Extended Generalized Shuffle Networks: Sufficient Conditions for Strictly Nonblocking Operation

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Received August 1997; accepted October 1997

Abstract: Since Clos gave the first construction of a strictly nonblocking multistage interconnection network, only a few other constructions have been proposed in almost a half-century. In this paper, we introduce a constructive class of networks which utilizes crossbars of virtually any size and for which the sizes can vary from stage to stage. The interconnection between stages is a generalized shuffle pattern. We derive sufficient conditions for strictly nonblocking operation and suggest the potential for wide application of these networks by providing several special case results.
Networks 33: 269–291, 1999

1. INTRODUCTION

This work was initially motivated during the process of attempting to design efficient photonic nonblocking switching networks. Various photonic switching constraints suggested a need for flexible design capabilities. It became apparent that it would be useful to be able to design networks composed of various numbers of stages of diverse functionality switching modules of (nearly) arbitrary sizes and shapes.

This led to the development of a broad class of networks, which we call extended generalized shuffle (EGS) networks, whose members demonstrate the diversity just mentioned while at the same time satisfy a simple interstage interconnection rule. (Formal definitions of these networks and other terminology used in this introduction follow in Section 2.) The generality of EGS networks is such that they include all three-stage Clos networks [2] and all the so-called baseline networks [5] (Omega, Banyan, etc.). Also, although photonic switching constraints prompted this work, our results are technology independent and, thus, EGS networks should in no way be considered as being limited to photonic applications.

EGS networks can exhibit numerous interesting attributes including multistage modularity, fault tolerance, and elegant path hunting and control algorithms. In this paper, we primarily focus on sufficient conditions for which EGS networks are strictly nonblocking for point-to-point connections. As will be presently seen, this is a rather large undertaking in itself and will serve to introduce the reader to some fundamental properties of EGS networks.

The results concerning nonblocking networks are theoretically interesting and, in addition, they serve as points of departure for the design of low probability blocking networks. This is important because strictly nonblocking networks are seldom implemented in practice due to cost and performance considerations. One interesting result to be presented is that the expressions that give the conditions for nonblocking operation are global as opposed to being specific to particular switching modules, that is,

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 CCC 0028-3045/99/040269-23

nowhere in these expressions does there exist the ratio of the number of inlets to the number of outlets of any switching module in the network. The implication is that for the most part nonblocking networks can be designed with arbitrarily sized switching modules.

The nonblocking conditions to be derived will be given in general terms of the number of inlets and outlets on the network, the number of inlets and the number of outlets on the switching modules in each stage of the network, and the number of stages in the network. For academic comparison, we show that a strictly nonblocking EGS network with N inlets and N outlets can be constructed with $O(N(\log N)^2)$ crosspoints. This is asymptotically as good as any construction we know of [1], but, perhaps, a more important consideration is whether practically sized EGS networks can be efficiently and conveniently constructed for low (but nonzero) probabilities of blocking while meeting the constraints of the technology being used. We think that the answer to this question is usually yes. In the next section, we formalize via definitions much of what has been casually mentioned in this Introduction.

Some aspects of our approach may look similar to that of Lea and Shyy [3, 4], but there are a number of important fundamental differences. First of all, their construction applies only to k-extra-stage, $0 \leq k \leq \log_2 N - 1$, networks using 2×2 crossbars in which the first N stages are a Banyan network and the last k stages are a mirror image of the first k stages (and other networks isomorphic to these). Our construction imposes no constraints on either the size of the crossbars or the number of stages. In addition, we employ the same general interconnection definition for all stages. Consequently, the channel graphs of their networks have a special structure and are relatively easy to analyze. Our channel graphs are very general and thus demand a different and more comprehensive analysis.

2. TERMINOLOGY AND DEFINITIONS

As noted in the Introduction, this work was motivated by constraints imposed by photonic technology on the design of switching networks. We begin by considering the switching modules from which a switching network is constructed, followed by the logical association and interconnection of these modules. We then discuss strictly nonblocking operation, including the concepts of paths and blocked paths.

2.1. Nonblocking Operation

We are interested in networks that are *strictly nonblocking* for *point-to-point connections.* A point-to-point connection is one that connects a single inlet and a single outlet. (This is distinguished, e.g., from a multipoint connection in which a single inlet may be connected to one or more outlets simultaneously.) A network which allows only point-to-point connections is said to be strictly nonblocking if any idle inlet can be connected to any idle outlet, regardless of the other point-to-point connections existing in the network. Unless otherwise indicated, from this point on, we will refer to such networks simply as nonblocking networks.

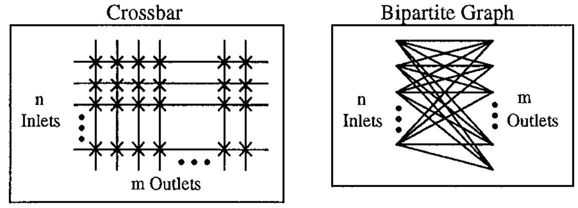

Fig. 1. Switching module representations.

2.2. Switching Modules

A switching module is defined as a device having two sets of terminals denoted inlets and outlets, plus a set of operational states such that for every inlet/outlet pair there exists at least one operational state in which that inlet/outlet pair are connected and at least one operational state in which that inlet/outlet pair are not connected. No operational states connect inlets to inlets or outlets to outlets. Furthermore, an operational state does not necessarily isolate connected inlet/outlet pairs, that is, a given inlet can be simultaneously connected to more than one outlet, and at the same time, a given outlet can be simultaneously connected to more than one inlet.

Thus, by our definition, a switching module is a device that has the capability to connect and disconnect all inlet/outlet pairs but not necessarily independently. In a subsequent paper, we will consider various functional types of switching modules, corresponding to a different connect/disconnect capability. However, in this paper, we limit our consideration to that of conventional ''crossbar''-type switching modules.

Consider an $n \times m$ switching module (n inlets and m outlets) represented equivalently by either an $n \times m$ crossbar or a complete bipartite graph having two vertex sets corresponding to the n inlets and m outlets, respectively, and the nm edges corresponding to the nm crosspoints in the $n \times m$ crossbar. An existing edge in the bipartite graph corresponds to a closed crosspoint in the crossbar, and a removed edge in the bipartite graph corresponds to an open crosspoint in the crossbar (see Fig. 1).

In this type of switching module, crosspoints may be opened or closed individually. This is the normal assumption for a crossbar switch and results in a nonblocking module since any inlet/outlet pair can be connected and at the same time isolated from any other inlet or outlet by requiring that the only closed crosspoint in any row

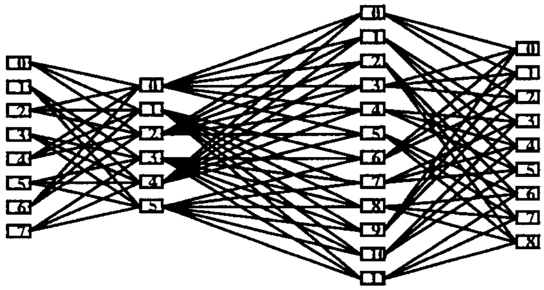

Fig. 2. EGS network η.

or column is the one at the intersection of the inlet row and outlet column of the inlet/outlet pair to be connected.

2.3. Multistage Interconnection Network (MIN)

A multistage interconnection network (MIN) is an interconnection of *stages* of switching modules. A *stage* is a set of identical switching modules. Let S_i, $i = 1, \ldots, s$, denote the ith stage of an s-stage MIN, where S_i contains r_i modules, each having n_i inlets and m_i outlets. The $N = r_1 n_1$ inlets of the switching modules of S_1 are the N inlets of the MIN and the $M = r_s m_s$ outlets of the switching modules of S_s are the M outlets of the MIN. For $i = 2, 3, \ldots, s$, the inlets of the switching modules of S_i are connected by links *only* to outlets of the switching modules of S_{i-1}, and for $i = 1, 2, \ldots, s - 1$, the outlets of the switching modules of S_i are connected by links *only* to inlets of the switching modules of S_{i+1}. So that all of these inlets and outlets can be connected, we require that $r_i m_i = r_{i+1} n_{i+1}$, for $1 \le i \le s - 1$.

2.4. Extended Generalized Shuffle (EGS) Network

An EGS network is simply a MIN with a particularly specified interconnection pattern. Formally, an EGS network is defined as a MIN in which, for $i = 1, 2, \ldots, s$, S_i = the integer set $\{0, 1, \ldots, r_i - 1\}$, and for $i = 1, 2, \ldots, s - 1$, the switching module $\alpha \in S_i$ is connected to switching module $\beta \in S_{i+1}$ if and only if $\beta \in \{[\alpha m_i + o_i]_{\mathrm{mod}\, r_{i+1}} : o_i \in \{0, 1, \ldots, m_i - 1\}\}$. (Our results will obviously apply to any isomorphic networks.)

Figures 2 and 3 depict two isomorphic EGS networks. The only difference between the two is that in Figure 2 all switching modules in each stage are positioned in numerical order and in Figure 3 they are not.

2.5. Paths

Define L_k, $k = 1, 2, \ldots, s - 1$, to be the set of links connecting the outlets of S_k to the inlets of S_{k+1}. We denote $\lambda \in L_k$ as a *stage-k link* and say that λ is *incident* to both the stage-k and $k + 1$ switching modules that it connects. For $1 \le i < j \le s$, let $w \in S_i$ and $z \in S_j$. A set of $j - i$ links, one from each L_k, $i \le k < j$, is said to satisfy the *$w \sim z$ chain condition* if the stage-i link is incident to w; for $k = i, i + 1, \ldots, j - 2$, the stage-$k$ and -$k + 1$ links are incident to the same stage-$k + 1$ switching module, and the stage-$j - 1$ link is incident to z. A *path* between w and z is defined as a set of $j - i$ links that satisfies the *$w \sim z$ chain condition*, plus the switching modules that these links connect.

Any two nonidentical sets of $j - i$ links (i.e., two sets differing by at least one link) that satisfy the $w \sim z$ chain condition comprise two different paths between w and z. Thus, different paths between a given pair of switching modules may have some, but not all, links in common and, of course, they may also have no links in common. The same is true for different paths between different pairs of switching modules. The total number P of paths between w and z is equal to the number of $j - i$-link sets satisfying the $w \sim z$ chain condition, where no two of these sets are identical.

For network inlet x appearing on a stage-1 switching module w and for network outlet y appearing on the stage-s switching module z, we say that a set of $s - 1$ links satisfying the $w \sim z$ chain condition comprises a path between x and y as well as a path between w and z. The *channel graph* $L(x, y)$ of input x and output y is defined as the union of all paths between x and y.

2.6. Blocked Paths

A connection from inlet x to outlet y via some path p is established by operating the appropriate crosspoint in each of the s switching modules (one for each stage) that successively connects x, the $s - 1$ links of p, and y. Any link so involved in a connection is said to be *busy*. A *blocked path* between a given input/output pair is one that contains at least one link that is busy from some other connection. A path is *idle* if it contains no such busy links.

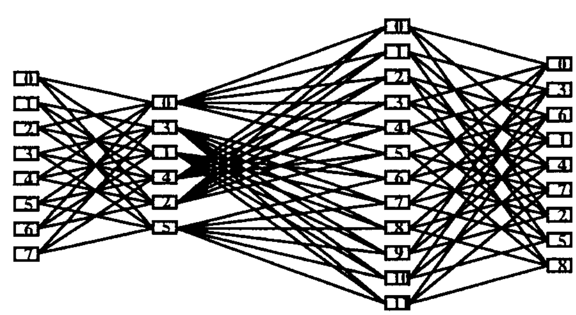

Fig. 3. EGS network isomorphic to network η.

3. THE FUNDAMENTAL PRINCIPLE OF STRICTLY NONBLOCKING NETWORKS

In Section 2.1, we defined a strictly nonblocking network as one in which any idle inlet/outlet pair can be connected regardless of the other existing connections in the network. In terms of paths, this means that there always exists at least one idle path between any idle input/output pair. Since the number of idle paths between an idle input/output pair is simply the total number of paths between that pair minus the number of blocked paths between that pair, we have the following condition which represents the fundamental principle of strictly nonblocking networks:

Strictly Nonblocking Necessary Condition:
The number of paths between any inlet/outlet pair must exceed the maximum number of paths which can be blocked between that pair.

If the numbers of paths between all inlet/outlet pairs differ by only a small amount and if the maximum numbers of paths which can be blocked between all inlet/outlet pairs also differ by only a small amount, then the following condition proves useful:

Strictly Nonblocking Sufficient Condition:
The minimum number of paths between any inlet/outlet pair exceeds the maximum number of paths which can be blocked between any inlet/outlet pair.

For the sake of brevity, we define

PATHS = the minimum number of paths between any inlet/outlet pair.
BLOCKED PATHS = the maximum number of paths which can be blocked between any inlet/outlet pair,

which yields

Strictly Nonblocking Sufficient Condition:
PATHS > BLOCKED PATHS.

Thus, if we can count (determine expressions for) both PATHS and BLOCKED PATHS for EGS networks, we will have ascertained generalized sufficient conditions for these networks to be nonblocking.

The logic to determine PATHS is relatively straightforward but the logic to determine BLOCKED PATHS is multileveled and rather intricate. To aid the reader in following this process, we provide a logic map (Fig. 4). We will continually be referring to this map throughout most of the rest of this paper. The various logic modules on the map are labeled identically with corresponding results from the text for ease of reference. The arrows indicate flows of logic progression. Thus, a given logic module can only be considered after providing results for the modules which direct arrows to the module in question. Our ultimate objective is to provide results for the row of modules second from the bottom, that is, modules (4.2a), (9.1a), (9.1b), and (9.1c), because these are the components of PATHS > BLOCKED PATHS.

The results for module (2.3) and the two modules (2.4) have already been provided via the definitions of a MIN and an EGS network in Section 2. We next consider modules (4.1i) and (4.2a) to determine a lower bound on the number of paths between any inlet/outlet pair.

4. A LOWER BOUND ON PATHS

While developing an expression for a lower bound on PATHS, we will find it useful to provide a more general result on the number of paths between switching modules in any two stages of an EGS network.

4.1. Numbers of Paths Between Switching Modules in Different Stages

We will need the following preliminary results:

Definition 4.1a. $\lfloor x \rfloor$ denotes the largest integer $\leq x$ and is called the *floor function* of x. $\lceil x \rceil$ denotes the smallest integer $\geq x$ and is called the *ceiling function* of x.

Definition 4.1b. $x_{\text{mod}m}$ denotes the smallest nonnegative remainder when dividing nonnegative integer x by positive integer m. The equivalent mathematical formulation is $x_{\text{mod}m} \equiv x - m\lfloor x/m \rfloor$.

Lemma 4.1c. For nonnegative integers w and x and positive integers y and z, if z divides wy, then $(w(x_{\text{mod}y}))_{\text{mod}z} = (wx)_{\text{mod}z}$.

Proof. For integers a and b with $0 \leq a$ and $0 \leq b \leq y - 1$, write $x = ay + b$. Then, $(w(x_{\text{mod}y}))_{\text{mod}z} = (wb)_{\text{mod}z}$. Also, $(wx)_{\text{mod}z} = (way + wb)_{\text{mod}z} = (wb)_{\text{mod}z}$, because z divides wy. ■

Lemma 4.1d. For any s consecutive integers and some positive integer m, if each integer d of the s integers is replaced with m consecutive integers, given by md, $md + 1, \ldots, md + m - 1$, the resulting ms integers are consecutive.

Proof. Consider any two consecutive integers, d and $d + 1$, in the original group of s. The smallest integer replacing $d + 1$ is $md + m$ which is consecutive with $md + m - 1$, the largest integer replacing d. ■

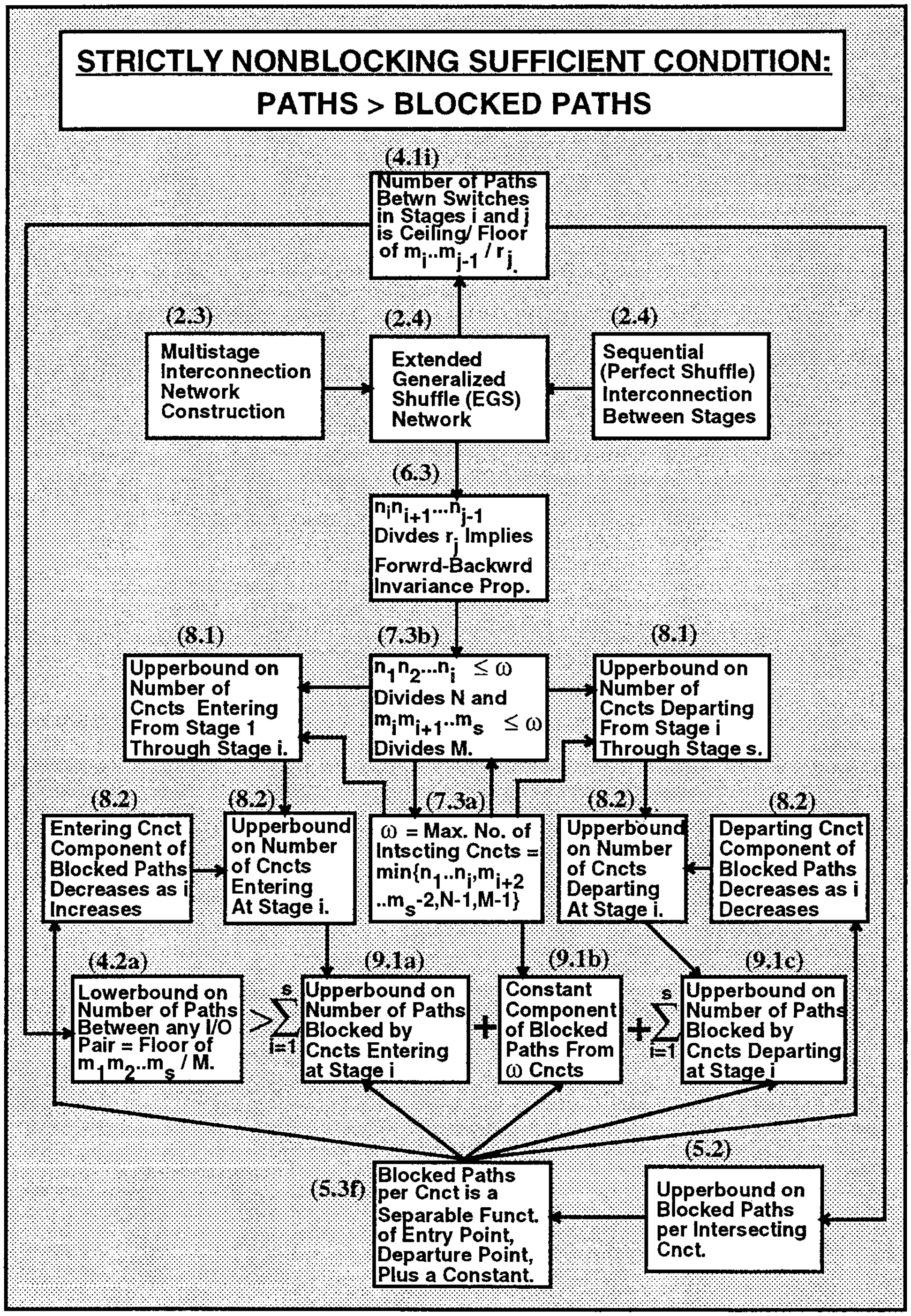

Fig. 4. The logic map for strictly nonblocking EGS networks.

Definition 4.1e. Define $N_{i,j} \equiv \prod_{k=i}^{j} n_k$ and $M_{i,j} \equiv \prod_{k=i}^{j} m_k$.

Lemma 4.1f. In a MIN, for $1 \leq i < j \leq s$, $r_i M_{i,j-1} = r_j N_{i+1,j}$.

Proof. By induction on the stage number. According to the definition of a MIN, $r_i m_i = r_{i+1} n_{i+1}$ for $1 \leq i \leq s - 1$ and, thus, the lemma is true for $j = i + 1$. We now show that if it is true for $j = i + t$ it is true for $j = i + t + 1$.

We assume that $r_i M_{i,i+t-1} = r_{i+t} N_{i+1,i+t}$. Again, by the definition of a MIN, we have that $r_{i+t} m_{i+t} = r_{i+t+1} n_{i+t+1}$ or $m_{i+t} = (r_{i+t+1} n_{i+t+1})/r_{i+t}$. Thus, we have $r_i M_{i,i+t-1} m_{i+t} = r_i M_{i,i+t} = (r_{i+t} N_{i+1,i+t} r_{i+t+1} n_{i+t+1})/r_{i+t} = r_{i+t+1} N_{i+1,i+t+1}$, which proves the lemma. ∎

Definition 4.1g. For positive integer k, $I(k)$ denotes the integer set $\{0, 1, \ldots, k-1\}$.

Lemma 4.1h. In an EGS network, for $1 \le i < j \le s$, switching module w in stage i has paths to a multiset $F_{i,j}(w)$ of switching modules in stage j given by

$$F_{i,j}(w) = \{(wM_{i,j-1} + d)_{\mathrm{mod} r_j} : d \in I(M_{i,j-1})\}.$$

Proof. By induction on the stage number. By definition of an EGS network, switching module w in stage i is connected (has paths) to a multiset $F_{i,i+1}(w)$ of switching modules in stage $i+1$ given by $F_{i,i+1}(w) = \{(wm_i + o_i)_{\mathrm{mod} r_{i+1}} : o_i \in I(m_i)\}$. Thus, the lemma is true for $j = i + 1$. We now show that if it is true for stage $j = i + t$ it is true for stage $j = i + t + 1$.

We assume that $F_{i,i+t}(w) = \{(wM_{i,i+t-1} + d)_{\mathrm{mod} r_{i+t}} : d \in I(M_{i,i+t-1})\}$. By definition of an EGS network, each of the switching modules in this multiset will have paths to m_{i+t} switching modules in stage $i + t + 1$, resulting in a multiset in that stage given by

$$\begin{aligned}\{[m_{i+t}(wM_{i,i+t-1} + d)_{\mathrm{mod} r_{i+t}} + o_{i+t}]_{\mathrm{mod} r_{i+t+1}} \\ : d \in I(M_{i,i+t-1}),\ o_{i+t} \in I(m_{i+t})\} \\ = \{[wM_{i,i+t} + m_{i+t}d + o_{i+t}]_{\mathrm{mod} r_{i+t+1}} \\ : d \in I(M_{i,i+t-1}),\ o_{i+t} \in I(m_{i+t})\},\end{aligned}$$

by Definition 4.1e and Lemma 4.1c (which applies because r_{i+t+1} divides $m_{i+t}r_{i+t}$ according to the definition of a MIN in Section 2.3). Now, the operation $m_{i+t}d + o_{i+t}$ is the same as that described in Lemma 4.1d. Thus, that lemma applies and the smallest integer in the resulting $m_{i+t}M_{i,i+t-1} = M_{i,i+t}$ consecutive integer set is zero, occurring when d and o_{i+t} are both zero. Our multiset thus becomes $\{[wM_{i,i+t} + d]_{\mathrm{mod} r_{i+t+1}} : d \in I(M_{i,i+t})\} = F_{i,i+t+1}(w)$, which proves the lemma. ■

Theorem 4.1i. In an EGS network, for $1 \le i < j \le s$, the number of paths between any stage-i switching module (or any stage-$i-1$ link) and any stage-j switching module (or any stage-j link) is either $\lfloor M_{i,j-1}/r_j \rfloor = \lfloor N_{i+1,j}/r_i \rfloor$ or $\lceil M_{i,j-1}/r_j \rceil = \lceil N_{i+1,j}/r_i \rceil$.

Proof. First, we note that dividing both sides of the equation in Lemma 4.1f by $r_i r_j$ yields $M_{i,j-1}/r_j = N_{i+1,j}/r_i$. Next, by Lemma 4.1h, we have $F_{i,j}(w) = \{[wM_{i,j-1} + d]_{\mathrm{mod} r_j} : d \in I(M_{i,j-1})\}$. The expression $[wM_{i,j-1} + d]$ (via substitution of all of the possible values of d) represents a set of $M_{i,j-1}$ consecutive integers. Write $M_{i,j-1} = br_j + c$, where b and c are integers with $0 \le b$ and $0 \le c \le r_j - 1$. Since no two integers in a set of $a \le r_j$ consecutive integers belong to the same congruence class modulo r_j, we must have that in any br_j consecutive integers there are exactly b integers in each congruence class modulo r_j and that in any $br_j + c$ consecutive integers there are $b + 1$ integers in each of c different congruence classes and b integers in each of the other $r_j - c$ congruence classes. These b and $b + 1$ occurrences of different congruence classes correspond directly to b and $b + 1$ paths from switching module w in stage i to different switching modules in stage j. Since $b = \lfloor M_{i,j-1}/r_j \rfloor$ and $b + 1 = \lceil M_{i,j-1}/r_j \rceil$, we have proved Theorem 4.1i for the switching module cases. The result for links follows immediately from the consideration that the only paths between a stage-$i-1$ link and a stage-j link are those which utilize (on a one-to-one basis) the paths between the stage i and stage-j switching modules to which these links are incident, respectively. ■

4.2. Lower Bound on the Number of Paths Between Any Inlet/Outlet Pair

By substituting 1 for i and s for j in Theorem 4.1i, we have immediately that the number of paths between any stage-1 switching module and any stage-s switching module (or, equivalently, the number of paths between any inlet/outlet pair) is either $\lfloor M_{1,s-1}/r_s \rfloor = \lfloor N_{2,s}/r_1 \rfloor$ or $\lceil M_{1,s-1}/r_s \rceil = \lceil N_{2,s}/r_1 \rceil$, which is the same as $\lfloor M_{1,s}/M \rfloor = \lfloor N_{1,s}/N \rfloor$ or $\lceil M_{1,s}/M \rceil = \lceil N_{1,s}/N \rceil$, because $r_s m_s = M$, the total number of network outlets, and $r_1 n_1 = N$, the total number of network inlets. Since these are the only two possible values, a lower bound on PATHS is provided by simply choosing the floor function expression and we have just proved

Theorem 4.2a. PATHS $= \lfloor M_{1,s}/M \rfloor = \lfloor N_{1,s}/N \rfloor$. (The minimum number of paths between any inlet/outlet pair in an EGS network $= \lfloor M_{1,s}/M \rfloor = \lfloor N_{1,s}/N \rfloor$.)

An obvious observation is that if M divides $M_{1,s}$ or, equivalently, if N divides $N_{1,s}$, then $\lfloor M_{1,s}/M \rfloor = \lfloor N_{1,s}/N \rfloor = \lceil M_{1,s}/M \rceil = \lceil N_{1,s}/N \rceil = M_{1,s}/M = N_{1,s}/N$, and this is the single value for the number of paths between any inlet/outlet pair. In most practical networks (for uniformity's sake), this will likely be the case. However, our subsequent theoretical results will not require this condition. We will be able to establish nonblocking conditions for EGS networks which do not have the same number of paths between all inlet/outlet pairs.

5. THE EFFECT OF A SINGLE INTERSECTING CONNECTION

We have just determined an expression for PATHS and must now consider BLOCKED PATHS in our search for the Strictly Nonblocking Sufficient Condition: PATHS > BLOCKED PATHS. To start this process, we will initially limit our considerations to the number of paths between a given inlet/outlet pair which can be blocked by a single additional connection in an EGS network. This corresponds to Module 5.2 in Figure 4.

5.1. Channel Graphs and Intersecting Connections

Recall from Section 2.5 that the channel graph $L(x, y)$ of inlet x and outlet y is defined as the union of all paths between x and y. $L(x, y)$ thus comprises sets of stage-k links, $1 \le k \le s - 1$. Any of these links may also be in the channel graph(s) of other inlet–outlet pairs. Thus, we may find that one or more links of $L(x, y)$ are busy due to existing connections in the network. Any such existing connection that utilizes one or more links of $L(x, y)$ is said to *intersect* or be an *intersecting connection of* $L(x, y)$. We will frequently shorten this terminology to *intersecting connection* when the reference to $L(x, y)$ is clear.

A link which is common to an intersecting connection and $L(x, y)$ is called an *intersecting link.* The set of intersecting links associated with a given intersecting connection are consecutive in stage number. To see why this is so, assume to the contrary that for $i < k < j$ the stages-i and -j links of an intersecting connection are intersecting links but the stage-k link is not. We then have an immediate contradiction because there is a path from x to the stage-i link [from the definition of $L(x, y)$], a path from the stage-i link to the stage-j link that includes the stage-k link (this being part of the overall path of the intersecting connection), and, finally, a path from the stage-j link to y [again from the definition of $L(x, y)$]. Thus, the stage-k link is part of a path from x to y and must be included in $L(x, y)$.

For an intersecting connection, if i is the smallest stage number intersecting link and $j - 1$ is the largest, we say that the intersecting connection *enters* the channel graph $L(x, y)$ at stage i and *departs* at stage j, corresponding to the switching module stage numbers defining the boundaries of the intersecting links. We denote such an intersecting connection as $C(i, j)$.

5.2. Paths Blocked by a Single Intersecting Connection

In an EGS network, let us identify the stage-k link ($i \le k \le j - 1$) of $C(i, j)$ as λ_k. Let B_k represent the number of paths from inlet x to λ_k. From Theorem 4.1i, we know that B_k is either $\lfloor N_{1,k}/N \rfloor$ or $\lceil N_{1,k}/N \rceil$. Similarly, let F_{k+1} represent the number of paths from λ_k to outlet y. Again, by Theorem 4.1i, we know that F_{k+1} is either $\lfloor M_{k+1,s}/M \rfloor$ or $\lceil M_{k+1,s}/M \rceil$.

Recall from Section 2.6 that a *blocked path* between x and y is one that contains at least one link that is busy from some other connection. An intersecting link is such a link. Consider the stage-i intersecting link λ_i of intersecting connection $C(i, j)$. There are B_i paths from x to λ_i and F_{i+1} paths from λ_i to y. So, there are $B_i F_{i+1}$ paths in $L(x, y)$ that include λ_i and all of these paths are blocked by λ_i.

Next, we consider how many additional paths are blocked by λ_{i+1}, the stage-$i + 1$ intersecting link of $C(i, j)$. Using the same logic as above, we have that λ_{i+1}, considered singly, blocks $B_{i+1}F_{i+2}$ paths. However, this expression includes some paths already blocked by λ_i, namely, all those paths containing both λ_i and λ_{i+1}. Since there are B_i paths from inlet x to λ_i and F_{i+2} paths from λ_{i+1} to outlet y, there are $B_i F_{i+2}$ paths in $L(x, y)$ that contain both λ_i and λ_{i+1}. Thus, the total number of paths in $L(x, y)$ blocked by λ_i and λ_{i+1} is given by $B_i F_{i+1} + B_{i+1}F_{i+2} - B_i F_{i+2}$.

The logic for three or more contiguous intersecting links is similar to that for two links in that we want to determine how many additional paths are blocked by each additional intersecting link. Theoretically, for $k \ge i + 2$, when we compute the number of additional paths blocked by λ_k, we can start with B_kF_{k+1} (the number of paths blocked by λ_k, considered singly) and then subtract the number of blocked paths included in this expression that have already been tabulated for λ_i through λ_{k-1}, namely, all the paths containing both λ_k and any of the intersecting links λ_i through λ_{k-1}. However, at this point, we introduce a simplifying approximation by subtracting only the $B_{k-1}F_{k+1}$ paths containing both λ_k and λ_{k-1}. As we shall subsequently see, this approximation greatly facilitates our analysis and, fortunately, has very little effect on the sufficient nonblocking conditions for most practical EGS networks.

The reason this latter point is true is that usually there are very few, if any, blocked paths that do not contain λ_{k-1} but do contain λ_k and any of the links λ_i to λ_{k-2}. The only way any such blocked paths could exist would be if there were at least one path from any of the links λ_i through λ_{k-2} to λ_k that did not include λ_{k-1}. By Theorem 4.1i, this cannot be true unless $M_{i,k-1}/r_k = N_{i+1,k}/r_i > 1$. (Later, we will consider this situation in producing a refinement on the sufficient nonblocking condition.)

Since the described approximation counts all blocked paths at least once, it may be used in the determination of an upper bound on the total number of paths blocked by a single intersecting connection. From above, we have that $B_i F_{i+1} + B_{i+1}F_{i+2} - B_i F_{i+2}$ paths are blocked by λ_i and λ_{i+1}, and for $i + 2 \le k \le j - 1$, we will add $B_kF_{k+1} - B_{k-1}F_{k+1}$ blocked paths for each λ_k. Thus, an upper bound on the number of paths blocked by $C(i, j)$ is given by $\sum_{k=i}^{j-1} B_kF_{k+1} - \sum_{k=i}^{j-2} B_kF_{k+2}$, which can be rewritten as $\sum_{k=i}^{j-2} B_k(F_{k+1} - F_{k+2}) + B_{j-1}F_j$ or as $\sum_{k=i+1}^{j-1} (B_k - B_{k-1})F_{k+1} + B_i F_{i+1}$. We know there is a path from λ_k to λ_{k+1} because these links are both included in $C(i, j)$. Therefore, $(F_{k+1} - F_{k+2})$ is nonnegative because every path from λ_{k+1} to y is necessarily a part of some path from λ_k to y. Furthermore, since the number of paths between any two entities in a network can never be less than zero, F_j is also nonnegative. Thus, it is evident that $\Sigma_{k=i}^{j-2} B_k(F_{k+1} - F_{k+2}) + B_{j-1}F_j$ has a maximum value if each B_l is replaced by $\lceil N_{1,l}/N \rceil$. Applying similar logic, we find that $\Sigma_{k=i+1}^{j-1} (B_k - B_{k-1})F_{k+1} + B_i F_{i+1}$ has a maximum value if each F_l is replaced by $\lceil M_{l,s}/M \rceil$. Thus, we have the result that $B[C(i, j)]$, an upper bound on the number of paths blocked by $C(i, j)$, is given by

$$B[C(i,j)] = \sum_{k=i}^{j-1} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \sum_{k=i}^{j-2} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left\lceil \frac{M_{k+2,s}}{M} \right\rceil . \tag{5.2a}$$

5.3. Some Simplifying Observations

Let us consider for a moment how we might use the information in Eq. (5.2a). This expression clearly indicates that the number of paths blocked by a single intersecting connection depends on its entry and departure points. Thus, we will eventually have to consider the numbers of connections that can enter and depart a channel graph at the various stages in the network. Our task would seem to be extremely difficult if we needed to consider all of the possible ways that various connections could intersect a channel graph. Fortunately, this will not be necessary as we now explain.

Let us rewrite Eq. (5.2a) by adding and subtracting the remaining terms through stage s for both summations. The result is

$$\begin{aligned} B[C(i,j)] &= \sum_{k=i}^{s} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left\lceil \frac{M_{k+1,s}}{M} \right\rceil \\ &- \sum_{k=i}^{s} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \\ &- \sum_{k=j}^{s} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left\lceil \frac{M_{k+1,s}}{M} \right\rceil \\ &+ \sum_{k=j-1}^{s} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \\ &= \sum_{k=i}^{s} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) \\ &+ \left\lceil \frac{N_{1,j-1}}{N} \right\rceil \left\lceil \frac{M_{j+1,s}}{M} \right\rceil \\ &- \sum_{k=j}^{s} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right). \end{aligned} \tag{5.3a}$$

(We should note that here and throughout the paper we adopt the usual convention that, in the event that a summation or product series does not exist because the beginning index exceeds the terminating index, the summation has a value of 0 and the product has a value of 1. Similarly, individual terms in a summation or product series are given values of 0 or 1, respectively, if those individual terms do not exist. This convention achieves the obvious desired result of not changing summation or product totals if nonexistent terms are encountered.)

There are three terms in the third line of Eq. (5.3a). The first term is a function of various network parameters and the entry stage i. The last two terms are functions of various network parameters and the departure stage j. Thus, the blocked-path contribution of an intersecting-connection entry point is independent of the departure point and vice versa. This means that any two feasible sets of connections that have identical distributions of entry points and departure points will have identical upper bounds on the total number of paths blocked, as determined by Eq. (5.3a). This is an extremely important and fortunate result because it removes the need to associate the entry and departure points of a given intersecting connection. We need only be concerned with entry and departure point distributions and not the specifics of the connection sets yielding these distributions.

We can simplify Eq. (5.3a) via some relevant definitions and observations. Define t to be the largest stage number satisfying $\lceil N_{1,t}/N \rceil = 1$ and define u to be the smallest stage number satisfying $\lceil M_{u,s}/M \rceil = 1$. (Note that since $N_{1,1} = n_1 \le N$ and $M_{s,s} = m_s \le M$ both t and u exist.) According to the definition of t, we have that $\lceil N_{1,t+1}/N \rceil > 1$. This implies that all inlets have at least one path to every stage-$t + 1$ switching module. Now, if the value of i in $C(i, j)$ were greater than $t + 1$, then, according to the definition of $C(i, j)$, its stage-$t + 1$ link (λ_{t+1}) would <u>not</u> intersect $L(x, y)$. But λ_{t+1} <u>must</u> intersect $L(x, y)$ because there is at least one path from x to all stage-$t + 1$ links and there is a path from λ_{t+1} to the stage-i link of $C(i, j)$ [both links being part of $C(i, j)$] and a path from the stage-i link of $C(i, j)$ to y [because $C(i, j)$ intersects $L(x, y)$]. Therefore, $C(i, j)$ must have $i \le t + 1$. Using similar logic, we can determine that $j \ge u - 1$.

The term $\Sigma_{k=j}^{s} \lceil N_{1,k}/N \rceil (\lceil M_{k+1,s}/M \rceil - \lceil M_{k+2,s}/M \rceil)$ in Eq. (5.3a) has $u \le j + 1 = \min\{k + 1\} < \min\{k + 2\}$. Thus, in this term, all $\lceil M_{k+1,s}/M \rceil = \lceil M_{k+2,s}/M \rceil = 1$ and, therefore, the entire term equals 0. Additionally, the second term of Eq. (5.3a) has $\lceil M_{j+1,s}/M \rceil = 1$ because $u \le j + 1$. So, we now have

$$B[C(i,j)] = \sum_{k=i}^{s} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) + \left\lceil \frac{N_{1,j-1}}{N} \right\rceil . \tag{5.3b}$$

Next, we add and subtract terms for $k = i - 1$ in the summation, yielding

$$B[C(i,j)] = \sum_{k=i-1}^{s} \left\lceil \frac{N_{1,k}}{N} \right\rceil \times \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) + \left\lceil \frac{N_{1,j-1}}{N} \right\rceil - \left\lceil \frac{N_{1,i-1}}{N} \right\rceil \times \left(\left\lceil \frac{M_{i,s}}{M} \right\rceil - \left\lceil \frac{M_{i+1,s}}{M} \right\rceil \right). \tag{5.3c}$$

Now, since $i - 1 \le t$ and $\lceil N_{1,k}/N \rceil = 1$ for all $k \le t$, we have

$$B[C(i,j)] = \sum_{k=i-1}^{t} \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) + \sum_{k=t+1}^{s} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) + \left\lceil \frac{N_{1,j-1}}{N} \right\rceil + \left\lceil \frac{M_{i+1,s}}{M} \right\rceil - \left\lceil \frac{M_{i,s}}{M} \right\rceil. \tag{5.3d}$$

In the summation $\sum_{k=i-1}^{t} (\lceil M_{k+1,s}/M \rceil - \lceil M_{k+2,s}/M \rceil)$, alternate terms cancel each other, except for the first and last terms, yielding $\lceil M_{i,s}/M \rceil - \lceil M_{t+2,s}/M \rceil$. Since the first of these two terms cancels the last term in Eq. (5.3d), we have

$$B[C(i,j)] = \left\lceil \frac{M_{i+1,s}}{M} \right\rceil + \left\lceil \frac{N_{1,j-1}}{N} \right\rceil + \sum_{k=t+1}^{s} \left\lceil \frac{N_{1,k}}{N} \right\rceil \times \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) - \left\lceil \frac{M_{t+2,s}}{M} \right\rceil. \tag{5.3e}$$

Now, for $k \ge u - 1$, we have $\lceil M_{k+1,s}/M \rceil = \lceil M_{k+2,s}/M \rceil = 1$ and, therefore, Eq. (5.3e) finally becomes

$$B[C(i,j)] = \left\lceil \frac{M_{i+1,s}}{M} \right\rceil + \left\lceil \frac{N_{1,j-1}}{N} \right\rceil + \sum_{k=t+1}^{u-2} \left\lceil \frac{N_{1,k}}{N} \right\rceil \times \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) - \left\lceil \frac{M_{t+2,s}}{M} \right\rceil. \tag{5.3f}$$

5.4. Discussion

Equation (5.3f) is a very important intermediate result in our development of sufficient conditions for nonblocking EGS networks. When we arrived at Eq. (5.3a), we were able to demonstrate the separability of the blocked-path contributions of the entry point and departure point of an intersecting connection. This was because every term depended on either i or j, but not on both. In Eq. (5.3f), again no term depends on both i and j, and the terms that depend on them individually are greatly simplified. In addition, in Eq. (5.3f), we (surprisingly?) find terms that depend on neither i nor j. This means that for any given EGS network there is a computable constant to be added to the upper-bound calculation for blocked paths for every intersecting connection. For many practical EGS networks, this constant will be found to have a value of minus 1, thus making Eq. (5.3f) much easier to use than might be first evident.

5.5. Examples

At this juncture, it would probably be helpful to the reader if we paused to consider some examples of the number of paths between inlets and outlets in an EGS network and the numbers of these paths which can be blocked due to other connections. After these examples, we will continue our development of nonblocking conditions.

Figure 5(a) illustrates a channel graph $L(x, y)$ of Network η shown in Figure 2. Recall that $L(x, y)$ is defined as the union of all paths between some inlet x and outlet y. In this instance, x is an inlet appearing on switch 4 in stage 1 and y is an outlet appearing on switch 2 in stage 4. [Stage numbers are indicated in parentheses along the top of Fig. 5(a).] $L(x, y)$ results from deleting all switches and links in Figure 2 that are not part of any path between x and y.

In Section 4.2, we observed that if M divides $M_{1,s}$ then there is a single value for the number of paths between any inlet/outlet pair as given by $M_{1,s}/M$. In network η, $M_{1,s} = 3 \times 6 \times 3 \times m_4 = 54m_4$ and $M = 9m_4$. Therefore, $M_{1,s}/M = 54m_4/9m_4 = 6$, which is the number of paths that we observe between x and y above. We can uniquely identify a particular path with a quadruplet of switch module numbers (one number for each module successively employed in the path from stages 1 through 4). The six paths between x and y are thus given by (4, 0, 0, 2), (4,

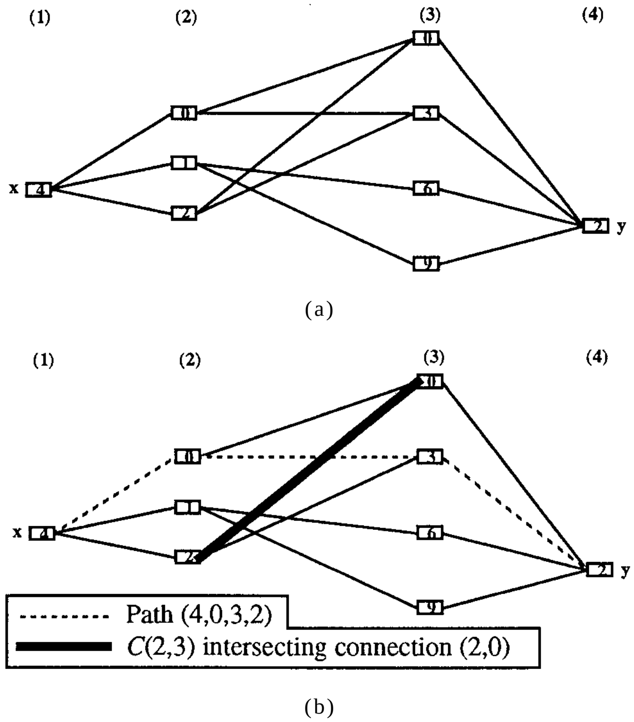

Fig. 5. (a) A channel graph $L(x, y)$ of network η. (b) A channel graph $L(x, y)$ of network η.

0, 3, 2), (4, 1, 6, 2), (4, 1, 9, 2), (4, 2, 0, 2), and (4, 2, 3, 2).

An intersecting connection $C(i, j)$ of $L(x, y)$ can enter $L(x, y)$ at stage $i = 1, 2$, or 3 and depart at stage $j = 2, 3, 4$ with the additional proviso that $i < j$. Thus, there are six possibilities for $C(i, j)$, namely, $C(1, 2)$, $C(1, 3)$, $C(1, 4)$, $C(2, 3)$, $C(2, 4)$, and $C(3, 4)$. Note that such identifiers do not indicate specific intersecting connections but rather refer to any intersecting connection entering at stage i and departing at stage j. We may uniquely identify a particular intersecting connection of type $C(i, j)$ in a fashion similar to the path identifiers in the previous paragraph, that is, with the switch module numbers employed successively from stages i to j. Thus, for example, the $C(2, 3)$ intersecting connections are (0, 0), (0, 3), (1, 6), (1, 9), (2, 0), and (2, 3). Figure 5(b) highlights path (4, 0, 3, 2) and $C(2, 3)$ intersecting connection (2, 0). [We note that (4, 0, 3, 2) could alternatively be interpreted as a $C(1, 4)$ intersecting connection between an inlet other than x and an outlet other than y on the stage-1 switch 4 and the stage-4 switch 2, respectively.]

Next, we will consider $B[C(i, j)]$ [an upper bound on the number of paths blocked by $C(i, j)$] as given by Eq. (5.3f). In Section 5.3, we defined t to be the largest stage number satisfying $\lceil N_{1,t}/N \rceil = 1$ and u to be the smallest stage number satisfying $\lceil M_{u,s}/M \rceil = 1$.

From Figure 2, we see that $N = 8n_1$, $N_{1,2} = 4n_1$, and $N_{1,3} = 3 \times 4n_1$ and, therefore, that $t = 2$. Similarly, we have $M = 9m_4$, $M_{3,4} = 3m_4$, and $M_{2,4} = 6 \times 3m_4$ and, therefore, that $u = 3$. Thus, the summation term in Eq. (5.3f) is equal to 0 because its lowest indexing value $t + 1 = 3$ is greater than its highest indexing value $u - 2 = 1$. Additionally, the last term in Eq. (5.3f) has a value of 1 because $M_{t+2,4} = M_{4,4} = m_4 < 4m_4 = M$. Therefore, Eq. (5.3f) reduces to

$$B[C(i, j)] = \left\lceil \frac{M_{i+1,4}}{M} \right\rceil + \left\lceil \frac{N_{1,j-1}}{N} \right\rceil - 1. \quad (5.5a)$$

A sample calculation:

$$B[C(1, 3)] = \left\lceil \frac{M_{2,4}}{M} \right\rceil + \left\lceil \frac{N_{1,2}}{N} \right\rceil - 1$$

$$= \left\lceil \frac{6 \times 3m_4}{9m_4} \right\rceil + \left\lceil \frac{4n_1}{8n_1} \right\rceil - 1 = 2 + 1 - 1 = 2.$$

Referring to Figure 5(a), it is easily verified that the example $C(1, 3)$ intersecting connection (4, 1, 6) blocks the two paths (4, 1, 6, 2) and (4, 1, 9, 2). It is also straightforward to verify in Figure 5(a) that every $C(1, 3)$ intersecting connection blocks exactly two paths, thus substantiating our above calculation that $B[C(1, 3)] = 2$.

Since $B[C(i, j)]$ is an upper bound on the number of paths blocked by $C(i, j)$, we have, in general, the possibility that one or more $C(i, j)$ intersecting connections blocks fewer than $B[C(i, j)]$ paths (in $L(x, y)$. As an example of this, we first calculate that

$$B[C(1, 4)] = \left\lceil \frac{M_{2,4}}{M} \right\rceil + \left\lceil \frac{N_{1,3}}{N} \right\rceil - 1$$

$$= \left\lceil \frac{6 \times 3m_4}{9m_4} \right\rceil + \left\lceil \frac{3 \times 4n_1}{8n_1} \right\rceil - 1$$

$$= 2 + 2 - 1 = 3.$$

In Figure 5(b), we can observe that the $C(1, 4)$ intersecting connection (4, 0, 3, 2) does indeed block the maximum of three paths, namely, (4, 0, 0, 2), (4, 0, 3, 2), and (4, 2, 3, 2). However, the $C(1, 4)$ intersecting connection (4, 1, 6, 2) blocks only two paths: (4, 1, 6, 2) and (4, 1, 9, 2).

Having completed these examples of paths and blocked paths, we now return to the general development of nonblocking conditions. We next consider a very important attribute of EGS networks that is crucial in the ability to construct efficient nonblocking networks.

6. THE FORWARD–BACKWARD INVARIANCE PROPERTY

We will use the following preliminary results:

6.1. Two Lemmas

Lemma 6.1a. If y is a positive integer, then $\lfloor \lfloor x \rfloor / y \rfloor = \lfloor x/y \rfloor$.

Proof. Represent $x = ay + b$, where a is an integer and $0 \le b < y$. Since $0 \le \lfloor b \rfloor \le b < y$, we have $\lfloor \lfloor b \rfloor / y \rfloor = \lfloor b/y \rfloor = 0$. Thus,

$$\left\lfloor \frac{\lfloor x \rfloor}{y} \right\rfloor = \left\lfloor \frac{\lfloor ay + b \rfloor}{y} \right\rfloor = \left\lfloor \frac{ay + \lfloor b \rfloor}{y} \right\rfloor$$
$$= a + \left\lfloor \frac{\lfloor b \rfloor}{y} \right\rfloor = a + \left\lfloor \frac{b}{y} \right\rfloor$$
$$= \left\lfloor \frac{ay + b}{y} \right\rfloor = \left\lfloor \frac{x}{y} \right\rfloor . \qquad \blacksquare$$

Lemma 6.1b. In a MIN, for $1 \le i < h < j \le s$, (1) if $M_{i,j-1}$ divides r_j, then $M_{i,h-1}$ divides r_h, and (2) if $N_{i+1,j}$ divides r_i, then $N_{h+1,j}$ divides r_h.

Proof. For Case (1), applying Lemma 4.1f twice we have

$$\frac{r_h}{M_{i,h-1}} = \frac{r_i}{N_{i+1,h}} = \frac{r_j N_{i+1,j}}{M_{i,j-1} N_{i+1,h}} ,$$

which equals $(r_j/M_{i,j-1})N_{h+1,j}$, the product of two integers. Similarly, for Case (2), applying Lemma 4.1f twice we have

$$\frac{r_h}{N_{h+1,j}} = \frac{r_j}{M_{h,j-1}} = \frac{r_i M_{i,j-1}}{N_{i+1,j} M_{h,j-1}} ,$$

which equals $(r_i/N_{i+1,j})M_{i,h-1}$, the product of two integers. $\blacksquare$

6.2. The Forward–Backward Invariance Theorem

Theorem 6.2. In an EGS network, for $1 \le i < j \le s$, if $M_{i,j-1}$ divides r_j (or, equivalently, $N_{i+1,j}$ divides r_i), then for $w \ne v$, either $F_{i,j}(w) = F_{i,j}(v)$ or $F_{i,j}(w) \cap F_{i,j}(v) = \emptyset$.

Proof.

$$F_{i,j}(w) = \{ (wM_{i,j-1} + d)_{\mathrm{mod} r_j} : d \in I(M_{i,j-1}) \}$$

(from Lemma 4.1h)

$$= wM_{i,j-1} + d - r_j \left\lfloor \frac{wM_{i,j-1} + d}{r_j} \right\rfloor$$

(from Definition 4.1b)

$$= wM_{i,j-1} + d - r_j \left\lfloor \frac{w + d/M_{i,j-1}}{r_j/M_{i,j-1}} \right\rfloor$$
$$= wM_{i,j-1} + d - r_j \left\lfloor \frac{\lfloor w + d/M_{i,j-1} \rfloor}{r_j/M_{i,j-1}} \right\rfloor$$

(by Lemma 6.1a)

$$= wM_{i,j-1} + d - r_j \left\lfloor \frac{w}{r_j/M_{i,j-1}} \right\rfloor$$
$$= wM_{i,j-1} + d - r_j \left\lfloor \frac{wM_{i,j-1}}{r_j} \right\rfloor$$
$$= \{ (wM_{i,j-1})_{\mathrm{mod} r_j} + d : d \in I(M_{i,j-1}) \} .$$

Now consider a partition of S_i composed of $r_i/N_{i+1,j}$ subsets $(R_0, R_1, \ldots, R_{r_i/N_{i+1,j}-1})$ each of cardinality $N_{i+1,j}$, where subset R_a is given by

$$R_a = \left\{ \frac{br_i}{N_{i+1,j}} + a : b \in I(N_{i+1,j}) \right\} \quad \text{for } a \in I\left(\frac{r_i}{N_{i+1,j}} \right) .$$

Then,

$$F_{i,j}(R_a) = \{ (R_a M_{i,j-1})_{\mathrm{mod} r_j} + d : d \in I(M_{i,j-1}) \}$$
$$= \left(\frac{br_i M_{i,j-1}}{N_{i+1,j}} + aM_{i,j-1} \right)_{\mathrm{mod} r_j} + d$$
$$= (br_j + aM_{i,j-1})_{\mathrm{mod} r_j} + d$$
$$= (aM_{i,j-1})_{\mathrm{mod} r_j} + d$$
$$= \{ aM_{i,j-1} + d : d \in I(M_{i,j-1}) \} ,$$

because

$$aM_{i,j-1} < \left(\frac{r_i}{N_{i+1,j}} \right) M_{i,j-1} = r_j .$$

Thus, $F_{i,j}(R_a)$ is invariant to the selection of $b \in I(N_{i+1,j})$, that is, $F_{i,j}(\alpha) = F_{i,j}(\beta)$ for any two elements α and β of R_a.

Now, for $a', a'' \in I(r_i/N_{i+1,j})$, where $a' + 1 \le a''$, the largest value of $F_{i,j}(R_{a'}) = a'M_{i,j-1} + (M_{i,j-1} - 1)$ and the smallest value of $F_{i,j}(R_{a''}) = a''M_{i,j-1} + 0 \ge (a' + 1)M_{i,j-1}$, which is larger than the largest value of $F_{i,j}(R_{a'})$. Thus, none of the values of $F_{i,j}(R_{a'})$ can be the same as any of the values of $F_{i,j}(R_{a''})$.

The theorem is now proven because, for $w \ne v$, if w and v are in the same subset R_a, then $F_{i,j}(w) = F_{i,j}(v)$. Also, if w and v are in different subsets, then $F_{i,j}(w) \cap F_{i,j}(v) = \emptyset$. $\blacksquare$

6.3. Discussion

For ease of discussion, in any EGS network in which, for $1 \le i < j \le s$, $M_{i,j-1}$ divides r_j (or, equivalently, $N_{i+1,j}$

divides r_i), we will say that the *division condition* holds for stages i through j. From Lemma 6.1b, it follows immediately that, if the division condition holds for stages i through j, it also holds for stages g through h, where $i \le g < h \le j$.

From Theorem 4.1, we know that the number of paths between any stage-i switching module and any stage-j switching module is either $\lfloor M_{i,j-1}/r_j \rfloor$ or $\lceil M_{i,j-1}/r_j \rceil$. If $M_{i,j-1}$ divides r_j, then $M_{i,j-1} \le r_j$ and the maximum number of paths between any stage-i switching module and any stage-j switching module is one. Therefore, in Theorem 6.2, $F_{i,j}(w)$ will have only single occurrences of any element. We thus recast the essence of Theorem 6.2 as follows:

The Forward–Backward Invariance Property:

In any EGS network in which the division condition holds for stages i through j, there exists a partition P_i of S_i composed of $r_i/N_{i+1,j} = r_j/M_{i,j-1}$ subsets, each of cardinality $N_{i+1,j}$, and a partition P_j of S_j composed of $r_i/N_{i+1,j} = r_j/M_{i,j-1}$ subsets, each of cardinality $M_{i,j-1}$, where, for each subset ρ of P_i, there is a corresponding subset σ of P_j, such that (1) each switching module in ρ has a single path to every switching module in σ (and no paths to any other switching modules in stage j) and (2) each switching module in σ has a single path to every switching module in ρ (and no paths to any other switching modules in stage i).

The crucial importance of the Forward–Backward Invariance Property in the construction of nonblocking networks will become evident in the next few sections.

6.4. Example

Figure 6 illustrates an EGS network and highlights all of the paths of a subset pair (ρ, σ) of 9 switching modules from the second stage and 4 switching modules from the fourth stage.

The Forward–Backward Invariance Property applies here for stages 2 through 4 because $m_2 m_3 = 2 \times 2 = 4$ which divides $r_4 = 12$ (or, equivalently, $n_3 n_4 = 3 \times 3 = 9$ which divides $r_2 = 27$). We note in passing that the Forward–Backward Invariance Property may hold for some successive stages in a network and not hold for others. For example, since $m_1 m_2 = 6 \times 2 = 12$ does not divide $r_3 = 18$ and $m_3 m_4 = 2 \times 5 = 10$ does not divide $r_5 = 15$, the property does not hold for stages 1 through 3 for stages 3 through 5, but as we have just seen, it does hold for stages 2 through 4.

7. AN UPPER BOUND ON THE NUMBER OF INTERSECTING CONNECTIONS OF A CHANNEL GRAPH

In Section 5.3, we established an expression for an upper bound on the number of paths blocked by a single intersecting connection in which the blocked-path contributions of the intersecting-connection entry point and departure point are independent of each other. Thus, to determine an upper bound on the number of paths blocked by a group of intersecting connections, we need only be concerned with their entry and departure point distributions. (We do not care which entry point and departure point are associated with any particular intersecting connection.) We will, however, need to know the maximum number of connections that can intersect channel graph $L(x, y)$. We identify this value as ω.

7.1. The Role of the Forward–Backward Invariance Property

Consider an intersecting connection between some inlet $x' \neq x$ and some outlet $y' \neq y$ that enters $L(x, y)$ at stage i, where $1 \le i \le k \le s$. Since every stage-i switching module in $L(x, y)$ has at least one path to at least one stage-k switching module in $L(x, y)$, there is at least one path from x' to some stage-k switching module in $L(x, y)$. Thus, if there are no paths from a given inlet to any stage-k switching modules of $L(x, y)$, then that inlet cannot be part of an intersecting connection that enters $L(x, y)$ from stages 1 through k.

Now suppose that the division condition (and, hence, the Forward–Backward Invariance Property) holds for stages 1 through k. Then, the stage-1 switch, on which idle inlet x appears, is a member of a subset ρ (of cardinality $N_{2,k}$) of S_1, where each switching module in ρ has a single path to every switching module in some subset σ of S_k (and no paths to any other switching modules in stage k). Thus, all stage-k switching modules in $L(x, y)$ must be members of σ. Additionally, each switching module in σ has a single path to every switching module in ρ (and no paths to any other switching modules in stage 1). Therefore, since each of the switching modules of ρ has n_1 inlets, all of the stage-k switching modules in $L(x, y)$ have paths to the same set of $n_1 \times N_{2,k} = N_{1,k}$ inlets (and paths to no other inlets). Removing x from consideration, we have that at most $N_{1,k} - 1$ inlets have paths to any stage-k switches of $L(x, y)$. Using this in conjunction with the result from the previous paragraph gives $N_{1,k} - 1$ as an upper bound on the total number of intersecting connections that can enter $L(x, y)$ from stages 1 through k. [An additional observation, which we will use a little later, is that none of these $N_{1,k} - 1$ inlets can be associated with an intersecting connection that enters after stage k. If such an intersecting connection did exist, its stage-k switching module d would not be a part of $L(x, y)$. But since all of the $N_{1,k}$ inlets (including x) have paths to the same stage-k switching modules, there is a path from x to d. Also, there is also a path from d to y via the portion of the intersecting connection from d to the entry point and thence via $L(x, y)$. Thus, there

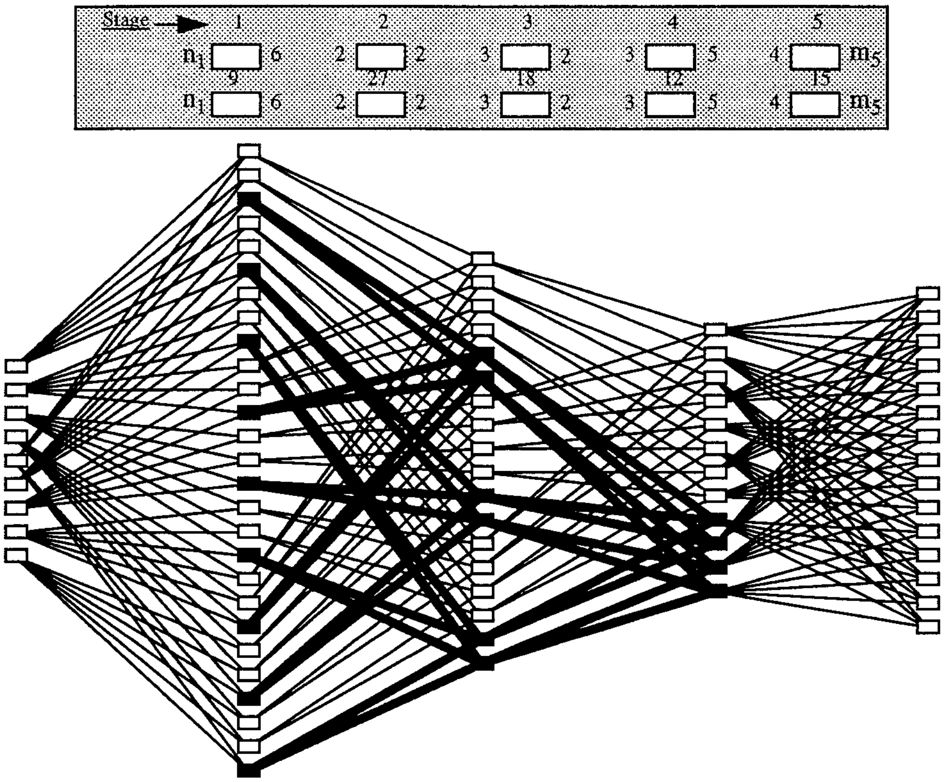

Fig. 6. Forward–Backward Invariance Property (for stages 2 through 4).

is a path from x to y that includes d and so d must be in $L(x, y)$, a contradiction.]

Two important consequences follow from these observations: First, the analysis is simplified because the numbers of switching modules in stages 1 through k in $L(x, y)$ do not have to be considered. Second, and far more importantly, the maximum number of intersecting connections entering from stages 1 through k is potentially much smaller than might otherwise be the case if the Forward–Backward Invariance Property did not limit the number of inlets having paths to any stage-k switches of $L(x, y)$. This is one of the main reasons that we can design efficient nonblocking EGS networks.

For the moment, we will proceed on the assumption that the division condition holds, wherever we would like it to hold, throughout a network. After establishing an expression for an upper bound on the number of intersecting connections (ω), we will be able to replace this rather nebulous assumption with a precise sufficient condition.

7.2. Entering and Departing Connections

We have from above that at most $N_{1,k} - 1$ intersecting connections can enter $L(x, y)$ from stages 1 through k and, using similar logic, we can establish that at most $M_{k,s} - 1$ intersecting connections can depart from stages k through s. Let us momentarily focus on stage k, $(1 \le k \le s - 2)$, to understand its role in establishing ω.

By definition, an intersecting connection must utilize one or more links of $L(x, y)$. This implies that an intersecting connection which enters at stage i cannot depart any sooner than stage $i + 1$, that is, it must at least utilize a stage-i link. Thus, we have that at most $N_{1,k} - 1$ intersecting connections can utilize stage-1 links through stage-k links of $L(x, y)$ (as limited by the maximum number of connections that can enter from stages 1 through k) and at most $M_{k+2,s} - 1$ intersecting connections can utilize stage-$k + 1$ links through stage-$s - 1$ links of $L(x, y)$ (as limited by the maximum number of connections that can depart from stages $k + 2$ through s). An upper bound on the number of intersecting connections is obtained by assuming that the $N_{1,k} - 1$ entering connections and the $M_{k+2,s} - 1$ departing connections are disjoint, that is, all of the $N_{1,k} - 1$ entering connections depart at or before stage $k + 1$ and all of the $M_{k+2,s} - 1$ departing connections enter at or after stage $k + 1$.

Thus, considering only the perspective of stage k, there are at most $(N_{1,k} - 1) + (M_{k+2,s} - 1) = N_{1,k} + M_{k+2,s} - 2$ connections that can intersect $L(x, y)$. This expression is valid for $(1 \le k \le s - 2)$, since it assumes that intersecting connections can enter as early as stage k and depart as late as stage $k + 2$. Considering all of the possible values for k, we have that at most $\min_{1 \le k \le s-2}\{N_{1,k} + M_{k+2,s} - 2\}$ connections can intersect $L(x, y)$.

7.3. An Expression for ω Plus a Sufficient Condition

Since the number of intersecting connections clearly cannot exceed $\min\{N-1, M-1\}$, we immediately have the following expression for ω:

$$\omega = \min_{1 \le k \le s-2} \{N_{1,k} + M_{k+2,s} - 2, N-1, M-1\}. \quad (7.3a)$$

The development of this expression assumed that the division condition held wherever it was necessary to hold. Where is it possible for the division condition to not hold and yet maintain the validity of the arguments producing expression (7.3a)? We claim it is not required for stages 1 through l, for any l satisfying $N_{1,l} > \omega$, nor for stages r through s, for any r satisfying $M_{r,s} > \omega$.

First of all, for the case where $\omega = \min\{N-1, M-1\}$, the division condition is not required to hold anywhere, because the arguments which employ it yield a larger value for ω than $\min\{N-1, M-1\}$. Next, consider the case where $\omega = \min_{1 \le k \le s-2}\{N_{1,k} + M_{k+2,s} - 2\}$. Since $N_{1,k}$ is monotonically nondecreasing in k, the value of k yielding ω must be less than any l satisfying $N_{1,l} > \omega$, and, thus, we do not require the division condition to hold for stages 1 through l. Similarly, since $M_{k,s}$ is monotonically nonincreasing in k, the value of $k+2$ yielding ω must be greater than any r satisfying $M_{r,s} > \omega$.

Thus, we have the following condition which is sufficient for expression (7.3a) to be valid:

Condition 7.3b. $N_{1,k}$ divides N for $N_{1,k} \le \omega$ and $M_{k,s}$ divides M for $M_{k,s} \le \omega$.

7.4. A Review

Let us pause for a moment to highlight some of our results so far and to see what remains in our quest to establish the strictly nonblocking sufficient condition: PATHS > BLOCKED PATHS. (Fig. 4 may be helpful during this discussion.)

Theorem 4.2a has given us an expression for PATHS. Expression (5.3f) provides an upper bound on blocked paths per intersecting connection as a separable function of entry point, departure point, plus a constant. Condition 7.3b gives a sufficient condition for which expression (7.3a) yields ω, an upper bound on intersecting connections.

What remains is to allocate these ω intersecting connections according to entry points and departure points and then to combine these allocations with expression (5.3f) to calculate an upper bound on the number of blocked paths. The aforementioned allocations must be such that they produce the largest possible number of blocked paths, subject to the constraints imposed by Condition 7.3b and ω.

8. ALLOCATION OF THE ENTRY AND DEPARTURE POINTS OF THE INTERSECTING CONNECTIONS

We first determine the allocations assuming that Condition 7.3b holds and that there are ω intersecting connections. Then, we will justify the assumption of ω intersecting connections by formally establishing the intuitive notion that the maximum number of blocked paths is monotonically increasing in the number of intersecting connections, that is, any number $d < \omega$ of intersecting connections cannot maximally block as many paths as can ω intersecting connections.

8.1. Aggregate Numbers of Entering and Departing Connections

In Section 7.1, we established that at most $N_{1,k} - 1$ inlets can be associated with intersecting connections that enter $L(x, y)$ from stages 1 through k, if the division condition holds for these stages. Assuming that Condition 7.3b holds, the division condition will hold for any k such that $N_{1,k} \le \omega$. For any other value of k (i.e., when $N_{1,k} > \omega$), there are at most ω intersecting connections that can enter from stages 1 through k. Therefore, we have that at most $\min\{N_{1,k} - 1, \omega\}$ intersecting connections can enter from stages 1 through k. Using similar logic, we find that at most $\min\{M_{k,s} - 1, \omega\}$ intersecting connections can depart from stages k through s, assuming Condition 7.3b holds.

8.2. Maximizing the Number of Blocked Paths

We now have expressions for the maximum aggregate numbers of entering and departing intersecting connections from stages 1 through k and stages k through s, respectively. What we need next is the allocation of these intersecting connections on a per stage basis such that we have an upper bound on the total number of blocked paths. This result will follow immediately upon inspection of expression (5.3f).

In this expression, the separable component of the number of blocked paths associated with each intersecting connection entering at stage i is given by $\lceil M_{i+1,s}/M \rceil$, which is a monotonically nonincreasing function in i. Therefore, an upper bound on the component of the number of blocked paths associated with entry points will be achieved if each stage i has as many as possible entering intersecting connections, given that each stage $k < i$ also meets this objective. Thus, we will assume that $\min\{N_{1,1} - 1, \omega\}$ intersecting connections enter at stage 1, $\min\{N_{1,2} - 1, \omega\} - \min\{N_{1,1} - 1, \omega\}$ intersecting connections enter at stage 2, etc., so that we have, in general, that $\min\{N_{1,i} - 1, \omega\} - \min\{N_{1,i-1} - 1, \omega\}$ intersecting connections enter at stage i.

In expression (5.3f), the separable component of the number of blocked paths associated with each intersecting connection departing at stage j is given by $\lceil N_{1,j-1}/N \rceil$, which is a monotonically nondecreasing function in j. Therefore, an upper bound on the component of the number of blocked paths associated with departure points will be achieved if each stage j has as many as possible departing intersecting connections, given that each stage $k > j$ also meets this objective. This leads to an upper bound of $\min\{M_{j,s} - 1, \omega\} - \min\{M_{j+1,s} - 1, \omega\}$ intersecting connections departing at stage j. [We should note that the third component term in expression (5.3f) is a function of neither i nor j and so does not affect this discussion.]

8.3. Justifying the Assumption of ω Intersecting Connections

In Sections 8.1 and 8.2, we developed expressions for the allocation of entry and departure points that gives an upper bound on the number of blocked paths assuming that there are a total of ω intersecting connections. We now show that the assumption of ω intersecting connections is correct, that is, there is no number $d < \omega$ of intersecting connections that can maximally block as many paths as can ω intersecting connections. We do this by proving that for any given intersecting connection the upper bound on the number of blocked paths (as given by expression 5.3f) is always positive.

Our first observation is that the sum of the first two terms of expression (5.3f), that is, $\lceil M_{i+1,s}/M \rceil + \lceil N_{1,j-1}/N \rceil$, always has a value of at least 2, since both of these terms always have a value of at least 1. Thus, we need to show that the remaining portion of expression (5.3f), that is, $\sum_{k=t+1}^{u-2} \lceil N_{1,k}/N \rceil (\lceil M_{k+1,s}/M \rceil - \lceil M_{k+2,s}/M \rceil) - \lceil M_{t+2,s}/M \rceil$, never has a value less than minus one.

CASE 1. $t + 1 \geq u - 1$.

For this case, the first term (the summation term) has a value of zero because the beginning index exceeds the terminating index. Additionally, the second term equals one, since $t + 2 \geq u$, and u (as defined in Section 5.3) is the smallest stage number satisfying $\lceil M_{u,s}/M \rceil = 1$. Thus, the first term minus the second term equals minus one.

CASE 2. $t + 1 \leq u - 2$.

First, we note that $\lceil M_{k+1,s}/M \rceil - \lceil M_{k+2,s}/M \rceil$ is never negative because $\lceil M_{k+1,s}/M \rceil$ is monotonically nonincreasing in k. Next, for all k within the limits of the summation, we have that $\lceil N_{1,k}/N \rceil \geq \lceil N_{1,t+1}/N \rceil \geq 2$, because (as defined in Section 5.3) t is the largest stage number satisfying $\lceil N_{1,t}/N \rceil = 1$. Therefore,

$$\sum_{k=t+1}^{u-2} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right)$$

$$- \left\lceil \frac{M_{t+2,s}}{M} \right\rceil \geq \sum_{k=t+1}^{u-2} 2 \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right)$$

$$- \left\lceil \frac{M_{t+2,s}}{M} \right\rceil = 2 \left\lceil \frac{M_{t+2,s}}{M} \right\rceil - 2 \left\lceil \frac{M_{u,s}}{M} \right\rceil$$

$$- \left\lceil \frac{M_{t+2,s}}{M} \right\rceil \quad \text{(from alternate term cancellation)}$$

$$= \left\lceil \frac{M_{t+2,s}}{M} \right\rceil - 2 \left\lceil \frac{M_{u,s}}{M} \right\rceil = \left\lceil \frac{M_{t+2,s}}{M} \right\rceil$$

$$- 2 \geq 2 - 2 = 0 \quad \text{(by definitions of } u \text{ and } t).$$

This proves that expression (5.3f) always has a positive value, thus justifying our assumption of ω intersecting connections.

9. AN UPPER BOUND ON BLOCKED PATHS

We are now in a position to combine the results of Section 8 with expression (5.3f) to produce the three component terms of BLOCKED PATHS in Figure 4.

9.1. Expressions for the Three Components of BLOCKED PATHS

From Section 8.2, we have $(\min\{N_{1,i} - 1, \omega\} - \min\{N_{1,i-1} - 1, \omega\})$ intersecting connections entering at stage i, and from expression (5.3f), we have $\lceil M_{i+1,s}/M \rceil$ as the separable blocked path contribution per intersecting connection entering at stage i. Multiplying these two terms together and summing for all i, we have

$$\sum_{i=1}^{s} (\min\{N_{1,i} - 1, \omega\} - \min\{N_{1,i-1} - 1, \omega\}) \left\lceil \frac{M_{i+1,s}}{M} \right\rceil \tag{9.1a}$$

as an upper bound on the intersecting connection entry point contribution of number of blocked paths.

From Section 8.2, we have $(\min\{M_{i,s} - 1, \omega\} - \min\{M_{i+1,s} - 1, \omega\})$ intersecting connections departing at stage i, and from expression (5.3f), we have $\lceil N_{1,i-1}/N \rceil$ as the separable blocked path contribution per intersecting connection departing at stage i. Multiplying these two terms together and summing for all i, we have

$$\sum_{i=1}^{s} \left(\min\{M_{i,s} - 1, \omega\} - \min\{M_{i+1,s} - 1, \omega\}\right) \left\lceil \frac{N_{1,i-1}}{N} \right\rceil \tag{9.1b}$$

as an upper bound on the intersecting connection departure point contribution of number of blocked paths.

The third component term of BLOCKED PATHS is simply the product of ω and the remaining portion of expression (5.3f) (a constant value for every intersecting connection). Thus, we have

$$\omega\left(\sum_{k=t+1}^{u-2} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) - \left\lceil \frac{M_{t+2,s}}{M} \right\rceil \right) \tag{9.1c}$$

for this component.

We have finally accounted for all of the elements of Figure 4. Also, since the sum of expressions (9.1a), (9.1b), and (9.1c) is an upper bound on the total number of paths blocked in $L(x, y)$, we have now established expressions for both PATHS and BLOCKED PATHS. Thus, we are ready to state the main result of this paper—a theorem that gives sufficient conditions for EGS networks to be strictly nonblocking for point-to-point connections. Later, we will consider a refinement to our logic that has the potential to reduce the upper bound on the total number of blocked paths for many practical EGS networks. (This refinement is not indicated on Fig. 4.)

10. THE MAIN RESULT—A STRICTLY NONBLOCKING THEOREM FOR EGS NETWORKS

We have already proved the theorem that we are about to state. That is what the logic map in Figure 4 is all about. We will review this logic after presenting the theorem. We will also discuss some of the attributes and ramifications of the theorem.

Additionally, we will show that the theorem includes (as a small but important subset) strictly nonblocking three-stage Clos networks. We will derive some important special case versions of the theorem and provide several examples.

10.1. A Strictly Nonblocking Theorem

Theorem 10.1. An EGS network is strictly nonblocking for point-to-point connections if $N_{1,k}$ divides N for $N_{1,k} \leq \omega$ and $M_{k,s}$ divides M for $M_{k,s} \leq \omega$, where

$$\omega = \min_{1 \leq k \leq s-2} \{N_{1,k} + M_{k+2,s} - 2, N - 1, M - 1\},$$

and if

$$\left\lfloor \frac{M_{1,s}}{M} \right\rfloor > \sum_{i=1}^{s} \left(\min\{N_{1,i} - 1, \omega\} - \min\{N_{1,i-1} - 1, \omega\}\right) \left\lceil \frac{M_{i+1,s}}{M} \right\rceil + \sum_{i=1}^{s} \left(\min\{M_{i,s} - 1, \omega\} - \min\{M_{i+1,s} - 1, \omega\}\right) \left\lceil \frac{N_{1,i-1}}{N} \right\rceil + \omega\left(\sum_{k=t+1}^{u-2} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) - \left\lceil \frac{M_{t+2,s}}{M} \right\rceil \right),$$

where t is the largest stage number satisfying $\lceil N_{1,t}/N \rceil = 1$ and u is the smallest stage number satisfying $\lceil M_{u,s}/M \rceil = 1$.

Review of Proof. According to the Strictly Nonblocking Sufficient Condition in Section 3.0, we needed to show that the minimum number of paths between any inlet/outlet pair exceeds the maximum number of paths which could be blocked between any inlet/outlet pair. In Section 4 (specifically Theorem 4.2a), we established that there are at least $\lfloor M_{1,s}/M \rfloor$ paths between any inlet/outlet pair. Most of the rest of the paper up to this point has been involved in the process of determining an upper bound on the number of paths which can be blocked between any inlet/outlet pair.

Two results were especially crucial in this endeavor: The first was the formulation of expression (5.3f) as an upper bound on the number of paths blocked by a single intersecting connection. In developing this expression, a simplifying assumption was introduced for those intersecting connections composed of three or more intersecting links. With this simplification, all blocked paths were counted at least once (instead of exactly once), and,

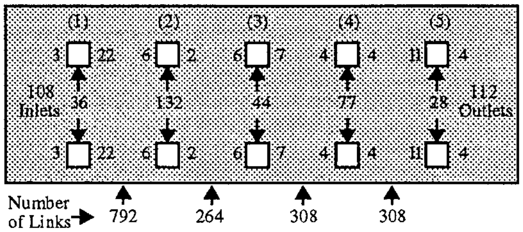

Fig. 7. Nonblocking EGS network ρ.

thus, a valid upper bound remained. For most practical networks, the overcount is either zero or very small, preserving the utility of the approximation. However, the most important aspect resulting from this simplifying assumption was that the upper bound became a separable function of the intersecting connection entry point and departure point, plus a constant. This meant that we did not have to associate the entry point and departure point of any particular intersecting connection. We needed only to determine the numbers of entering and departing intersecting connections at each stage, such that we maintained an upper bound on the total number of blocked paths.

This led to our second crucial result, namely, the Forward–Backward Invariance Theorem (Theorem 6.2), in which we established that the Forward–Backward Invariance Property holds for stages i through j if the division condition holds for stages i through j. The importance of this result was that it limited the number of connections that could intersect channel graph $L(x, y)$, assuming that the division condition held where necessary.

Utilizing this information, we next established ω [expression (7.3a)], an upper bound on the number of intersecting connections. We then allocated ω entry points and departure points, guaranteeing that we would maintain an upper bound on the number of blocked paths which result from multiplying the numbers of entering and departing intersecting connections at each stage with their corresponding blocked-path components from expression (5.3f).

A Note on Sufficiency. We have made two assumptions in this proof that may result in overcounting the maximum possible number of blocked paths. Thus, we can assert the sufficiency, but not the necessity, of the given conditions. However, in many (most?) practical networks, the stated upper bounds will either be exact or in error by only a small amount. Let us consider these two assumptions to better understand when overcounting is likely to occur:

The first assumption is the simplification mentioned above regarding the formulation of expression (5.3). It was noted in Section 5.2 that this approximation can only overcount blocked paths if there are at least two paths between a pair of intersecting links of an intersecting connection. This will generally not happen unless there are a relatively large number of stages in the network.

The second assumption is that all of the blocked paths of all of the intersecting connections are different. This assumption is implicit in our process of simply adding the various blocked path components (for ω intersecting connections) to arrive at a total number of blocked paths. We must be careful here to distinguish that we are not talking about the likelihood of all counted blocked paths being different, but whether or not this is even possible. As with the first assumption, we are more likely to have overcounted in a network with a large number of stages. Indeed, a little later, we will describe a refinement that shows that we have necessarily overcounted in some networks.

10.2. Observations

The first thing that we note about Theorem 10.1 is its generality. The conditions are in terms of the number of inlets and number of outlets (N and M) of the network, the number of inlets and number of outlets (n_i and m_i) on the switching modules in each stage i of the network, and the number of stages (s) of the network. There are no individual constraints on any of these items. There is no required relationship between N and M. There is no symmetry required in the network. There is no relationship required between the size of the switching modules in one stage and any other stage.

Perhaps the most important aspect of the theorem is that nowhere does there appear the ratio of any n_i to any m_i. The implication is that for the most part nonblocking networks can be designed with arbitrarily sized switching modules. Another way of viewing this situation is that the conditions for nonblocking operation are global as opposed to being specific to particular switching modules. The intuitive reason for this is that the nonblocking conditions were established for the network as a whole and not for subsets thereof, that is, nothing was defined or constructed recursively. (There are no imbedded strictly nonblocking subsets in these networks.)

Due in part to its generality, Theorem 10.1 is rather complex. However, much of the complexity disappears for networks exhibiting various constraints, uniformity, and/or symmetry. We will present these simplifications later.

10.3. An Example Nonblocking Network

Figure 7 depicts a five-stage EGS network ρ that has a different number of inlets than outlets and different size switching modules in each stage. We will evaluate the various expressions in Theorem 10.1 to see if this network is strictly nonblocking. We choose this nonuniform and

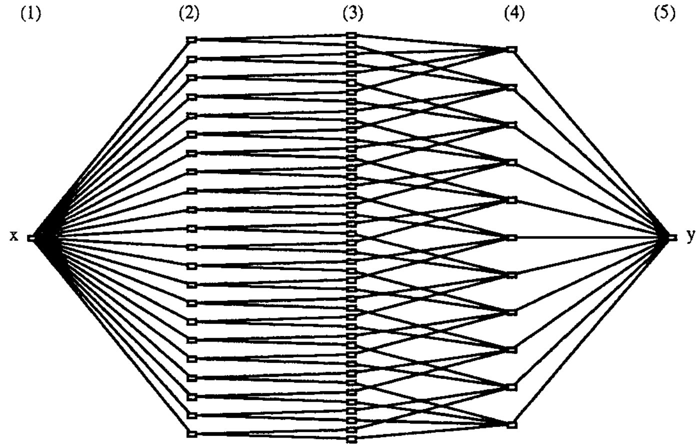

Fig. 8. A channel graph $L(x, y)$ of network ρ.

nonsymmetric network as our first example so that we may better understand the general capability of this theorem. In the spirit of nonuniformity, note also the numbers of links between stages in this network.

The relevant parameters of the network are $N = 108$, $M = 112$, n_1 through $n_5 = 3, 6, 6, 4$, and 11, respectively, m_1 through $m_5 = 22, 2, 7, 4$, and 4, respectively, and $s = 5$.

We first calculate $\lfloor M_{1,s}/M \rfloor$, the minimum number of paths between any inlet/outlet pair. We have

$$\left\lfloor \frac{M_{1,s}}{M} \right\rfloor = \left\lfloor \frac{22 \times 2 \times 7 \times 4 \times 4}{112} \right\rfloor = \lfloor 44 \rfloor = 44.$$

Therefore, there are exactly 44 paths between every inlet/outlet pair. Figure 8 depicts the channel graph $L(x, y)$ of some inlet x and some outlet y of network ρ. (The reader may wish to verify that there are indeed 44 paths between x and y in this channel graph.)

Next, we need to determine ω, the maximum number of connections that can intersect $L(x, y)$. We have

$$\begin{aligned}\omega &= \min_{1 \le k \le s-2} \{N_{1,k} + M_{k+2,s} - 2, N - 1, M - 1\}\\ &= \min\{N_{1,1} + M_{3,5} - 2, N_{1,2} + M_{4,5} - 2,\\ &\quad N_{1,3} + M_{5,5} - 2, 107, 111\}\\ &= \min\{3 + 112 - 2, 18 + 16 - 2,\\ &\quad 108 + 4 - 2, 107, 111\}\\ &= \min\{113, 32, 110, 107, 111\} = 32.\end{aligned}$$

This limitation on the number of intersecting connections results when our indexing variable $k = 2$. Let us review the logic employed here. When $k = 2$, we are adding the maximum number of intersecting connections that can <u>enter</u> $L(x, y)$ in stages 1 and 2 to the maximum number of intersecting connections that can <u>depart</u> $L(x, y)$ in stages 4 and 5. This addition produces an upper bound by implicitly assuming that the entering and departing intersecting connections are all different, that is, all of the intersecting connections entering at stages 1 and 2 depart at or before stage 3 and all of the intersecting connections departing at stages 4 and 5 enter at or after stage 3. If this is not so, then one or more of the intersecting connections entering at stages 1 or 2 must depart at stages 4 or 5, thus resulting in fewer than 32 total intersecting connections.

Our result that $\omega = 32$ is provisional at this point. We must additionally verify that $N_{1,k}$ divides N for $N_{1,k} \le \omega$ and $M_{k,s}$ divides M for $M_{k,s} \le \omega$. Since $N_{1,3} = 108 = N$ and $M_{3,5} = 112 = M$, these division conditions are satisfied and our result for ω is valid. We should note here that if both division conditions had not been satisfied, we would be unable to establish the nonblocking status of network ρ because of a failure to satisfy all of the conditions of Theorem 10.1.

Next, we determine stage numbers t and u. Since $N_{1,3} = 108 = N < 432 = N_{1,4}$, we have $t = 3$. Also, since $M_{3,5} = 112 = M < 224 = M_{2,5}$, we have $u = 3$. Therefore, $t + 1 > u - 2$ and so

$$\omega\left(\sum_{k=t+1}^{u-2} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right)\right.$$

$$\left. - \left\lceil \frac{M_{t+2,s}}{M} \right\rceil \right) = 32\left(0 - \left\lceil \frac{M_{5,5}}{M} \right\rceil \right)$$

$$= 32\left(- \left\lceil \frac{4}{112} \right\rceil \right) = -32.$$

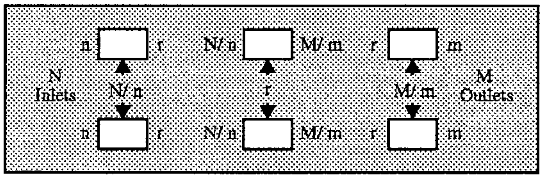

Fig. 9. General three-stage Clos network.

Our last step is to determine if

$$\left\lfloor \frac{M_{1,s}}{M} \right\rfloor = 44 > \sum_{i=1}^{s} (\min\{N_{1,i} - 1, \omega\} - \min\{N_{1,i-1} - 1, \omega\}) \left\lceil \frac{M_{i+1,s}}{M} \right\rceil + \sum_{i=1}^{s} \left((\min\{M_{i,s} - 1, \omega\} - \min\{M_{i+1,s} - 1, \omega\}) \left\lceil \frac{N_{1,i-1}}{N} \right\rceil \right) - 32.$$

Evaluating the right side of this expression, we have

$$\begin{aligned}
&[(2 - 0) \times 2 + (17 - 2) \times 1 + (32 - 17) \times 1 \\
&\quad + (32 - 32) \times 1 + (32 - 32) \times 1] \\
&\quad + [(32 - 32) \times 1 + (32 - 32) \times 1 \\
&\quad + (32 - 15) \times 1 + (15 - 3) \times 1 + (3 - 0) \times 4] \\
&\quad - 32 = (2 \times 2 + 15 \times 1 + 15 \times 1) \\
&\quad + (17 \times 1 + 12 \times 1 + 3 \times 4) - 32 \\
&\qquad = 34 + 41 - 32 = 43 < 44.
\end{aligned}$$

Thus, we have at most 43 of the 44 paths between x and y that can be blocked and so network ρ is strictly nonblocking.

This calculation assumed that two intersecting connections entered at stage 1, 15 at stage 2, and 15 at stage 3. It also assumed that 17 departed at stage 3, 12 at stage 2, and 3 at stage 1. Part of the power of Theorem 10.1 is that the determination of an upper bound on the number of blocked paths does not require these entrance and departure points be pairwise associated with specific intersecting connections.

Earlier we noted the nonuniformity of the numbers of links between stages in network ρ. We mention it again to underscore the generality that it suggests. Observe, for example, that the minimum number of links occurs between stages 2 and 3. Thus, we do not require the number of links to be uniform or increasing as we move toward the center of the network.

This example was chosen to illustrate various points. It is not likely that anyone would ever choose to construct such a network. However, it is interesting to note that, even with such ''strange'' switching module sizes, we have produced a reasonably efficient network. Using the number of crosspoints as the conventional complexity metric, we calculate $3 \times 22 \times 36 + 6 \times 2 \times 132 + 6 \times 7 \times 44 + 4 \times 4 \times 77 + 11 \times 4 \times 28 = 8272$ as compared to $108 \times 112 = 12096$ for a single-stage crossbar switch.

10.4. Corollary to Theorem 10.1

For many practical EGS networks, it will be the case that $\min_{1 \le k \le s-2}\{N_{1,k} + M_{k+2,s} - 2\} \le \min\{N - 1, M - 1\}$. For such networks, we have the following corollary of Theorem 10.1:

Corollary 10.4. If $\omega = \min_{1 \le k \le s-2}\{N_{1,k} + M_{k+2,s} - 2\}$, then for the value of k giving ω, Theorem 10.1 becomes

$$\left\lfloor \frac{M_{1,s}}{M} \right\rfloor > \sum_{i=1}^{k} (N_{1,i} - N_{1,i-1}) \left\lceil \frac{M_{i+1,s}}{M} \right\rceil + \sum_{i=k+2}^{s} (M_{i,s} - M_{i+1,s}) \left\lceil \frac{N_{1,i-1}}{N} \right\rceil.$$

Proof. For the value of k giving ω, we must have $k \le t$ and $k + 2 \ge u$, for, otherwise, $\omega = N_{1,k} + M_{k+2,s} - 2 \ge N_{1,k} - 1 > N - 1$ and/or $\omega = N_{1,k} + M_{k+2,s} - 2 \ge M_{k+2,s} - 1 > M - 1$, contradicting our assumption on ω. It follows that $u - t \le 2$. Hence,

$$\sum_{k=t+1}^{u-2} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) - \left\lceil \frac{M_{t+2,s}}{M} \right\rceil = - \left\lceil \frac{M_{t+2,s}}{M} \right\rceil = -1.$$

The right side of (10.1) can be written as

$$\begin{aligned}
&\sum_{i=1}^{k} [(N_{1,i} - 1) - (N_{1,i-1} - 1)] \left\lceil \frac{M_{i+1,s}}{M} \right\rceil \\
&\quad + \sum_{i=k+1}^{t+1} (\min\{N_{1,i} - 1, \omega\} - \min\{N_{1,i-1} - 1, \omega\}) \\
&\quad + \sum_{i=k+2}^{s} [(M_{i,s} - 1) - (M_{i+1,s} - 1)] \left\lceil \frac{N_{1,i-1}}{M} \right\rceil \\
&\quad + \sum_{i=u-1}^{k+1} (\min\{M_{i,s} - 1, \omega\} - \min\{M_{i+1,s} - 1, \omega\}) \\
&\quad - \omega = \sum_{i=1}^{k} (N_{1,i} - N_{1,i-1}) \left\lceil \frac{M_{i+1,s}}{M} \right\rceil
\end{aligned}$$

$$+ \min\{N_{1,t+1} - 1, \omega\} - \min\{N_{1,k} - 1, \omega\}$$
$$+ \sum_{i=k+2}^{s} (M_{i,s} - M_{i+1,s}) \left\lceil \frac{N_{1,i-1}}{N} \right\rceil$$
$$+ \min\{M_{k-1,s} - 1, \omega\} - \min\{M_{k+2,s} - 1, \omega\}$$
$$- \omega = \sum_{i=1}^{k} (N_{1,i} - N_{1,i-1}) \left\lceil \frac{M_{i+1,s}}{M} \right\rceil + \omega$$
$$- (N_{1,k} - 1) + \sum_{i=k+2}^{s} (M_{i,s} - M_{i+1,s}) \left\lceil \frac{N_{1,i-1}}{N} \right\rceil$$
$$+ \omega - (M_{k+2,s} - 1) - \omega$$
$$= \sum_{i=1}^{k} (N_{1,i} - N_{1,i-1}) \left\lceil \frac{M_{i+1,s}}{M} \right\rceil + \sum_{i=k+2}^{s} (M_{i,s} - M_{i+1,s}) \left\lceil \frac{N_{1,i-1}}{N} \right\rceil ,$$

because $(N_{1,k} - 1) + (M_{k+2,s} - 1) = \omega$.

11. SOME SPECIAL CASES OF THEOREM 10.1

Using Theorem 10.1 and/or Corollary 10.4, we develop expressions for three important special-case classes of networks. The first of these are the three-stage Clos networks.

11.1. Three-Stage Clos Networks

We will show that three-stage Clos networks are a subset of the class of networks covered by Theorem 10.1. Figure 9 depicts the generalized Clos network in which each switching module in the first stage is connected by a link to each switching module in the second stage and, similarly, each switching module in the second stage is connected by a link to every switching module in the third stage. The Clos network thus satisfies our definition of a MIN. It also satisfies the definition of an EGS network in that links from switching modules in stage i may be thought of as being connected consecutively to switching modules in stage $i + 1$.

We begin by calculating $\lfloor M_{1,s}/M \rfloor$, the minimum number of paths between any inlet/outlet pair. We have

$$\left\lfloor \frac{M_{1,s}}{M} \right\rfloor = \left\lfloor \frac{r \times (M/m) \times m}{M} \right\rfloor = r.$$

Next, we need to determine ω, the maximum number of intersecting connections. We have

$$\omega = \min_{1 \le k \le s-2} \{N_{1,k} + M_{k+2,s} - 2, N - 1, M - 1\}$$
$$= \min\{N_{1,1} + M_{3,3} - 2, N - 1, M - 1\}$$
$$= \min\{n + m - 2, N - 1, M - 1\}.$$

We will verify that $N_{1,k}$ divides N for $N_{1,k} \le \omega$ and $M_{k,s}$ divides M for $M_{k,s} \le \omega$ for any of these three possible values of ω. Since $N_{1,k}$ is product of integers, if $N_{1,k}$ divides N for $\omega \le N_{1,k}$, then $N_{1,j}$, $(j < k)$, must also divide N for $N_{1,j} \le \omega$. Now, since $\omega < N$ and $N_{1,2} = n \times N/n = N$, the first division condition is satisfied for any possible value of ω. Similarly, the second division condition is satisfied by the fact that $\omega < M$ and $N_{2,3} = (M/m) \times m = M$.

Next, we determine stage numbers t and u. Since $N_{1,2} = N$, $t \ge 2$, and since $M_{2,3} = M$, $u \le 2$. Therefore, $t \ge 2 \ge u \Rightarrow t + 1 > u - 2$ and $t + 2 > u$. Therefore,

$$\omega\left(\sum_{k=t+1}^{u-2} \left\lceil \frac{N_{1,k}}{N} \right\rceil \left(\left\lceil \frac{M_{k+1,s}}{M} \right\rceil - \left\lceil \frac{M_{k+2,s}}{M} \right\rceil \right) - \left\lceil \frac{M_{t+2,s}}{M} \right\rceil \right) = \omega(0 - 1) = -\omega.$$

Next, for $i = 1, 2$, or 3, we have $\lceil M_{i+1,s}/M \rceil = 1$, because $M_{i+1,s}$ is monotonically nonincreasing in i and $M_{2,3} = M$. Similarly, for $i = 1, 2$, or 3, we have $\lceil N_{1,i-1}/N \rceil = 1$ because $N_{1,i-1}$ is monotonically nondecreasing in i and $N_{1,2} = N$. Therefore,

$$\sum_{i=1}^{s} (\min\{N_{1,i} - 1, \omega\} - \min\{N_{1,i-1} - 1, \omega\}) \left\lceil \frac{M_{i+1,s}}{M} \right\rceil$$
$$= \sum_{i=1}^{s} (\min\{N_{1,i} - 1, \omega\} - \min\{N_{1,i-1} - 1, \omega\})$$
$$= \min\{N_{1,3} - 1, \omega\} - \min\{N_{1,0} - 1, \omega\}$$
(via alternate term cancellation)
$$= \min\{rN, \omega\} - \min\{0, \omega\} = \omega$$

and, similarly,

$$\sum_{i=1}^{s} (\min\{M_{i,s} - 1, \omega\} - \min\{M_{i+1,s} - 1, \omega\}) \left\lceil \frac{N_{1,i-1}}{N} \right\rceil$$
$$= \sum_{i=1}^{s} (\min\{M_{i,s} - 1, \omega\} - \min\{M_{i+1,s} - 1, \omega\})$$
$$= \min\{M_{1,3} - 1, \omega\} - \min\{M_{4,3} - 1, \omega\}$$
(via alternate term cancellation)
$$= \min\{rM - 1, \omega\} - \min\{0, \omega\} = \omega.$$

Having computed all of the PATHS and BLOCKED

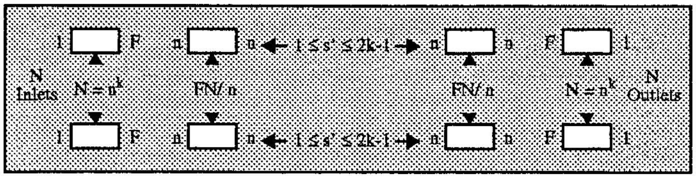

Fig. 10. Generalized uniform network $U(n)$.

PATHS terms, we have $r > \omega + \omega - \omega = \omega = \min\{n + m - 2, N - 1, M - 1\}$ so $r \geq \min\{n + m - 1, N, M\}$.

The usual stated condition for three-stage Clos networks to be strictly nonblocking is $r \geq n + m - 1$. Our more general result includes special cases in which $r < n + m - 1$ can suffice.

Let us now consider another important special case, that is, when nearly all of the switching modules are identical and the number of inlets and outlets are equal.

11.2. Uniform Networks

A generalized *uniform network* is depicted in Figure 10 and defined as follows: For $(n \geq 2)$, a uniform network $U(n)$ is an EGS network with $N = n^k$ inlets and outlets and $s = s' + 2$ stages of switching modules. The first stage of the network is composed of N $1 \times F$ switching modules. The next s' stages, $(1 \leq s' \leq 2k - 1)$, are each composed of NF/n $n \times n$ switching modules. The last stage is composed of N $F \times 1$ switching modules.

11.2.1. Determining ω

In a $U(n)$, since $n_1 = 1$ and $n_i = n$ for $2 \leq i \leq s - 1$, $N_{1,i} = n^{i-1}$ for $1 \leq i \leq s - 1$. Similarly, we have $M_{i,s} = n^{s-t}$, for $2 \leq i \leq s$. Therefore, $\min_{1\leq i\leq s-2}\{N_{1,i} + M_{i+2,s} - 2\} = \min_{1\leq i\leq s-2}\{n^{i-1} + n^{s-i-2} - 2\}$. We differentiate this expression with respect to i and for the moment consider i to be real. The result is $n^{i-1}(\ln n) - n^{s-i-2}(\ln n)$, which we set equal to zero and solve for i, yielding $i = (s - 1)/2$.

If s is odd, we have an integer solution for i, which, when substituted in $(n^{i-1} + n^{s-i-2} - 2)$, yields $n^{(s-1)/2-1} + n^{s-(s-1)/2-2} - 2 = n^{(s-3)/2} + n^{(s-3)/2} - 2 = 2n^{(s-3)/2} - 2$. In a uniform network, s has a maximum value of $2k + 1$, and since this expression is monotonically increasing in s, its maximum value is $2n^{(2k+1-3)/2} - 2 < 2n^{(2k-2)/2} = (2/n)n^k \leq n^k = N$ (because $n \geq 2$). Therefore,

$$\omega = 2n^{(s-3)/2} - 2 \quad (\text{for } s \text{ odd}). \qquad (11.2a)$$

For s even, we subtract $(n^{i-1} + n^{s-i-2} - 2)$ from $(n^{(i+1)-1} + n^{s-(i+1)-2} - 2)$, yielding $(n - 1)[n^{i-1} - n^{s-i-3}]$, which is easily shown to have a negative value if $i < (s/2) - 1$ and a positive value if $i > (s/2) - 1$. Therefore, the two possibilities for a minimum value of $(n^{i-1} + n^{s-i-2} - 2)$ are when $i = (s/2) - 1$ or $(s/2)$. Checking these two cases, we get the same result $(n + 1)n^{(s-4)/2} - 2$. Substituting $2k$ for s (the maximum value when s is even), we get $(n + 1)n^{k-2} - 2 < (n + 1)n^{k-2} = ((n + 1)/n^2)n^k < n^k = N$ (because $n \geq 2$). Therefore,

$$\omega = (n + 1)n^{(s-4)/2} - 2 \quad (\text{for } s \text{ even}). \qquad (11.2b)$$

Expressions (11.2a) and (11.2b) are provisional pending verification of the two requisite division conditions of 10.1. This is easy in the case of a uniform network because for any i $(1 \leq i \leq k + 1)$ $N_{1,i} = n^{i-1}$ divides $n^k = N > \omega$ and, similarly, for any i $(s - k \leq i \leq s)$ $M_{i,s} = n^{s-i}$ divides $n^k = N > \omega$.

We have now established that the value of ω satisfies the condition for Corollary 10.4 to apply (for both odd and even values of s) and thus we can use it in the remaining analysis of uniform networks.

11.2.2. Equal Entry and Departure Components

We observe that $N_{1,j} = n^{j-1} = M_{s-j+1,s}$ for $2 \leq j \leq s - 1$. Then, for s odd, it is easily verified that the first (second, etc.) term in $\sum_{i=1}^{(s-1)/2} (N_{1,i} - N_{1,i-1})\lceil M_{i+1,s}/M\rceil$ equals the last (second to last, etc.) term in $\sum_{i=(s+3)/2}^{s} (M_{i,s} - M_{i+1,s})\lceil N_{1,i-1}/N\rceil$ and that these two summations both have $(s - 3)/2$ terms. For $i = 1$, $N_{1,i} = N_{1,i-1} = 1$ and thus Corollary 10.4 becomes

$$\lfloor Fn^{s-k-2}\rfloor > 2\sum_{i=2}^{(s-1)/2} (n^{i-1} - n^{i-2})\lceil n^{s-i-k-1}\rceil. \qquad (11.2c)$$

For s even, we write Corollary 10.4 as

$$\lfloor Fn^{s-k-2}\rfloor > \sum_{i=1}^{s/2} (n^{i-1} - n^{i-2})\lceil n^{s-i-k-1}\rceil + \sum_{i=(s/2)+2}^{s} (n^{i-1} - n^{i-2})\lceil n^{s-i-k-1}\rceil.$$

The first summation has one more term than the second. We separate the $i = s/2$ term and then, as above, the first (second, etc.) term in the first summation equals the last (second to last, etc.) term in the second summation. This gives

$$\lfloor Fn^{s-k-2}\rfloor > 2\sum_{i=2}^{s/2-1}(n^{i-1}-n^{i-2})\lceil n^{s-i-k-1}\rceil + (n^{s/2-1}-n^{s/2-2})\lceil n^{s/2-k-1}\rceil. \quad (11.2d)$$

11.2.3. Simplifying the Entry Point Blocked Path Expression

We consider two cases:

CASE 1. $3 \le s \le k+3$.

In this case, $s-i-k-1 \le (k+3)-i-k-1 = 2-i$, so $\lceil n^{s-i-k-1}\rceil = 1$ for $2 \le i \le s$. Thus, (11.2c(d)) becomes

$$\lfloor Fn^{s-k-2}\rfloor > 2(n^{(s-3)/2}-1) \quad (\text{for } 3 \le s \le k+3). \quad (11.2e)$$

$$\lfloor Fn^{s-k-2}\rfloor > n^{s/2-1}+n^{s/2-2}-2 \quad (\text{for } 3 \le s \le k+3). \quad (11.2f)$$

CASE 2. $k+4 \le s \le 2k+1$.

We note that if $i < s-k-1$ then $\lceil n^{s-i-k-1}\rceil = n^{s-i-k-1}$ and if $i \ge s-k-1$ then $\lceil n^{s-i-k-1}\rceil = 1$. Therefore, we split the summations in (11.2c(d)) and they become

$$\lfloor Fn^{s-k-2}\rfloor > 2(s-k-3)(n^{s-k-2}-n^{s-k-3}) + 2n^{(s-3)/2}-2n^{s-k-3} \quad (11.2g)$$

(for $k+4 \le s \le 2k+1$) and

$$\lfloor Fn^{s-k-2}\rfloor > 2(s-k-3)(n^{s-k-2}-n^{s-k-3}) + n^{s/2-1}+n^{s/2-2}-2n^{s-k-3} \quad (11.2h)$$

(for $k+4 \le s \le 2k+1$).

11.2.4. Combining the Various Cases

It can easily be verified that $\lfloor[(s-k-3)(n-1)-1](n^{s-k-3})\rfloor = -1$ if $s \le k+3$ and $=[(s-k-3)(n-1)-1](n^{s-k-3})$ if $s \ge k+4$. Given that $\ddot{s}$ denotes the remainder when dividing s by 2, it can also be easily verified that $2n^{\ddot{s}/2}+(n-1)(1-\ddot{s}) = 2n^{\ddot{s}/2}$ for s odd and $=n+1$ for s even. Subsequently, it can be verified that the following expression satisfies all four expressions (11.2e, 11.2f, 11.2g, and 11.2h):

$$\lfloor Fn^{s-k-2}\rfloor > 2\left\lfloor[(s-k-3)(n-1)-1]\frac{n^{s-3}}{N}\right\rfloor + n^{(s-4)/2}[2n^{\ddot{s}/2}+(n-1)(1-\ddot{s})]. \quad (11.2i)$$

We can solve expression (11.2i) for F and thus explicitly indicate a sufficient condition for F to produce a strictly nonblocking $U(n)$. Let I represent the right side of expression (11.2i) and note that I is an integer since both terms composing I are always integers. Thus, we need $\lfloor(Fn^{s-2})/N\rfloor = \lfloor Fn^{s-k-2}\rfloor > I$ or, equivalently, $\lfloor Fn^{s-k-2}\rfloor \ge I+1$.

Now, since $Fn^{s-k-2} \ge \lfloor Fn^{s-k-2}\rfloor$, we must have $Fn^{s-k-2} \ge I+1$, implying that $F \ge (I+1)n^{-s+k+2} > \lceil(I+1)n^{-s+k+2}\rceil - 1$ and so $F \ge \lceil(I+1)n^{-s+k+2}\rceil$. Using this expression for F, we have $\lfloor Fn^{s-k-2}\rfloor \ge \lfloor\lceil(I+1)n^{-s+k+2}\rceil n^{s-k-2}\rfloor \ge \lfloor(I+1)n^{-s+k+2}n^{s-k-2}\rfloor = \lfloor(I+1)\rfloor = I+1$, as required.

Substituting the right side of expression (11.2i) for I in $F \ge \lceil(I+1)n^{-s+k+2}\rceil$, we get

$$F \ge \lceil n^{-s+k+2}(2\lfloor[(s-k-3)\times(n-1)-1](n^{s-k-3})\rfloor+1)\rceil + n^{k-s/2}[2n^{\ddot{s}/2}+(n-1)(1-\ddot{s})] \quad (11.2j)$$

as an explicit sufficient condition for F to produce a strictly nonblocking $U(n)$.

11.3. *U*(2) Networks

$U(2)$ networks are important special cases of $U(n)$ networks. This is because 2×2 switching modules are fundamental building blocks for many types of networks reported in the literature. By substituting the value of 2 for n in expression (11.2j), we immediately have the following result:

A $U(2)$ network is strictly nonblocking if

$$F \ge \lceil 2^{-s+k+2}(2\lfloor(s-k-4)(2^{s-k-3})\rfloor+1)\rceil + 2^{k-s/2}(2\times 2^{\ddot{s}/2}+1-\ddot{s}). \quad (11.3a)$$

We can decompose expression (11.3a) into four simpli-

TABLE I. *F* values for strictly nonblocking *U*(2) networks

	$3 \le s \le \log_2 N + 2$	$\log_2 N + 3 \le s \le 2\log_2 N + 1$
s even	$F \ge 3N/2^{s/2} - N/2^{s-2}$	$F \ge 3N/2^{s/2} - \log_2 N + s - 3$
s odd	$F \ge 2^{1.5}N/2^{s/2} - N/2^{s-2}$	$F \ge 2^{1.5}N/2^{s/2} - \log_2 N + s - 3$

fied specific versions corresponding to the value of s relative to k and the parity of s. The logic and manipulations required to achieve these results are quite straightforward and are thus omitted. The refinement to be described next section can improve on some of the values in Table I.

12. A REFINEMENT

In Section 10.1, we mentioned the possibility of overcounting the number of blocked paths. In this section, we develop specific conditions that guarantee that the sum of expressions (9.1a), (9.1b), and (9.1c) does indeed overcount the maximum number of blocked paths. We will not attempt to quantify this overcount (perhaps the topic of another paper), but we will use to advantage the fact that an overcount has occurred.

By Theorem 4.1i, for $1 \le i \le j \le s$, the number of paths between any stage-$(i-1)$ link and any stage-j link is either $\lfloor M_{i,j-1}/r_j \rfloor$ or $\lceil M_{i,j-1}/r_j \rceil$. Consider a $U(2)$ network and let $i = 3$ and $j = s - 2$. Then, $r_j = F2^{k-1}$ and $M_{i,j-1} = 2^{s-5}$. So if $2^{s-k-4} \ge F$, then $\lfloor M_{i,j-1}/r_j \rfloor \ge 1$ and there is at least one path between every stage-2 link and every stage-$(s-2)$ link.

Our upper bound on the maximum number of blocked paths assumes that all the paths blocked by entering and departing intersecting connections are different paths. In this case, however, if there is at least one path between every stage-2 link and every stage-$(s-2)$ link, then that path will be counted twice in the totalling of blocked paths contributed by stage-2 link intersecting connections and stage-$(s-2)$ intersecting connections. [Both the stage-2 link and stage-$(s-2)$ link will count this path as a blocked path.]

Let $s = 2k - 1$. Then, from Table I, we have $F \ge 2^{1.5}N/2^{s/2} - k + s - 3 = k$. (Remember that $k = \log_2 N$.) From Section 11.2.1, we have $(s-1)/2$ as the value of k giving ω. It can be verified that $(k-1)2^{k-3}$ gives the number of blocked paths in Corollary 10.4 (for $k \ge 3$). If an overcount by one has occurred, we need $F2^{k-3} > (k-1)2^{k-3} - 1$. If $F = k - 1$, this inequality will be satisfied. Also, from above, $2^{s-k-4} = 2^{2k-1-k-4} = 2^{k-5} > F = k - 1$ will be satisfied for $k \ge 8$.

So, in a $U(2)$ network with $k \ge 8$ and $s = 2k - 1$ stages, a value of $F = \log_2 N - 1$ is sufficient for a strictly nonblocking network. This value is one better than that of Cantor [1] or of Shyy and Lea [4]. (Note that our value of s includes the first and last fan-out and fan-in stages and so $s = 2k - 1$ means only $2k - 3$ stages of 2×2 switch modules.)

With $F = k - 1$, each of the $2k - 3$ stages has $(k-1)2^{k-1}$ 2×2 switch modules each with four crosspoints. So, the total crosspoint count is $(2k-3)4(k-1)2^{k-1} = 2(2k-3)(k-1)N$, thus substantiating our $O(N(\log N)^2)$ construction claim in the Introduction.

13. CONCLUSION

EGS networks are interesting because they are very general, efficient, and can be constructed for a strictly nonblocking operation. They include as special cases numerous existing classes of networks. The regularity of the interconnection pattern admits to very useful mathematical analyses. This extends to routing algorithms and high-speed control mechanisms, both of which are beyond the scope of this already-lengthy paper.

REFERENCES

[1] D. Cantor, On non-blocking switching networks, Networks 1 (1972), 367–377.

[2] C. Clos, A study of non-blocking switching networks, Bell Syst Tech J 32 (1953), 406–424.

[3] C.-T. Lea, Multi-$\log_2 N$ networks and their applications in high-speed electronic and photonic switching systems, IEEE Trans Commun C-38 (1990), 1740–1749.

[4] D.-J. Shyy and C.-T. Lea, $\mathrm{Log}_2(N, m, p)$ strictly nonblocking networks, IEEE Trans Commun C-39 (1991), 1502–1510.

[5] C.L. Wu and T.Y. Feng, On a class of multistage interconnection networks, IEEE Trans Commun C-29 (1980), 694–702.