 2. The resulting network, which is topologically equivalent to BL, is $N_2(3, f_1, f_2, f_3)$ with

$$f_1 = (0, 1, 2, 3), \quad f_2 = (0, 2, 1), \quad \text{and } f_3 = (0, 2)$$

(see Fig. 3). Note that if one interchanges the switching elements $(x_3, 0, 1)_i$ with $(x_3, 1, 0)_i$ for $i = 2, 3$ the resulting network is the baseline network.

Example 5. Suppose that $N_2(n, f_1, f_2, \ldots, f_n)$ is an Omega-equivalent network Δ and $N_2(n, f'_1, f'_2, \ldots, f'_n)$ is another Omega-equivalent network Δ'. Then, $\Delta \oplus \Delta'$ is $N_2(n, f_1, f_2, \ldots, f_n, f'_1, f'_2, \ldots, f'_n)$.

Example 6. Suppose that $N_2(n, f_1, f_2, \ldots, f_n)$ is an Omega-equivalent network Δ and $1 \leq k \leq n$. Hwang et al. [5] defined four k-extra-stage networks as follows: $\Delta_F(k)$ [respectively, $\Delta_L(k)$] is Δ together with k extra stages identical to the (respectively, last) first k stages of Δ, and $\Delta_{F^{-1}}(k)$ [respectively, $\Delta_{L^{-1}}(k)$] is Δ together with k extra stages identical to the mirror image of the first (respectively, last) k stages of Δ. Then,

$$\Delta_F(k) \text{ is } N_2(n, f_1, f_2, \ldots, f_n, f_1, f_2, \ldots, f_k);$$

$$\Delta_{F^{-1}}(k) \text{ is } N_2(n, f_1, f_2, \ldots, f_n, f_k^{-1}, f_{k-1}^{-1}, \ldots, f_1^{-1});$$

$$\Delta_L(k) \text{ is } N_2(n, f_1, f_2, \ldots, f_n, f_{n-k+1}, f_{n-k+2}, \ldots, f_n);$$

$\Delta_{L^{-1}}(k)$ is $N_2(n, f_1, f_2, \ldots, f_n, f_n^{-1}, f_{n-1}^{-1}, \ldots, f_{n-k+1}^{-1})$.

3. TOPOLOGICAL EQUIVALENCE

The main effort of this paper was to establish methods for determining if two bit permutation networks $N_d(n, f_1, f_2, \ldots, f_m)$ and $N_d(n, f'_1, f'_2, \ldots, f'_m)$ are topologically equivalent or graphically isomorphic. The following theorem is the foundation of our theory:

Theorem 1. *If there exist permutations* $g_0, g_1, \ldots, g_m$ *on* $\{0, 1, \ldots, n\}$ *such that* $g_i(0) = 0$ *for* $0 \le i \le m$ *and* $f'_i = g_{i-1}^{-1} \circ f_i \circ g_i$ *for* $1 \le i \le m$, *then* $N_d(n, f_1, f_2, \ldots, f_m)$ *is isomorphic to* $N_d(n, f'_1, f'_2, \ldots, f'_m)$.

Proof. Consider the bijection g from the vertex set of $N_d(n, f_1, f_2, \ldots, f_m)$ to the vertex set of $N_d(n, f'_1, f'_2, \ldots, f'_m)$ defined by

$$(x_n, x_{n-1}, \ldots, x_1)_i \xrightarrow{g} (x_{g_i(n)}, x_{g_i(n-1)}, \ldots, x_{g_i(1)})_i \quad \text{for } 0 \le i \le m.$$

In other words, $g((x_n, x_{n-1}, \ldots, x_1)_i) = (x'_n, x'_{n-1}, \ldots, x'_1)_i$ whenever $x'_j = x_{g_i(j)}$ for $1 \le j \le n$.

To see that these two networks are isomorphic, we only need to check that g is edge-preserving. Suppose that e is an edge from $(x_n, x_{n-1}, \ldots, x_1)_{i-1}$ to $(y_n, y_{n-1}, \ldots, y_1)_i$ in $N_d(n, f_1, f_2, \ldots, f_m)$, that is, $y_j = x_{f_i(j)}$ for $1 \le j \le n$. Let

$$g((x_n, x_{n-1}, \ldots, x_1)_{i-1}) = (x'_n, x'_{n-1}, \ldots, x'_1)_{i-1}, \quad \text{i.e.,} \quad x'_j = x_{g_{i-1}(j)} \quad \text{for } 1 \le j \le n,$$

and

$$g((y_n, y_{n-1}, \ldots, y_1)_i) = (y'_n, y'_{n-1}, \ldots, y'_1)_i, \quad \text{i.e.,} \quad y'_j = y_{g_i(j)} \quad \text{for } 1 \le j \le n.$$

Then,

$$y'_j = y_{g_i(j)} = x_{f_i \circ g_i(j)} = x_{g_{i-1} \circ f'_i(j)} = x'_{f'_i(j)} \quad \text{for } 1 \le j \le n.$$

Thus, there exists an edge from $(x'_n, x'_{n-1}, \ldots, x'_1)_{i-1}$ to $(y'_n, y'_{n-1}, \ldots, y'_1)_i$. Conversely, an edge in $N_d(n, f'_1, f'_2, \ldots, f'_m)$ also corresponds to an edge in $N_d(n, f_1, f_2, \ldots, f_m)$. ∎

As a quick application of Theorem 1, consider *BL* in Figure 2 as $N_2(3, f_1, f_2, f_3)$, with $f_1 = (0, 1, 2, 3)$, $f_2 = (0, 1, 2)$, and $f_3 = (0, 1)$ and the network in Example 4 as $N_2(3, f'_1, f'_2, f'_3)$, with $f'_1 = (0, 1, 2, 3)$, $f'_2 = (0, 2, 1)$, and $f'_3 = (0, 2)$. Then, $g_0 = (2, 3)$ and $g_1 = g_2 = g_3 = (1, 2)$ show that these two networks are isomorphic.

Theorem 2. *Every bit permutation network* $N_d(n, f_1, f_2, \ldots, f_m)$ *is isomorphic to some bit permutation network* $N_d(n, f'_1, f'_2, \ldots, f'_m)$, *where* $f'_i = (0, k_i)$ *with* $k_i \in \{1, 2, \ldots, n\}$ *for* $1 \le i \le m$.

Proof. We shall prove by induction on j the following claim which implies the theorem:

Claim(j). $N_d(n, f_1, f_2, \ldots, f_m)$ is isomorphic to some $N_d(n, f'_1, f'_2, \ldots, f'_m)$, where $f'_i = (0, k_i)$ with $k_i \in \{1, 2, \ldots, n\}$ for $1 \le i \le j$.

Claim(0) is clearly true. Suppose that Claim$(j-1)$ holds. Consider a general $j \ge 1$. Let $k_j = f'_j(0)$. Then, $k_j \in \{1, 2, \ldots, n\}$. Let $g_j = (f'_j)^{-1} \circ (0, k_j)$ and all other g_i are identity permutations. It is easy to check that $g_i(0) = 0$ for $0 \le i \le m$. According to Theorem 1, $N_d(n, f'_1, f'_2, \ldots, f'_m)$, and, therefore, $N_d(n, f_1, f_2, \ldots, f_m)$, is isomorphic to $N_d(n, f''_1, f''_2, \ldots, f''_m)$, where $f''_i = g_{i-1}^{-1} \circ f'_i \circ g_i$ for $1 \le i \le m$. In particular, $f''_i = f'_i = (0, k_i)$ for $1 \le i \le j-1$, and $f''_j = f'_j \circ g_j = f'_j \circ (f'_j)^{-1} \circ (0, k_j) = (0, k_j)$. This gives Claim$(j)$. ∎

So, basically, we only need to consider the networks with $f_i = (0, k_i)$. For convenience, we shall use $N_d(n, k_1, k_2, \ldots, k_m)$ as a short notation for the network $N_d(n, f_1, f_2, \ldots, f_m)$, with $f_i = (0, k_i)$ and $k_i \in \{1, 2, \ldots, n\}$ for $1 \le i \le m$.

Our next step is to determine when two networks are isomorphic.

Theorem 3. *If* g *is a permutation on* $\{0, 1, 2, \ldots, m\}$ *with* $g(0) = 0$, *then the network* $N_d(n, k_1, k_2, \ldots, k_m)$ *is isomorphic to* $N_d(n, g(k_1), g(k_2), \ldots, g(k_m))$.

Proof. The theorem following from Theorem 1 and the fact that $g \circ (0, k_i) \circ g^{-1} = (0, g(k_i))$. Note that we apply Theorem 1 by choosing all g_i as g. ∎

For any sequence $(k_1, k_2, \ldots, k_m)$ over $\{1, 2, \ldots, n\}$ with $a = |\{k_1, k_2, \ldots, k_m\}|$, let $1 = i_1 < i_2 < \cdots < i_a$ be those indices such that $k_{i_r} \notin \{k_1, k_2, \ldots, k_{i_r-1}\}$ for $1 \le r \le a$. Choose any permutation g on $\{1, 2, \ldots, n\}$ with the property that $g(k_{i_r}) = r$ for $1 \le r \le a$. Then, $(k_1^*, k_2^*, \ldots, k_m^*) = (g(k_1), g(k_2), \ldots, g(k_m))$ is a sequence with the property that $\{k_1^*, k_2^*, \cdots k_m^*\} = \{1, 2, \ldots, a\}$ and for any $1 \le r \le a$, $\{k_1^*, k_2^*, \ldots, k_{i_r}^*\} = \{1, 2, \ldots, r\}$ when i_r is the minimum index with $k^*_{i_r} = r$. Such a sequence is called a *canonical sequence* over $\{1, 2, \ldots, n\}$.

Corollary 4. *Any $N_d(n, f_1, f_2, \ldots, f_m)$ is isomorphic to $N_d(n, k_1, k_2, \ldots, k_m)$ for some canonical sequence over $\{1, 2, \ldots, n\}$.*

Theorem 5. *If $a = |\{k_1, k_2, \ldots, k_m\}|$, then $N_d(n, k_1, k_2, \ldots, k_m)$ has d^{n-a} connected components.*

Proof. First, we claim that for any two vertices $x = (x_n, x_{n-1}, \ldots, x_1)_i$ and $y = (y_n, y_{n-1}, y_{n-1}, \ldots, y_1)_j$ such that $x_k = y_k$ for all $k \not\in \{k_1, k_2, \ldots, k_m\}$ there exists a path joining them.

For the case in which $i = 0$ and $j = m$, $x = x^{(0)} \to x^{(1)} \to \cdots \to x^{(m)} = y$ is a desired path, where $x^{(i)}$ is a vertex in stage i that is obtained from $x^{(i-1)}$ by replacing its k_i-th coordinate with y_{k_i}. For the general case, consider the vertices $x^* = (x_n, x_{n-1}, \ldots, x_1)_0$ and $y^* = (y_n, y_{n-1}, \ldots, y_1)_m$. By the above construction, there exists an x^*–y^* path. It is clear that $(x_n, x_{n-1}, \ldots, x_1)_i \to (x_n, x_{n-1}, \ldots, x_1)_{i-1} \to \cdots \to (x_n, x_{n-1}, \ldots, x_1)_0 = x^*$ is an x–x^* path. Similarly, there exists a y^*–y path. Thus, x and y are joined by a path.

On the other hand, a move from any vertex to its neighbor never changes its k-th coordinate for $k \notin \{k_1, k_2, \ldots, k_m\}$. Thus, two vertices with different values at the k-th coordinate for some $k \notin \{k_1, k_2, \cdots k_m\}$ are not in the same connected component of the network. Therefore, the network has d^{n-a} connected components. ∎

Theorem 6. *For any two canonical sequences $(k_1, k_2, \ldots, k_m)$ and $(k'_1, k'_2, \ldots, k'_m)$ over $\{1, 2, \ldots, n\}$, the network $N_d(n, k_1, k_2, \ldots, k_m)$ is isomorphic to $N_d(n, k'_1, k'_2, \ldots, k'_m)$ if and only if $(k_1, k_2, \ldots, k_m) = (k'_1, k'_2, \ldots, k'_m)$.*

Proof. The two networks are clearly isomorphic if the two sequences are equal.

Conversely, suppose that $(k_1, k_2, \ldots, k_m) \neq (k'_1, k'_2, \ldots, k'_m)$. Let i be the minimum index such that $k_i \neq k'_i$, say $k_i < k'_i$ and $k_r = k'_r$ for $1 \leq r \leq i - 1$. Since the two sequences are canonical, $i \geq 2$. By the definition, there exists some $1 \leq r \leq i - 1$ such that $k_r = k'_r \in \{k_i, k'_i\}$. Choose a maximum such r. Then, $|\{k_r, k_{r+1}, \ldots, k_i\}| = a \neq a' = \{k'_r, k'_{r+1}, \ldots, k'_i\}$. Note that $N_d(n, k_r, k_{r+1}, \ldots, k_i)$ has d^{m-a} connected components while $N_d(n, k'_r, k'_{r+1}, \ldots, k'_i)$ has $d^{m-a'}$ connected components. Hence, the two networks are not isomorphic. ∎

Remark. By Theorem 6, the $N_d(n, k_1, k_2, \ldots, k_m)$ in Corollary 4 is unique. Such a network is called the *canonical representation* of $N_d(n, f_1, f_2, \ldots, f_m)$.

A condition slightly weaker than the $P(i, j)$ characterization follows from Theorem 6:

Corollary 7. *An $(n + 1)$-stage network with the unique path property and satisfying the $P(i, i + 1)$ property for $i = 0, 1, \ldots, n$, is in the Omega-equivalent class if and only if it also satisfies the $P(0, j)$ property for $j = 1, 2, \ldots, n$.*

Proof. By satisfying the $P(i, i + 1)$ property, the network is a bit permutation network. By satisfying the $P(0, j)$ property, the number of components is increasing in j, which implies that $k_j \neq k_i$ for all $i < j$. It follows that $(k_1, k_2, \ldots, k_n)$ are all distinct, but there is only one such canonical sequence. Corollary 7 follows from Theorem 6 immediately. ∎

Theorem 8. *The number of equivalent classes among $(m + 1)$-stage d-nary bit permutation networks is*

$$\sum_{t=1}^{n} \sum_{i=0}^{t} \frac{1}{t!} (-1)^{t-i} \binom{t}{i} i^m.$$

Proof. By Corollary 4 and Theorem 6, we only need to count the set $C(m, t)$ of canonical sequences of length m with t distinct elements for $1 \leq t \leq n$. Denote by $P(m, t)$ the set of all partitions of $\{1, 2, \ldots, m\}$ into t nonempty subsets. Define a mapping h from $C(m, t)$ to $P(m, t)$ by $h(k_1, k_2, \ldots, k_m) = \{C_1, C_2, \ldots, C_t\}$, where $C_i = \{j : k_j = i\}$. It is easy to see that h is one-to-one. On the other hand, for any partition $\{C_1, C_2, \ldots, C_t\}$ in $P(m, t)$, we may assume that $\min C_1 \leq \min C_2 \leq \cdots \leq \min C_t$. Let $k = (k_1, k_2, \ldots, k_m)$, where $k_i = j$ for $i \in C_j$. Then, $h(k) = \{C_1, C_2, \ldots, C_t\}$. So, h is onto. Therefore, $|C(m, t)| = |P(m, t)|$, which by definition is the Sterling number of the second kind $S(m, t)$ (see [10]). It is well known that

$$S(m, t) = \frac{1}{t!} \sum_{i=0}^{t} (-1)^{t-i} \binom{t}{i} i^m.$$

Summing over t, we obtain Theorem 8. ∎

4. ALGORITHM AND APPLICATIONS

By the theorems in Section 3, a bit permutation network $N_d(n, f_1, f_2, \ldots, f_m)$ is topologically equivalent to its canonical representation $N_d(n, k_1, k_2, \ldots, k_m)$, whose topology is determined by the canonical sequence $(k_1, k_2, \ldots, k_m)$. We shall summarize an efficient algorithm from the proofs of Theorems 2 and 3. We then apply it to determine the equivalence among the networks mentioned in the examples of Section 2.

Algorithm. Find the canonical representation of a bit permutation network.
Input. A bit permutation network $N_d(n, f_1, f_2, \ldots, f_m)$.
Output. The canonical sequence $(k_1^*, k_2^*, \ldots, k_m^*)$ for $N_d(n, f_1, f_2, \ldots, f_m)$.
Method.

```
f̄_0 = (0); f̄_{m+1} = (0);
for j = 1 to m do f̄_j = f_j;
for j = 1 to m do
    k_j = f̄_j(0);
    f̄_{j+1} = (0, k_j)∘f̄_j∘f̄_{j+1};
    end for;
for j = 1 to n do mark[j] = 0;
a = 0;
for j = 1 to m do
    if mark[k_j] = 0 then
        a = a + 1;
        mark[k_j] = a;
        k_j* = mark[k_j];
        end then;
    end for;
```

The time complexity of this algorithm is $O(mn)$.

Now, we use the above algorithm to get the canonical sequences of k-extra-stage Omega-equivalent networks.

Example 7. The following are the canonical sequences of $\Delta_F(k)$ for the six different Omega-equivalent networks Δ:

(a) For $\Delta = SE$, the sequence is $(1, 2, \ldots, m, 1, 2, \ldots, k)$.
(b) For $\Delta = SE^{-1}$, the sequence is $(1, 2, \ldots, m, 1, 2, \ldots, k)$.
(c) For $\Delta = BL$, the sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.
(d) For $\Delta = BL^{-1}$, the sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.
(e) For $\Delta = BY$, the sequence is $(1, 2, \ldots, m, 1, 2, \ldots, k)$.
(f) For $\Delta = BY^{-1}$, the sequence is $(1, 2, \ldots, m, 1, 2, \ldots, k)$.

In the type $\Delta_F(k)$, there are two nonequivalent classes.

Example 8. The following are the canonical sequences of $\Delta_{F^{-1}}(k)$ for the six different Omega-equivalent networks Δ.

(a) For $\Delta = SE$, the sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.
(b) For $\Delta = SE^{-1}$, the sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.
(c) For $\Delta = BL$, the sequence is $(1, 2, \ldots, m, k, k - 1, \ldots, 1)$.
(d) For $\Delta = BL^{-1}$, the sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.
(e) For $\Delta = BY$, the sequence is $(1, 2, \ldots, m, k, k - 1, \ldots, 1)$.
(f) For $\Delta = BY^{-1}$, the sequence is $(1, 2, \ldots, m, k, k - 1, \ldots, 1)$.

In the type $\Delta_{F^{-1}}(k)$, there are two nonequivalent classes.

Example 9. The following are the canonical sequences of $\Delta_L(k)$ for the six different Omega-equivalent networks Δ.

(a) For $\Delta = SE$, the sequence is $(1, 2, \ldots, m, 1, 2, \ldots, k)$.
(b) For $\Delta = SE^{-1}$, the sequence is $(1, 2, \ldots, m, 1, 2, \ldots, k)$.
(c) For $\Delta = BL$, the sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.
(d) For $\Delta = BL^{-1}$, the sequence is $(1, 2, \ldots, m, k, k - 1, \ldots, 1)$.
(e) For $\Delta = BY$, the sequence is $(1, 2, \ldots, m, m - k + 1, m - k + 2, \ldots, m)$.
(f) For $\Delta = BY^{-1}$, the sequence is $(1, 2, \ldots, m, m - k + 1, m - k + 2, \ldots, m)$.

In the type $\Delta_L(k)$, there are four nonequivalent classes.

Example 10. The following are the canonical sequences of $\Delta_{L^{-1}}(k)$ for the six different Omega-equivalent networks Δ:

(a) For $\Delta = SE$, the sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.
(b) For $\Delta = SE^{-1}$, the sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.
(c) For $\Delta = BL$, the sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.
(d) For $\Delta = BL^{-1}$, the canonical sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.
(e) For $\Delta = BY$, the sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.
(f) For $\Delta = BY^{-1}$, the sequence is $(1, 2, \ldots, m, m, m - 1, \ldots, m - k + 1)$.

In the type $\Delta_{L^{-1}}(k)$, there is one equivalent class.

If $k = m$, then the k-extra-stage networks have only two nonequivalent classes among $\Delta_F(k)$, $\Delta_{F^{-1}}(k)$, $\Delta_L(k)$, and $\Delta_{L^{-1}}(k)$, namely, one characterized by sequence $(1, 2, \ldots, m, 1, 2, \ldots, m)$, and the other, by $(1, 2, \ldots, m, m, m-1, \ldots, 1)$. Since the Benes network $BL_{L^{-1}}(n)$ is in the second class, networks in the second class are rearrangable. Benes [2] conjectured that $SE_F(n)$ is rearrangeable. If the conjecture is true, then the networks in the first class are also rearrangeable.

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