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Partition polytopes over 1-dimensional points

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Abstract. We consider partitions of a finite set whose elements are associated with a single numerical attribute. For each partition we consider the vector obtained by taking the sums of the attributes corresponding to the elements in the parts (sets) of the partition, and we study the convex hulls of sets of such vectors. For sets of all partitions with prescribed number of elements in each set, we obtain a characterizing system of linear inequalities and an isomorphic representation of the face lattice. The relationship of the resulting class of polytopes to that of generalized permutahedra is explored.

Key words. partitions – polytopes – supermodular functions – system-assembly

1. Introduction

Partitioning of finite sets constitutes an important class of combinatorial optimization problems. Frequently, in partitioning problems, each element of the partitioned set is associated with a (fixed) number of numerical attributes, that is, a vector, and the evaluation of a partition depends on the vectors obtained by summing up the vectors that correspond to each part (set) of the partition. Applications of this model include graph partitioning [1], inventory grouping [3,4], location problems [4], hypothesis testing in statistics [4], storage allocation [5–7], group testing [9], and system reliability [10–12]. Here, we consider such partitioning problems where the associated vectors are one-dimensional, that is, each element of the partitioned set is associated with a <u>single</u> numerical attribute. Many of the above-mentioned applications fit this situation.

Consider the partitioning of the set $N = \{1, \ldots, n\}$ into p *parts* where each $i \in N$ is associated with a real number θ^i. Such a *partition* π is then associated with a p-vector θ^π whose j-th coordinate is the sum of the θ^i's over the indices i assigned to the j-th part of π. The *partition polytope* corresponding to a set of partitions Π, denoted $P^{(\Pi)}$, is then the convex hull of $\{\theta^\pi : \pi \in \Pi\}$. When Π is the set of all partitions with prescribed part sizes, we refer to $P^{(\Pi)}$ as a *single-shape partition polytope*. Our goal is to study these polytopes. The following paragraphs highlight some of our key findings.

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One of our results (Theorem 1) gives a representation of single-shape partition polytopes through explicit systems of linear inequalities. With Π as a set of partitions defined through the specification of the part-sizes, we show that x is in $P^{(\Pi)}$ if and only if x satisfies:

$$\sum_{j\in I} x_j \geq \min_{\pi\in\Pi} \sum_{j\in I} (\theta^\pi)_j \text{ for each } \emptyset \subset I \subset \{1,\ldots,p\}\,, \text{ and} \tag{1}$$

$$\sum_{j=1}^{p} x_j = \sum_{j=1}^{n} \theta^i \left(= \sum_{j=1}^{p} (\theta^\pi)_j \text{ for each partition } \pi \text{ having } p \text{ parts} \right)\;; \tag{2}$$

of course, the right hand-side of (1) and (2) is easily computable. This representation facilitates the solution of corresponding partitioning problems by linear programming and by methods for optimizing nonlinear objectives subject to linear constraints.

To put the above representation in perspective, we recall from the Main Theorem for Polytopes that the convex hull P of any finite subset B of R^p is the solution set of a system of linear inequalities. While the explicit identification of such a system for P is generally difficult, it is known that for each facet F of P there is a vector $c_F \in R^p$ with $F = \{x \in P : (c_F)^T x = \min_{\theta\in B}(c_F)^T\theta\}$ and P has the representation $[\cap_F\{x \in R^p : (c_F)^T x \geq \min_{\theta\in B}(c_F)^T\theta\}] \cap L$ where L is the smallest linear subspace containing B. Our result shows that for single-shape partition polytopes the c_F's can be taken as $0-1$ vectors. For a sufficient condition for (1)–(2) to provide a representation of other partition polytopes and for examples that demonstrate that such a representation need not hold for arbitrary classes of partitions see [8].

Another result (Theorem 3) is the discovery of an isomorphism of the face-lattice of single-shape partition polytopes and a family of chains. The structure we reveal is expressed both in terms of defining inequalities and in terms of defining vertices. The results put into focus the work of Barnes, Hoffman and Rothblum [2] about properties of vertices of partition polytopes; we prove the sufficiency of their necessary condition (obtained for multi-dimensional partition polytopes). In particular, we show that the direction of edges is always the difference of two unit vectors; see Sect. 4 for an important application of this property for assembly problems.

A third result (Theorem 7 and Corollary 2) is the placement of the class of single-shape partition polytopes within other families of polytopes in terms of the combinatorial structure of their faces and in terms of normal equivalence (see Sect. 2 for formal definitions). Surprisingly, when the θ^i's are distinct, each single-shape partition polytope is normally equivalent to the standard permutahedron (see Sect. 3 for a definition). But, we demonstrate that new polytopes are generated when repeated θ^i's are allowed.

Preliminaries about partitions and polytopes are summarized in Sect. 2. Our main results are stated in Sect. 3 and are proved in Sect. 5. An application of our results to optimal assembly of systems with the goal of maximizing reliability are discussed in Sect. 4.

2. Preliminaries: partitions and polytopes

Throughout we let n be a positive integer and $N \equiv \{1, \ldots, n\}$. A *(labeled) partition* is an ordered collection of sets $\pi = (\pi_1, \ldots, \pi_p)$ where $\pi_1, \ldots, \pi_p$ are disjoint, nonempty subsets of N whose union is N. In this case we refer to p as the *size* of π and to the sets $\pi_1, \ldots, \pi_p$ as the *parts of* π. Also, if the number of elements in the parts of the partition $\pi = (\pi_1, \ldots, \pi_p)$ are $n_1, \ldots, n_p$, respectively, we refer to $(n_1, \ldots, n_p)$ as the *shape* of π; of course, in this case $\sum_{j=1}^{p} n_j = |N| = n$. We sometimes refer to *p-partitions* or to $(n_1, \ldots, n_p)$*-partitions* as partitions of size p or of shape $(n_1, \ldots, n_p)$, respectively. A partition is called *consecutive* if its parts consist of consecutive integers, that is, if there is an enumeration of its parts, say $\pi_{j_1}, \ldots, \pi_{j_p}$, such that for $t = 1, \ldots, p$ and corresponding positive integers $n_{j_1}, \ldots, n_{j_p}$, $\pi_{j_t} = \{\sum_{s=1}^{t-1} n_{j_s} + 1, \ldots, \sum_{s=1}^{t} n_{j_s}\}$.

We assume that each element i in the given partitioned set N is associated with a real number θ^i and, without loss of generality, we assume that

$$\theta^1 \leq \theta^2 \leq \ldots \leq \theta^n . \tag{3}$$

We note that (3) implies that

$$\sum_{i=u+1}^{u+w} \theta^i \leq \sum_{i=v+1}^{v+w} \theta^i \text{ for nonnegative integer } u, v \text{ and } w \text{ with } u \leq v . \tag{4}$$

Further, if $u < v$ and $w > 0$, equality holds in (4) if and only if θ^i is a constant for $u + 1 \leq i \leq v + w$, thus, (4) holds strictly when the inequalities in (3) are strict, $u < v$ and $w > 0$.

We identify row and column vectors and use R^p to denote the set of either type of p-vectors. Also, we refer to the standard definitions for *affine*, *tangential*, *conic hulls* of subsets of R^p and for the *dimension* of convex sets, in particular, we use the notation aff C, tng C, cone C and dim C, respectively. Also, we refer to the standard topology in R^p and use the notation cl B for the *closure* of a subset B of R^p.

A *polytope* in R^p is the convex hull of finitely many points in R^p. The Main Theorem for Polytopes (see [16, Theorem 1.1, p. 29]) asserts that a subset of R^p is a polytope if and only if it is bounded and is the solution set of a system of linear inequalities.

For a subset $S \subseteq \{1, \ldots, n\}$ we define the *S-summation scalar* θ_S by $\theta_S \equiv \sum_{i \in S} \theta^i$. For a p-partition $\pi = (\pi_1, \ldots, \pi_p)$ we define the π*-summation-vector* θ_π by $\theta_\pi \equiv (\theta_{\pi_1}, \ldots, \theta_{\pi_p}) \in R^p$. Given a set Π of p-partitions, we define the Π*-partition polytope* by $P^{(\Pi)} \equiv \text{conv}\{\theta_\pi : \pi \in \Pi\} \subseteq R^p$.

Given a polytope P in R^p, we say that a linear inequality $\sum_{j=1}^{p} c_j x_j \leq \gamma$ is *valid* for P if $P \subseteq \{x \in R^p : \sum_{j=1}^{p} c_j x_j \leq \gamma\}$. A *face* of P is any set of the form $F = P \cap \{x \in R^p : \sum_{j=1}^{p} c_j x_j = \gamma\}$ where $\sum_{j=1}^{p} c_j x_j \leq \gamma$ is a valid inequality for P. Of course, the faces of P are themselves polytopes. A face F of P is called *proper* if $\emptyset \neq F \neq P$. Faces of dimension 0, 1 and $(\dim P) - 1$ are called *vertices*, *edges* and *facets*, respectively. At convenience, we refer to a *vertex* not only as a face of dimension zero, but also as the single element that such a face contains.

The next proposition summarizes useful properties of faces of polytopes; see [16, Propositions 2.2 and 2.3, pp. 52–53, and Theorem 2.7 and following discussion pp. 57–58].

Proposition 1. *Let P be a polytope in R^p. Then:*

(a) P is the convex hull of its vertices,
(b) intersections of faces of P are faces of P,
(c) each face of P is the intersection of facets of P,
(d) each proper face F of P is a facet of a face F' of P,
(e) the faces of a face F of P are exactly the faces of P that are contained in F, in particular, the vertices of F are the vertices of P that are contained in F,
(f) a face F' of P is strictly included in a face F of P if and only if $F' \subseteq F$ and $\dim F' < \dim F$,
(g) if P is a polytope with representation

$$\sum_{j=1}^{p} B_{kj} x_j \leq b_k \text{ for all } k \in \beta \tag{5}$$

where β is a finite index set, then each facet F of P has a representation of the form $F = \{x \in P : \sum_{j=1}^{p} C_{kj} x_j = c_r\}$ for some $k \in \beta$, and
(h) if $\dim P = 1$, then P has exactly two vertices.

Part (b) of Proposition 1 implies that, with inclusion as the partial order, the set of faces of a polytope P is a lattice, and we refer to this lattice as the *face-lattice* of P. We say that two polytopes are *combinatorically-equivalent* if there is a one-to-one dimension-preserving isomorphism of the face-lattice of one onto the face-lattice of the other, where by *isomorphism* we mean an inclusion-preserving map.

Let P be a polytope in R^p. For each nonempty face F of P, we define the *normal cone* of F, denoted N_F, by

$$N_F \equiv \{c \in R^p : F = \operatorname{argmax}_{x \in P} c^T x\}\,, \tag{6}$$

where $\operatorname{argmax}_{x \in P} c^T x$ refers to the set of maximizers of the function on P that maps $x \in P$ to $c^T x$. This definition differs from [16, p. 193] where N_F is defined by $\{c \in R^p : F \subseteq \operatorname{argmax}_{x \in P} c^T x\}$. The next proposition summarizes some facts about the normal cones; as the results refer to our non-standard definitions, we provide a proof in the Appendix.

Proposition 2. *Let P be a polytope in R^p. Then:*

(a) for every nonempty face F of P, N_F is a nonempty cone in R^p,
(b) $\{N_F : F$ is a face of $P\}$ is a partition of R^p,
(c) if F and G are two nonempty faces of P, then $F \subseteq G$ if and only if $\operatorname{cl} N_F \supseteq \operatorname{cl} N_G$,
(d) the map $F \to N_F$ is one-to-one, and
(e) for every nonempty face F of P, $\operatorname{tng} N_F = (\operatorname{tng} F)^{\perp}$, in particular, $\dim N_F = \dim(\operatorname{tng} F)^{\perp} = p - \dim F$.

The *normal fan* of a polytope $P \subseteq R^d$ is defined by $N(P) \equiv \{N_F : F$ is a nonempty face of $P\}$. Two polytopes are *normally equivalent* if their normal fans coincide. The next result shows that normal equivalence implies combinatorial equivalence.

Proposition 3. *Normal equivalence of polytopes implies combinatorial equivalence.*

Proof. Suppose P and Q are normally equivalent polytopes. Let R^p as the Euclidean space containing $N(P) = N(Q)$, then $P \subseteq R^p$ and $Q \subseteq R^p$. For a face F of P, $N_F \in N(P) = N(Q)$ implying that Q has a face, say H, with $N_H = N_F$; part (e) of Proposition 2 implies that $\dim H = p - \dim N_H = p - \dim N_F = \dim F$. As $N(P) = N(Q)$, the constructed dimension-preserving map of faces of P to faces of Q is onto; also, parts (d) and (c) of Proposition 2 show that this map is one-to-one and inclusion-preserving.

□

It can be shown that two normally equivalent polytopes have representations as the solution sets of systems of linear inequalities, say $Ax \leq b$ and $A'x \leq b'$, with $A = A'$ and identical parametrization of corresponding faces through tightening inequalities determined by subsets of the set of rows of $A = A'$. Thus, normal equivalence assures related algebraic representation beyond common combinatorial structure established in Proposition 3.

3. Statement of main results

We consider partitions with given shape; with the θ^i's given, the data then consists of an ordered list of positive integers $n_1, \ldots, n_p$ with $\sum_{j=1}^{p} n_j = n$. The set of partitions with shape $(n_1, \ldots, n_p)$ is then denoted $\Pi^{(n_1,\cdots,n_p)}$ and the corresponding partition polytope is denoted $P^{(n_1,\cdots,n_p)}$ and referred to as a *single-shape partition polytope*. We state our results about single-shape partition polytopes in the current section while proofs are provided in Sect. 5.

Single-shape partition polytopes with $n_j = 1$ for each j are called *generalized permutahedra*, the standard *permutahedron* in R^p corresponding to the case where $n_j = 1$ and $\theta^j = j$ for each j. These polytopes were first investigated by Schoute [15] (see also [16, pp.17–18 and 23]). Single-shape partition polytopes as defined herein constitute a specialization of the polytopes considered in [2] (obtained by restricting the partitioned vectors to be one-dimensional).

Henceforth, we assume that the list $(n_1, \ldots, n_p)$ is given and fixed. In particular, whenever we refer to a *partition* we mean an $(n_1, \ldots, n_p)$-partition, so, we avoid explicit reference to the shape of partitions through the prefix "$(n_1, \ldots, n_p)$-".

For a subset I of $\{1, \ldots, p\}$ we introduce the notation

$$n_I \equiv \sum_{j \in I} n_j \text{ , and} \tag{7}$$

$$\theta_{(I)} \equiv \sum_{i=1}^{n_I} \theta^i \ . \tag{8}$$

Also, let $C^{(n_1,\dots,n_p)}$ be the solution set of the system of linear inequalities given by

$$\sum_{j\in I} x_j \geq \theta_{(I)} \text{ for each nonempty subset } I \text{ of } \{1,\dots,p\}, \text{ and} \tag{9}$$

$$\sum_{j=1}^{p} x_j = \theta_{(\{1,\dots,p\})} = \sum_{i=1}^{n} \theta^i\,. \tag{10}$$

Theorem 1 (Representation of Single-Shape Partition Polytopes and their Vertices). *$P^{(n_1,\dots,n_p)} = C^{(n_1,\dots,n_p)}$ and the vertices of this polytope are precisely the θ_π's where π ranges over the consecutive partitions.*

Suppose $n_j = 1$ for each $j = 1,\dots,p$. In this case $n = p$, all partitions are consecutive, the θ_π's are the vectors obtained from coordinate-permutation of $\theta = (\theta^1,\dots,\theta^p)$, and Theorem 1 implies that the convex hull of these permuted vectors is the solution set of (9)–(10). This conclusion is an important result of Rado [14] asserting that the convex hull of all permutations of a given vector is the set of vectors that are majorized by that vector; see [13, Corollary B.3, p. 23]. The specialization of Theorem 1 with $n_j = 1$ and $\theta^j = j$ for $j = 1,\dots,p$, namely for the standard permutahedron, is due to Schoute [15]; see also [16, Ex. 0.3, p. 23].

For each $I \subseteq \{1,\dots,p\}$, let F_I be the subset of $C^{(n_1,\dots,n_p)}$ obtained by tightening the inequality corresponding to I in (9), that is,

$$F_I \equiv \{x \in C^{(n_1,\dots,n_p)} : \sum_{j\in I} x_j = \theta_{(I)}\}\,. \tag{11}$$

For each I, $C^{(n_1,\dots,n_p)} \subseteq \{x \in R^p : x_j \geq \theta_{(I)}\}$ and therefore F_I is a face of $C^{(n_1,\dots,n_p)}$. From part (b) of Proposition 1, intersections of F_I's are also faces of $C^{(n_1,\dots,n_p)}$. Further, parts (c) and (g) of Proposition 1 show that each face of $C^{(n_1,\dots,n_p)}$ is an intersection of F_I's. So, the faces of $C^{(n_1,\dots,n_p)}$ are precisely the intersections of F_I's.

A (possibly empty) sequence $I_1, I_2,\dots,I_k$ of subsets of $\{1,\dots,p\}$ is called a *chain* if $\emptyset \subset I_1 \subset I_2 \subset \dots \subset I_k \subset \{1,\dots,p\}$, in which case we refer to k as the *length* of the chain. Such a chain is usually augmented with $I_0 \equiv \emptyset$ and $I_{k+1} \equiv \{1,\dots,p\}$. We say that chain $I'_1, I'_2,\dots,I'_{k'}$ is a *subchain* of $I_1, I_2,\dots,I_k$ and that $I_1, I_2,\dots,I_k$ is a *superchain* of $I'_1, I'_2,\dots,I'_{k'}$ if $\{I'_1, I'_2,\dots,I'_{k'}\} \subseteq \{I_1, I_2,\dots,I_k\}$; we say that I'_1, $I'_2,\dots,I'_{k'}$ is a *proper subchain* of $I_1, I_2,\dots,I_k$ and that $I_1, I_2,\dots,I_k$ is a *proper superchain* of $I'_1, I'_2,\dots,I'_{k'}$ when the above inclusion is strict. The maximal length of a chain is $p-1$ and every chain has a superchain of length $p-1$.

For a chain $I_1, I_2,\dots,I_k$, we have that $\{I_t \setminus I_{t-1} : t = 1,\dots,k+1\}$ is a partition of $\{1,\dots,p\}$. In particular, if the length of the chain is $p-1$, each of the sets I_t I_{t-1} is a singleton and $\{I_t \setminus I_{t-1} : t = 1,\dots,p\} = \{\{j\} : 1 \leq j \leq p\}$. So, a chain of length $p-1$ defines an order on $\{1,\dots,p\}$ with integer j ranked in place t if $I_t \setminus I_{t-1} = \{j\}$, thus, such a chain defines a unique consecutive partition π (of N) where for $j = 1,\dots,p$

$$\pi_j \equiv \{n_{I_{t-1}} + 1,\dots,n_{I_t}\} \text{ for the unique index } t \text{ for which } I_t \setminus I_{t-1} = \{j\}\,.$$

Observing that a consecutive partition π with $\pi_{j_t} = \{\sum_{s=1}^{t-1} n_{j_s} + 1,\dots,\sum_{s=1}^{t} n_{j_s}\}$ for $t = 1,\dots,p$ corresponds uniquely to the chain $I_1 \equiv \{j_1\}$, $I_2 \equiv \{j_1, j_2\},\dots,I_{p-1} \equiv$

$\{j_1, \ldots, j_{p-1}\}$ (of length $p-1$), we have that the correspondence of chains of length $p-1$ into consecutive partitions is one-to-one and onto. We say that a consecutive partition π *is consistent with* a chain $I_1, \ldots, I_k$ if $I_1, \ldots, I_k$ is a subchain of the unique chain of length $p-1$ corresponding to π.

We say that a chain $I_1, I_2, \ldots, I_k$ is a *representing chain* of a subset F of R^p, if $F = \cap_{t=1}^k F_{I_t}$. The next theorem shows that each face F of $P^{(n_1,\cdots,n_p)}$ has a representing chain and uses such chains to characterizes F's faces, in particular, its vertices.

Theorem 2 (Chain-Representation of Faces). *A subset F of R^p is a nonempty face of $P^{(n_1,\cdots,n_p)}$ if and only if it has a representing chain, that is, there is a chain $I_1, \ldots, I_k$ with $F = \cap_{t=1}^k F_{I_t}$. Further:*

(a) if F and F' are nonempty faces of $P^{(n_1,\cdots,n_p)}$, then the following are equivalent:
  (i) $F' \subseteq F$,
  (ii) each representing chain of F has a superchain which is a representing chain of F', and
  (iii) some representing chain of F has a superchain which is a representing chain of F',

(b) if F is a nonempty face of $P^{(n_1,\cdots,n_p)}$ with representing chain $I_1, \ldots, I_k$ then a vertex v of $P^{(n_1,\cdots,n_p)}$ is in F if and only if there is a consecutive partition π which is consistent with $I_1, \ldots, I_k$ and has $v = \theta_\pi$, and

(c) if $I_1, \ldots, I_k$ is a chain, then $\cap_{t=1}^k F_{I_t} \neq \emptyset$.

A chain $I_1, I_2, \ldots, I_k$ is called *minimal* if no set can be dropped without affecting the intersection $\cap_{t=1}^k F_{I_t}$, that is, for $s = 1, \ldots, k$, $\cap_{t=1,t\neq s}^k F_{I_t} \neq \cap_{t=1}^k F_{I_t}$. Of course, subchains of minimal chains are minimal, and every chain $I_1, \ldots, I_k$ has a minimal subchain $I'_1, \ldots, I'_{k'}$ with $\cap_{t=1}^{k'} F_{I'_t} = \cap_{t=1}^k F_{I_t}$. We say that minimal chain $I'_1, \ldots, I'_{k'}$ *refines* minimal chain $I_1, \ldots, I_k$ if $I'_1, \ldots, I'_{k'}$ can be constructed from $I_1, \ldots, I_k$ by augmenting this chain with additional sets and then dropping sets which become superflous, formally $I'_1, \ldots, I'_{k'}$ refines $I_1, \ldots, I_k$ if there exists a chain $I''_1, \ldots, I''_{k''}$ which is a superchain of both $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ and $\cap_{t=1}^{k''} F_{I''_t} = \cap_{t=1}^{k'} F_{I'_t}$. We observe that the refining relationship is a partial order on the set of minimal chains.

Example 1. Suppose $P = n = 3$, $n_1 = n_2 = n_3 = 1$, $\theta^1 = 1$ and $\theta^2 = \theta^3 = 2$. The chains $I_1 = \{1, 2\}$ and $I'_1 = \{1\}$ are minimal chains and they represent the faces $\{x \in R^3 : x_1 + x_2 = 3, x_3 = 2\}$ and $\{1, 2, 2\}$, respectively, (see the forthcoming Lemma 9). Now, the chain $I''_1 = \{1\}$, $I''_2 = \{1, 2\}$ is a superchain of the above two minimal chains and $F_{I''_1} \cap F_{I''_2} = F_{I'_1}$. So, the chain I'_1 refines the chain I_1.

□

The next theorem explores minimal representing chains of faces of $P^{(n_1,\cdots,n_p)}$.

Theorem 3 (Minimal Chain Representation of Faces). *A subset F of R^p is a nonempty face of $P^{(n_1,\cdots,n_p)}$ if and only if there is a minimal chain $I_1, \ldots, I_k$ with $F = \cap_{t=1}^k F_{I_t}$, and the correspondence of nonempty faces of $P^{(n_1,\cdots,n_p)}$ onto minimal chains is one-to-one. Further, if F is a nonempty face of $P^{(n_1,\cdots,n_p)}$ corresponding to minimal chain $I_1, \ldots, I_k$ then* *(a) if* $I'_1, \ldots, I'_{k'}$ *is a chain with* $F = \cap_{t=1}^{k'} F_{I'_t}$*, then* $I_1, \ldots, I_k$ *is a subchain of* $I'_1, \ldots, I'_{k'}$*,*

(b) if F' *is a nonempty face of* $P^{(n_1,\cdots,n_p)}$*, then the following are equivalent:*

(i) $F' \subseteq F$*,*

(ii) F' *has a representing chain which is a superchain of* $I_1, \ldots, I_k$*,*

(iii) the minimal chain representing F' *refines* $I_1, \ldots, I_k$*,*

(c) a vertex v *of* $P^{(n_1,\cdots,n_p)}$ *is in* F *if and only if there is a consecutive partition* π *which is consistent with* $I_1, \ldots, I_k$ *and has* $v = \theta_\pi$*, in particular,* $F = \text{conv}\{\theta_\pi : \pi$ *is a partition which is consistent with* $I_1, \ldots, I_k\}$*, and*

(d) $\dim F \leq \dim P^{(n_1,\cdots,n_p)} - k$*.*

The following example demonstrates that a consecutive partition π need not be consistent with the minimal chain representing a face F for θ_π to be in F.

Example 1 (continued). With the earlier data $I_1 = \{1, 2\}$ is a minimal chain representing face $F = \{x \in R^3 : x_1 + x_2 = 3, x_3 = 2\}$. The partition $\pi = (\pi_1 = \{1\}, \pi_2 = \{3\}, \pi_3 = \{2\})$ is not consistent with this chain, but the vertex $\theta_\pi = (1, 2, 2)$ is in F. Of course, $(1, 2, 2)$ has the representation $\theta^{\pi'}$ for the partition $\pi' = (\pi'_1 = \{1\}, \pi'_2 = \{2\}, \pi'_3 = \{3\})$ which is consistent with the minmal chain $I_1 = \{1, 2\}$.

□

Property (a) of the minimal chain corresponding to a face F of $P^{(n_1,\cdots,n_p)}$ characterizes that chain as the common subchain of all representing chains of F, namely as the unique *minimal* representing chain for F. Property (b) shows that the correspondence of faces to minimal representing chains is an isomorphism of the face-lattice with set inclusion as the partial order onto the set of minimal chains with the "refining" partial order; in particular, we obtain a lattice structure for the minimal chains. Property (c) uses the minimal chain representing a face to characterize the vertices of that face. Finally, property (d) shows the length of the minimal chain corresponding to a face of $P^{(n_1,\cdots,n_p)}$ yields an upper bound on the dimension of that face. The following example demonstrates that, in general, these bounds are not necessarily tight.

Example 2. Suppose $\theta^1 = 1$, $\theta^2 = 2$, $\theta^3 = \theta^4 = \theta^5 = 3$, $p = 4$, $n_1 = 2$ and $n_2 = n_3 = n_4 = 1$, and let $P \equiv P^{(n_1,\cdots,n_4)}$. The minimal chain corresponding to vertex $F \equiv \{(3, 3, 3, 3)\}$ is $\{1\}$ with length is $k = 1$. As the forthcoming Corollary 1 shows that $\dim P = 3$ we have that $(\dim P) - k = 3 - 1 > 0 = \dim F$.

□

A chain $I_1, \ldots, I_k$ is called *maximal* if it has no proper superchain which is a representing chain of $F \equiv \cap_{t=1}^{k} F_{I_t}$. To study maximal chains, we define for each chain $I_1, \ldots, I_k$ the characteristic $T(I_1, \ldots, I_k)$ given by:

$$T(I_1, \ldots, I_k) \equiv \{t = 0, 1, \ldots, k : \text{where } |I_{t+1} \setminus I_t| \geq 2 \text{ and } \theta^i \text{ is constant for } n_{I_t} < i \leq n_{I_{t+1}}\}.$$

The next theorem explores maximal representing chains of faces of $P^{(n_1,\cdots,n_p)}$.

Theorem 4 (Maximal Chain Representation of Faces). *Each representing chain of a face F of $P^{(n_1,\cdots,n_p)}$ has a superchain which is a maximal representing chain of F. Further, if $I_1,\ldots,I_k$ is a representing chain of a face F of $P^{(n_1,\cdots,n_p)}$, then the following are equivalent:*

(a) $I_1,\ldots,I_k$ *is maximal,*
(b) $T(I_1,\ldots,I_k)=\emptyset$,
(c) $\dim F = p-1-k$, *and*
(d) $\operatorname{tng} F = \{z \in R^p : \sum_{j\in I_t\setminus I_{t-1}} z_j = 0 \quad \text{for } t=1,\ldots,k+1\}$.

Theorem 4 implies that all maximal representing chains of a given face F of $P^{(n_1,\cdots,n_p)}$ have a common length, namely $p-1-(\dim F)$. But, the following example illustrates that, unlike the situation for minimal representing chains examined in Theorem 3, no uniqueness is available for maximal representing chains.

Example 3. Consider Example 2. Representations of vertex $F \equiv \{(3,3,3,3)\}$ via a chain $I_1,\ldots,I_k$ as $F = \cap_{t=1}^k F_{I_t}$ are available through the minimal chain $\{1\}$ (with length 1) and any superchain of $\{1\}$; in particular, there are six such maximal superchains, e.g., $\{1\}$, $\{1,2\}$, $\{1,2,4\}$ and $\{1\}$, $\{1,3\}$, $\{1,3,4\}$, and all have length $3 = p-1-(\dim F)$.

□

Corollary 1 (θ^i's do not coincide). $\dim P^{(n_1,\cdots,n_p)} = p-1$ *if and only if either* $p=1$ *or not all* θ^i*'s coincide.*

For a subset I of $\{1,\ldots,p\}$ we let e^I be the vector in R^p with $(e^I)_j = 1$ if $j\in I$ and $(e^I)_j = 0$ if $j \in \{1,\ldots,p\}\setminus I$. Also, for a chain $I_1,\ldots,I_k$, let

$$N(I_1,\ldots,I_k) \equiv \{\sum_{t=1}^{k+1} \beta_t e^{I_t} : \beta_t < 0 \text{ for } t=1,\ldots,k \text{ and } \beta_{k+1}\in R \text{ (unrestricted)}\}\,. \tag{12}$$

We use this notation to explore representations of normal cones of faces of $P^{(n_1,\cdots,n_p)}$ defined through (6) with $P = P^{(n_1,\cdots,n_p)}$.

Theorem 5 (Representation of Normal Cones). *A representing chain $I_1,\ldots,I_k$ of a face F of $P^{(n_1,\cdots,n_p)}$ is both maximal and minimal if and only if $N(I_1,\ldots,I_k) = N_F$.*

We mention that Theorem 5 can be generalized to a face F whose minimal representing chain is not maximal. The representation for N_F is then substantially more complex and is given in terms of F's minimal chain; see the discussion following the proof of Theorem 5 in Sect. 5.

Theorem 6 (Vertices and Edges).

(a) For $v \in R^p$ the following are equivalent:
  (i) v is a vertex of $P^{(n_1,\cdots,n_p)}$,
  (ii) there is a consecutive partition π with $v = \theta_\pi$, and

(iii) *there is a chain of length* $p-1$, *say* $I_1, \ldots, I_{p-1}$, *with* $\{v\} = \cap_{t=1}^{p-1} F_{I_t}$.

(b) *For distinct vertices* v *and* v' *of* $P^{(n_1,\cdots,n_p)}$ *the following are equivalent:*

(i) $\operatorname{conv}\{v, v'\}$ *is an edge of* $P^{(n_1,\cdots,n_p)}$,

(ii) *there exist consecutive partitions* π *and* π' *such that* $\{v, v'\} = \{\theta_\pi, \theta_{\pi'}\}$, π *and* π' *coincide on all but exactly two parts, say the* j*-th and* k*-th part, and* $\pi_j \cup \pi_k = \pi'_j \cup \pi'_k$ *is a consecutive set of integers, and*

(iii) *there exist two chains of length* $p-1$, *say* $I_1, \ldots, I_{p-1}$ *and* $I'_1, \ldots, I'_{p-1}$ *such that* $\{v, v'\} = \{\cap_{t=1}^{p-1} F_{I_t}, \cap_{t=1}^{p-1} F_{I'_t}\}$ *and* $I_1, \ldots, I_{p-1}$ *and* $I'_1, \ldots, I'_{p-1}$ *have a common subchain of length* $p-2$.

Further, if the above equivalent conditions hold and j *and* k *are as in (ii), then* $v - v'$ *is a scalar multiple of* $(e^j - e^k)$.

The next theorem and corollary concern situations where the θ^i's are distinct, in particular, in such cases we have that the lattice of faces of $P^{(n_1,\cdots,n_p)}$ is isomorphic to the lattice of chains.

Theorem 7 (Distinct θ^i's). *Suppose the* θ^i*'s are distinct. Then for every nonempty face* F *of* $P^{(n_1,\cdots,n_p)}$ *there is a unique collection* $\{I_1, \ldots, I_k\}$ *of distinct subsets of* $\{1, \ldots, p\}$ *with* $F = \cap_{t=1}^{k} F_{I_t}$, *further, the subsets* $I_1, \ldots, I_k$ *are well-ordered under set-inclusion, that is, with possible relabeling* $I_1, \ldots, I_k$ *is a chain.*

□

Corollary 2 (Distinct θ^i's). *Suppose the* θ^i*'s are distinct. Then:*

(a) $\dim P^{(n_1,\cdots,n_p)} = p-1$.

(b) *A subset* F *of* R^p *is a nonempty face of* $P^{(n_1,\cdots,n_p)}$ *if and only if there is a chain* $I_1, \ldots, I_k$ *with* $F = \cap_{t=1}^{k} F_{I_t}$ *and the correspondence of nonempty faces of* $P^{(n_1,\cdots,n_p)}$ *to chains is one-to-one; in particular, a nonempty face of* $P^{(n_1,\cdots,n_p)}$ *has a unique maximal representing chain which is a minimal chain.*

(c) $P^{(n_1,\cdots,n_p)}$ *is normally equivalent to the standard permutahedron in* R^p *(defined in the second paragraph of this section).*

(d) *If* F *is a nonempty face of* $P^{(n_1,\cdots,n_p)}$ *corresponding to chain* $I_1, \ldots, I_k$ *then*

(i) *a nonempty face* F' *of* $P^{(n_1,\cdots,n_p)}$ *with corresponding chain* $I'_1, \ldots, I'_{k'}$ *is included in* F *if and only if* $I_1, \ldots, I_k$ *is a subchain of* $I'_1, \ldots, I'_{k'}$,

(ii) *for a consecutive partition* π, $\theta_\pi \in F$ *if and only if* π *is consistent with* $I_1, \ldots, I_k$, *in particular,* $F = \operatorname{conv}\{\theta_\pi : \pi$ *is a partition which is consistent with* $I_1, \ldots, I_k\}$,

(iii) $\dim F = p - 1 - k$, *and*

(iv) $\operatorname{tng} F = \{z \in R^p : \sum_{j \in I_t \setminus I_{t-1}} z_j = 0 \quad \textit{for } t = 1, \ldots, k\}$.

(e) *For* $v \in R^p$ *the following are equivalent:*

(i) v *is a vertex of* $P^{(n_1,\cdots,n_p)}$,

(ii) *there is a unique consecutive partition* π *with* $v = \theta_\pi$, *and*

(iii) *there is a unique chain* $I_1, \ldots, I_{p-1}$, *with* $\{v\} = \cap_{t=1}^{p-1} F_{I_t}$;

further, if the above equivalent conditions hold, $I_1, \ldots, I_{p-1}$ *is the chain corresponding to* π.

(f) *For a pair of vertices* v *and* v' *of* $P^{(n_1,\cdots,n_p)}$ *the following are equivalent:*

(i) conv$\{v, v'\}$ *is an edge of* $P^{(n_1,\ldots,n_p)}$,

(ii) there exist consecutive partitions π *and* π' *such that* $\{v, v'\} = \{\theta_\pi, \theta_{\pi'}, \pi$ *and* π' *coincide on all but exactly two parts, say the* j*-th and* k*-th part, and* $\pi_j \cup \pi_k = \pi'_j \cup \pi'_k$ *is a consecutive set of integers, and*

(iii) there exist two chains of length $p-1$*, say* $I_1, \ldots, I_{p-1}$ *and* $I'_1, \ldots, I'_{p-1}$ *such that* $\{v, v'\} = \{\cap_{t=1}^{p-1} F_{I_t}, \cap_{t=1}^{p-1} F_{I'_t}\}$ *and* $I_1, \ldots, I_{p-1}$ *and* $I'_1, \ldots, I'_{p-1}$ *have a common subchain of length* $p-2$*;*

further, if the above equivalent conditions hold and j *and* k *are as in (ii),* $v - v'$ *is a scalar multiple of* $(e^j - e^k)$.

The conclusions of Corollary 2 about the face structure of $P^{(n_1,\ldots,n_p)}$ when $n_j = 1$ and $\theta^j = j$ for $j = 1, \ldots, p$ are due to Schoute [15]; see also [16, pp. 17, 23].

Part (c) of Corollary 2 implies that for each positive integer p, all single-shape partition polytopes corresponding to data that includes distinct θ^i's and p as the size of the partitions are normally equivalent, in particular, they are combinatorically equivalent. This conclusion holds independently of the particular values of the θ^i's and of the values of the n_j's!

In view of the conclusion of part (c) of Corollary 2, one may speculate that each partition polytope is normally, or at least combinatorically, equivalent to a generalized permutahedron. The next example shows that this conjecture is false.

Example 4. Suppose $P = 3$, $n = 6$, $\theta^1 = \theta^2 = 1$ and $\theta^3 = \theta^4 = \theta^5 = \theta^6 = 2$. Consider the single-shape partition polytope corresponding to shape $(1, 2, 3)$. By Theorem 1, the vertices of this polytope are the θ_π's associated with the consecutive partitions. Table 1 lists the consecutive partitions and associated vertices – it demonstrates that there are exactly 4 vertices.

The generalized permutahedron in R^3 is determined by 3 parameters, say η_1, η_2 and η_3 where $\eta_1 \le \eta_2 \le \eta_3$; the permutahedron is then given by the solution set of the following linear inequality system (for example, see Theorem 1)

$$
\begin{array}{lll}
x_1 \ge \eta_1 & x_2 \ge \eta_1 & x_3 \ge \eta_1 \\
x_1 + x_2 \ge \eta_1 + \eta_2 & x_1 + x_3 \ge \eta_1 + \eta_2 & x_2 + x_3 \ge \eta_1 + \eta_2 \\
x_1 + x_2 + x_3 = \eta_1 + \eta_2 + \eta_3 & &
\end{array}
$$

Now, if the η_j's coincide, the generalized permutahedron is a single point. In the case where the η_j's do not coincide, that is, $\eta_1 < \eta_3$, we have a two-dimensional polytope (see Corollary 1) whose projection of the first two coordinates is the solution set of the following linear system:

$$\eta_1 \le x_1 \le \eta_3, \quad \eta_1 \le x_2 \le \eta_3, \quad \eta_1 + \eta_2 \le x_1 + x_2 \le \eta_2 + \eta_3 .$$

It is easily seen that this set is a sextahegon if $\eta_1 < \eta_2 < \eta_3$ and a triangle if either $\eta_1 = \eta_2 < \eta_3$ or $\eta_1 < \eta_2 = \eta_3$; in the former case the set has exactly six vertices and in the latter two cases it has exactly three vertices. In neither case is the set combinatorically equivalent to our single-shape partition polytope and by Proposition 3 it can neither be normally equivalent to it.

□

Table 1.

Determining Permutation of π_1, π_2 and π_3	The Consecutive Partition π_1	π_2	π_3	θ_π
π_1, π_2, π_3	{1}	{2, 3}	{4, 5, 6}	(1, 3, 6)
π_1, π_3, π_2	{1}	{5, 6}	{2, 3, 4}	(1, 4, 5)
π_2, π_1, π_3	{3}	{1, 2}	{4, 5, 6}	(2, 2, 6)
π_2, π_3, π_1	{6}	{1, 2}	{3, 4, 5}	(2, 2, 6)
π_3, π_1, π_2	{4}	{5, 6}	{1, 2, 3}	(2, 4, 4)
π_3, π_2, π_1	{6}	{4, 5}	{1, 2, 3}	(2, 4, 4)

We conclude this section with the observation that chains are in one-to-one correspondence with partitions of $\{1, \ldots, p\}$ with chain $I_1, I_2, \ldots, I_k$ corresponding to the partition $(I_1, I_2 \setminus I_1, \ldots, I_{k+1} \setminus I_k)$. Chain-inclusion then corresponds to partition-refinement. A complete alternative analysis can be carried out by focusing on partitions rather than on chains.

4. Optimal partitions with applications to system assembly

The definition of Schur Convexity was recently extended by Hwang and Rothblum [11] to asymmetric functions. Specifically, a real-valued function on R^p is called *quasi-convex along* a nonzero vector $d \in R^p$, or briefly *d-quasi-convex*, if the maximum of the function over every line-segment with direction d is attained at one of the two endpoints of that line-segment. With e^j for $j = 1, \ldots, p$ as the j-unit vector in R^p and $D \equiv \{e^j - e^k : j, k = 1, \ldots, p\}$, a real-valued function is called *asymmetric Schur convex* if it is d-quasi convex for every d in D. Theorem 2.3 of [11] then demonstrates that when such a function g is maximized over a polytope P all of whose edges have direction in D, then g attains a maximum over P at an extreme point; this is the case with P as the unit simplex in R^p or, by Theorem 6, with P as a single-shape partition polytope. In particular, as Theorem 1 shows that the extreme points of single-shape partition polytopes correspond to consecutive partitions, we obtain the following result:

Theorem 8. *Let g be an asymmetric Schur convex function on R^p and consider the real-valued function U on partitions defined by $U(\pi) = g(\theta^\pi)$. Then U attains a maximum over $\Pi^{(n_1,\ldots,n_p)}$ at a consecutive partition.*

□

We next demonstrate an application of Theorem 7 for problems of system assembly with the goal of maximizing reliability.

Consider a system having p modules as components. Each of these modules can be either *operative* or *inoperative*. The *state* of the system is determined by the set of operative modules and is represented by a vector $s \in \{0, 1\}^p$, where $s_i = 0$ if module i is inoperative and $s_i = 1$ if module i is operative. The operativeness of the system is determined by a *structure function* $J : \{0, 1\}^p \to \{0, 1\}$, i.e., the system is *inoperative* if it is in a state s with $J(s) = 0$ and the system is *operative* if it is in a state s with $J(s) = 1$. The system is called *coherent* if the structure function is *monotone*, that is, if $J(s) \leq J(s')$ for $s, s' \in \{0,1\}^p$ with $s \leq s'$. (We note that the standard definition of coherence has an added requirement which we do not need herein.)

The modules are assumed to be composed of parts which are functionally interchangeable, with module $j \in \{1, \ldots, p\}$ requiring exactly $n_j > 0$ parts. The modules are constructed in series, that is, a module is operative if and only if each of its parts is operative. All needed $n = \sum_j n_j$ parts are assumed to be available. An *assembly* for the system is an assignment of parts to the modules in a way that matches the requirements of each module; it corresponds to a partition with shape $(n_1, \ldots, n_p)$ and we identify assemblies and partitions.

The *reliability* of a part, a module and the system as a whole is the probability of being operative. We assume that positive reliabilities of the parts are given and that operativeness of the parts are stochastically independent. Also, the parts are enumerated in a weakly increasing order of their reliability. So, with θ^i as the log of the reliability of the i-th part we have that (3) is satisfied.

The reliability of a module depends on its composition. Given an assembly $\pi = (\pi_1, \ldots, \pi_p)$, the series structure of the modules implies that the reliability of module j is given by

$$\rho(\pi)_j \equiv \prod_{i \in \pi_j} \exp(\theta^i) = \exp\left(\sum_{i \in \pi_j} \theta^i\right) = \exp(\theta_{\pi_j}), \text{ for } j = 1, \ldots, p\,; \tag{13}$$

The reliability of the system as a whole depends on the way it is constructed. Let ρ be a vector whose coordinates $\rho_1, \ldots \rho_p$ are, respectively, the reliabilities of the modules. Then the system's reliability is the expectation of $J(\underline{s})$ where $\underline{s}$ is a random vector whose components have independent Binomial distributions with coefficients $\rho_1, \rho_2, \ldots, \rho_p$ and is given by

$$f(\rho) = \sum_{s \in \{0.1\}^p} J(s) \left[\prod_{\{j: s_j = 0\}} (1 - \rho_j)\right] \left[\left(\prod_{j: s_j = 1} \rho_j\right)\right]. \tag{14}$$

With $g : R^p \to R$ as the function defined for $\xi \in R^p$ by $g(\xi) = f(e^{\xi_1}, \ldots, e^{\xi_p})$, the system's reliability under assembly $\pi = (\pi_1, \ldots, \pi_p)$ is then given by

$$U(\pi) \equiv f\left[\rho(\pi)_1, \ldots, \rho(\pi)_p\right] = g(\theta_{\pi_1}, \ldots, \theta_{\pi_p}) = g(\theta_\pi)\,. \tag{15}$$

The goal is to find an assembly π that maximizes the system-reliability function $U(.)$.

As $f(\rho)$ is the expectation of a function of a random vector with independent coordinates, we get by conditioning on the positions of any pair of modules, say module j and module k, that $f(\rho)$ can be decomposed into

$$\begin{aligned} f(\rho) &= (1-\rho_j)(1-\rho_k)h_{00}(\rho) + (1-\rho_j)\rho_k h_{01}(\rho) + \rho_j(1-\rho_k)h_{10}(\rho) + \rho_j\rho_k h_{11}(\rho) \\ &= \rho_j\rho_k\,[h_{11}(\rho) - h_{01}(\rho) - h_{10}(\rho) + h_{00}(\rho)] + \rho_j\,[h_{10}(\rho) - h_{00}(\rho)] \\ &\quad + \rho_k\,[h_{10}(\rho) - h_{00}(\rho)] + [h_{00}(\rho) + h_{01}(\rho) + h_{10}(\rho)] \end{aligned} \tag{16}$$

with $h_{00}(\rho)$, $h_{01}(\rho)$, $h_{10}(\rho)$ and $h_{11}(\rho)$ not depending on ρ_j and ρ_k; further, system-coherence assures that $h_{00}(\rho) \leq h_{01}(\rho) \leq h_{11}(\rho)$ and $h_{00}(\rho) \leq h_{10}(\rho) \leq h_{11}(\rho)$. Consequently, for $\xi \in R^p$,

$$g(\xi) = e^{\xi_j+\xi_k} H_{00}(\xi) + e^{\xi_j} H_{01}(\xi) + e^{\xi_k} H_{10}(\xi) + H_{11}(\xi) \tag{17}$$

with the $H_{uv}(\xi)$'s not depending on ξ_j and ξ_k (for $u, v \in \{0, 1\}$); further, system-coherence implies that $H_{01}(\xi)$ and $H_{10}(\xi)$ are nonnegative and therefore that g is $(e^j - e^k)$-quasi-convex (in fact, g is then convex along any line with direction $e^j - e^k$).

We conclude that when the system is coherent, g is asymmetric Schur convex; thus, Theorem 7 implies that there exists a consecutive reliability-maximizing assembly. We note that an assembly is consecutive if there is a ranking of the modules so that for each pair of modules the parts that are assigned to the module with the higher ranking are at least as good as those assigned to the module with the lower ranking. For alternative approaches for studying the optimality of consecutive assemblies see [10] and [11].

5. Proofs

In this section we prove the results stated in Sections 3 and 4. For most of this section, till stated otherwise, we assume as in Sect. 3 that a list $(n_1, \ldots, n_p)$ is given and that all considered partitions are labeled and have shape $(n_1, \ldots, n_p)$; in particular, we omit the prefix "$(n_1, \ldots, n_p)$-".

We start by proving that the polytope $C^{(n_1,\ldots,n_p)}$ defined through (9)–(10) contains $P^{(n_1,\ldots,n_p)}$. Theorem 1 (proved below) asserts that, in fact, the two polytopes coincide.

Lemma 1. *For every partition π, $\theta_\pi \in C^{(n_1,\ldots,n_p)}$, in particular, $P^{(n_1,\ldots,n_p)} \subseteq C^{(n_1,\ldots,n_p)}$.*

Proof. Let π be a partition and I a subset of $\{1, \ldots, p\}$. As $\cup_{j\in I}\pi_j$ has $\sum_{j\in I} n_j = n_I$ elements, (3) implies that

$$\sum_{j\in I}(\theta_\pi)_j = \sum_{i\in\cup_{j\in I}\pi_j} \theta^i \geq \sum_{i=1}^{|\cup_{j\in I}\pi_j|} \theta^i = \sum_{i=1}^{n_I} \theta^i = \theta_{(I)}$$

with equality holding for $I = \{1, \ldots, p\}$, that is, $\theta_\pi \in C^{(n_1,\ldots,n_p)}$. As $C^{(n_1,\ldots,n_p)}$ is convex, we conclude that $P^{(n_1,\ldots,n_p)} = \text{conv}\{\theta_\pi : \pi \in \Pi^{(n_1,\ldots,n_p)}\} \subseteq C^{(n_1,\ldots,n_p)}$.

A real-valued function f on subsets of $\{1, \ldots, p\}$ is called *supermodular* if

$$f(I \cup J) + f(I \cap J) \geq f(I) + f(J) \text{ for every pair } I \text{ and } J \text{ of subsets of } \{1, \ldots, p\}\,; \tag{18}$$

the function f is called *strictly supermodular* if strict inequality holds whenever neither of the two sets is a subset of the other, that is, $I \not\subseteq J$ and $J \not\subseteq I$.

Lemma 2. *The function $\theta_{(.)}$ mapping $I \subseteq \{1, \ldots, p\}$ into $\theta_{(I)}$ given by (7)–(8) is supermodular; further, if I and J are subsets of $\{1, \ldots, p\}$ where $I \not\subseteq J$ and $J \not\subseteq I$, then $\theta_{(.)}$ satisfies (18) with equality if and only if θ^i is a constant for $n_{I\cap J} < i \leq n_{I\cup J}$.*

□

Proof. For subsets I and J of $\{1,\ldots,p\}$, $n_{I\cup J} = n_I + n_{J\setminus I}$, $n_J = n_{I\cap J} + n_{J\setminus I}$, and (4) with $u \equiv n_{I\cap J}$, $v \equiv n_I (\geq n_{I\cap J} = u)$ and $w = n_{J\setminus I} \geq 0$ implies that

$$\theta_{(I\cup J)} - \theta_{(I)} = \sum_{i=1}^{n_{I\cup J}} \theta^i - \sum_{i=1}^{n_I} \theta^i = \sum_{i=n_I+1}^{n_{I\cup J}} \theta^i = \sum_{i=n_I+1}^{n_I+n_{J\setminus I}} \theta^i$$
$$\geq \sum_{i=n_{I\cap J}+1}^{n_{I\cap J}+n_{J\setminus I}} \theta^i = \sum_{i=n_{I\cap J}+1}^{n_J} \theta^i = \sum_{i=1}^{n_J} \theta^i - \sum_{i=1}^{n_{I\cap J}} \theta^i = \theta_{(J)} - \theta_{(I\cap J)}\ ;$$

further, as $I \subsetneq J$ and $J \subsetneq I$ if and only if $n_{I\cap J} < n_I$ and $n_{J\setminus I} > 0$, respectively, we have from the comment following (4) that when $I \subsetneq J$ and $J \subsetneq I$, $\theta_{(.)}$ satisfies (18) with equality if and only if θ^i is a constant for $n_{I\cap J} < i \leq n_{I\cap J}$.

□

The next lemma establishes a key property of the F_I's (defined through (11)).

Lemma 3. *Let I and J be subsets of $\{1,\ldots,p\}$ with $F_I \cap F_J \neq \emptyset$. Then $F_I \cap F_J = F_{I\cup J} \cap F_{I\cap J}$; further, if $I \subsetneq J$ and $J \subsetneq I$, then θ^i is a constant for $n_{I\cap J} < i \leq n_{I\cup J}$.*

Proof. Let $x \in F_I \cap F_J$ (such x exists as it is assumed that $F_I \cap F_J \neq \emptyset$). Then

$$\theta_{(I)} + \theta_{(J)} = \sum_{j\in I} x_j + \sum_{j\in J} x_j = \sum_{j\in I\cup J} x_j + \sum_{j\in I\cap J} x_j \geq \theta_{(I\cup J)} + \theta_{(I\cap J)} \geq \theta_{(I)} + \theta_{(J)}\ ,$$

where the inequalities follow from (9) applied to $I \cup J$ and to $J \cap J$ and from Lemma 2. It follows that all of the above inequalities hold as equalities, thus, $\sum_{i\in I\cup J} x_i = \theta_{(I\cup J)}$, $\sum_{i\in I\cap J} x_i = \theta_{(I\cap J)}$ and $\theta_{(I\cup J)} + \theta_{(I\cap J)} = \theta_{(I)} + \theta_{(J)}$; as each vector in $F_I \cap F_J$ is in $F_{I\cup J} \cap F_{I\cap J}$, we have that $F_I \cap F_J \subseteq F_{I\cup J} \cap F_{I\cap J}$. To see the reverse inclusion let $y \in F_{I\cup J} \cap F_{I\cap J}$. Then

$$\theta_{(I)} + \theta_{(J)} \leq \sum_{j\in I} y_j + \sum_{j\in J} y_j = \sum_{j\in I\cup J} y_j + \sum_{j\in I\cap J} y_j = \theta_{(I\cup J)} + \theta_{(I\cap J)} = \theta_{(I)} + \theta_{(J)}\ ;$$

it follows that all of the above inequalities hold as equalities, thus, $\sum_{j\in I} y_j = \theta_{(I)}$ and $\sum_{j\in J} y_j = \theta_{(J)}$, that is, $y \in F_I \cap F_J$. So, the inclusion $F_{I\cup J} \cap F_{I\cap J} \subseteq F_I \cap F_J$ has also been established. Finally, if $I \subsetneq J$ and $J \subsetneq I$, then the established equality $\theta_{(I\cup J)} + \theta_{(I\cap J)} = \theta_{(I)} + \theta_{(J)}$ combines with Lemma 1 to show that θ^i is a constant for $n_{I\cap J} < i \leq n_{I\cup J}$.

□

Recall the one-to-one correspondence of chains of length $p-1$ and consecutive partitions observed in Sect. 3. The next lemma characterizes the summation vectors of consecutive partitions in terms of the corresponding chains.

Lemma 4. *Let $I_1, I_2, \ldots, I_{p-1}$ be a chain of length $p-1$ and let π be the corresponding consecutive partition. Then θ_π is the unique solution of the linear system*

$$\sum_{j\in I_t} x_j = \theta_{(I_t)} \text{ for } t = 1,\ldots,p\ ; \tag{19}$$

in particular, $\cap_{t=1}^{p-1} F_{I_t} = \{\theta_\pi\}$.

Proof. As $\{I_t \setminus I_{t-1} : t = 1, \ldots, p\} = \{\{j\} : 1 \le j \le p\}$, we have that for each $t = 1, \ldots, p$ there is a positive integer $j_t \in \{1, \ldots, p\}$ with $I_t \setminus I_{t-1} = \{j_t\}$. It follows that (19) can be written as

$$\sum_{s=1}^{t} x_{j_s} = \theta_{(I_t)} \text{ for } t = 1, \ldots, p\,;$$

this triangular linear system has a unique solution given by $x_{j_t} = \theta_{(I_t)} - \theta_{(I_{t-1})}$ for $t = 1, \ldots, p$. As π is the consecutive partition corresponding the chain $I_1, I_2, \ldots, I_{p-1}$, for each t, $\pi_{j_t} = \{\sum_{s=1}^{t-1} n_{j_s} + 1, \ldots, \sum_{s=1}^{t} n_{j_s}\}$ and $\theta_{\pi_{j_t}} = \theta_{(I_t)} - \theta_{(I_{t-1})} = x_{j_t}$; hence, $x = \theta_\pi$. As Lemma 1 assures that $\theta_\pi \in C$, it follows that $\cap_{t=1}^{p-1} F_{I_t} = \{\theta_\pi\}$.

□

Lemma 5. *Let $I_1, \ldots, I_k$ be a chain. Then there exists a consecutive partition which is consistent with $I_1, \ldots, I_k$; further, for each such consecutive partition π, $\theta_\pi \in \cap_{t=1}^{k} F_{I_t}$.*

Proof. Trivially, the chain $I_1, \ldots, I_k$ has a superchain of length $p-1$, say $I'_1, \ldots, I'_{p-1}$. As $I_1, \ldots, I_k$ is a subchain of $I'_1, \ldots, I'_{p-1}$, the consecutive partition corresponding to $I'_1, \ldots, I'_{p-1}$ is consistent with $I_1, \ldots, I_k$. Next, let π be a consecutive partition which is consistent with the chain $I_1, \ldots, I_k$. Then the (unique) chain of length $p-1$ corresponding to π, say $I'_1, \ldots, I'_{p-1}$, is a superchain of $I_1, \ldots, I_k$, and Lemma 4 implies that $\theta_\pi \in \cap_{t=1}^{p-1} F_{I'_t} \subseteq \cap_{t=1}^{k} F_{I_t}$.

□

The next lemma shows how to represent nonempty intersections of arbitrary F_I's to intersections corresponding to chains.

Lemma 6. *Let $I_1, \ldots, I_k$ be a chain and let $I'_1, \ldots, I'_{k'}$ be nonempty proper subsets of $\{1, \ldots, p\}$ with $F = (\cap_{t=1}^{k} F_{I_t}) \cap (\cap_{t=1}^{k'} F_{I'_t}) \neq \emptyset$. Then there exists a superchain of $I_1, \ldots, I_k$ which is a representing chain of F.*

Proof. It suffices to consider the case with $k' = 1$, in which case we let I stand for I'_1. So, $F \equiv (\cap_{t=1}^{k} F_{I_t}) \cap F_I \neq \emptyset$. For $t = 1, \ldots, k+1$, let $J_t \equiv I_{t-1} \cup [(I_t \setminus I_{t-1}) \cap I]$. We next prove by induction that for $s = 0, 1, \ldots, k$, $F = (\cap_{t=1}^{s} F_{J_t}) \cap F_{I_s \cup I} \cap (\cap_{t=s+1}^{k} F_{I_t})$. As $I_0 = \emptyset$, the case where $s = 0$ follows from the representation $F = F_I \cap (\cap_{t=1}^{k} F_{I_t})$. Next assume that the asserted representation holds for $0 \le s < k$. As $I_s \subseteq I_{s+1}$, we have that $(I_s \cup I) \cap I_{s+1} = I_s \cup [(I_{s+1} \setminus I_s) \cap I] = J_{s+1}$ and $(I_s \cup I) \cup I_{s+1} = I_{s+1} \cup I$; by the induction assumption $F_{I_s \cup I} \cap F_{I_s+1} \supseteq F \neq \emptyset$, hence, Lemma 3 implies that $F_{I_s \cup I} \cap F_{I_s+1} = F_{(I_s \cup I) \cap I_{s+1}} \cap F_{(I_s \cup I) \cup I_{s+1}} = F_{J_{s+1}} \cap F_{I_{s+1} \cup I}$ and therefore $F = (\cap_{t=1}^{s} F_{J_t}) \cap F_{I_s \cup I} \cap (\cap_{t=s+1}^{k} F_{I_t}) = (\cap_{t=1}^{s+1} F_{J_t}) \cap F_{I_{s+1} \cup I} \cap (\cap_{t=s+2}^{k} F_{I_t})$. Thus, the induction hypothesis has been established with $s+1$ replacing s. As $I_{k+1} = \{1, \ldots, p\}$, we have that $I_k \cup I = I_k \cup [(I_{k+1} \setminus I_k) \cap I] = J_{k+1}$ and the verified inductive hypothesis with $s = k$ proves that $F = \cap_{t=1}^{k+1} F_{J_t}$. We next observe that

$$\emptyset = I_0 \subseteq J_1 \subseteq I_1 \subseteq J_2 \subseteq \ldots \subseteq J_k \subseteq I_k \subseteq J_{k+1} \subseteq I_{k+1} = \{1, \ldots, p\}\,; \tag{20}$$

as $\cap_{t=1}^{k+1} F_{J_t} = F \subseteq \cap_{t=1}^{k} F_{I_t}$ it follows that the constructed superchain of $I_1, \dots, I_k$ is a representing chain of F.

□

Lemma 7. *A subset F of R^p is a nonempty face of $C^{(n_1,\cdots,n_p)}$ if and only if there exists a chain $I_1, \dots, I_k$ with $F = \cap_{t=1}^{k} F_{I_t}$.*

Proof. First assume that $F = \cap_{t=1}^{k} F_{I_t}$ where $I_1, \dots, I_k$ is a chain. Following the definition of the F_I's in (11) we already observed that intersections of F_I's are faces of $C^{(n_1,\cdots,n_p)}$. To see that $F = \cap_{t=1}^{k} F_{I_t} \neq \emptyset$, observe that the chain $I_1, \dots, I_k$ has a superchain of length $p-1$, say $I'_1, \dots, I'_{p-1}$; by Lemma 4, if π is the (unique) consecutive partition corresponding to $I'_1, \dots, I'_{p-1}$, then $\theta_\pi \in \cap_{t=1}^{p-1} F_{I'_t} \subseteq \cap_{t=1}^{k} F_{I_t}$. So, F is a nonempty face of $C^{(n_1,\cdots,n_p)}$.

Next assume that F is a nonempty face of $C^{(n_1,\cdots,n_p)}$. As observed following the definition of the F_I's in (11), there exist subsets $I'_1, \dots, I'_{k'}$ of $\{1, \dots, p\}$ with $\emptyset \neq F = \cap_{t=1}^{k'} F_{I'_t}$. As the empty chain is a representing chain of $C^{(n_1,\cdots,n_p)}$, Lemma 6 implies the existence a representing chain of $F \cap C^{(n_1,\cdots,n_p)} = F$ which is a superchain of the empty chain.

□

Proof of Theorem 1. By Lemma 1, $P^{(n_1,\cdots,n_p)} \subseteq C^{(n_1,\cdots,n_p)}$. We prove the reverse inclusion by showing that each vertex v of $C^{(n_1,\cdots,n_p)}$ is in $P^{(n_1,\cdots,n_p)}$; see part (a) of Proposition 1. Now, by Lemma 7, there exists a chain $I_1, \dots, I_k$ such that $\{v\} = \cap_{t=1}^{k} F_{I_t}$; this chain has a superchain of length $p-1$, say $I'_1, \dots, I'_{p-1}$. Let π be the (unique) consecutive partition corresponding to this chain. By Lemma 4, $\{\theta_\pi\} \in \cap_{t=1}^{p-1} F_{I'_t} \subseteq \cap_{t=1}^{k} F_{I_t} = \{v\}$, implying that $v = \theta_\pi \in P^{(n_1,\cdots,n_p)}$; in particular, v has a representation $v = \theta_\pi$ with π as a consecutive partition.

To complete the proof of Theorem 1, let π be a consecutive partition and let $I_1, \dots, I_{p-1}$ be the chain of length $p-1$ corresponding to π. By Lemma 4 and Lemma 7, $\{\theta_\pi\} = \cap_{t=1}^{p-1} F_{I_t}$ and this set is a face of $C^{(n_1,\cdots,n_p)}$; so, θ_π is a vertex of $C^{(n_1,\cdots,n_p)} = P^{(n_1,\cdots,n_p)}$.

□

Proof of Theorem 2. Lemma 7 shows that a subset F of R^p is a nonempty face of $P^{(n_1,\cdots,n_p)}$ if and only if it has a representing chain; in particular, this conclusion assures that for each chain $I_1, \dots, I_k$, $\cap_{t=1}^{k} F_{I_t}$ is nonempty, proving (c).

To establish (a), let F and F' be nonempty faces of $P^{(n_1,\cdots,n_p)}$. The implication (i) $\Rightarrow$ (ii) follows from Lemma 6 and the implication (ii) $\Rightarrow$ (iii) is immediate from the established existence of representing chains of faces. Finally, if $I_1, \dots, I_k$ is a representing chain of F and $I'_1, \dots, I'_{k'}$ is a superchain of $I_1, \dots, I_k$ which a representing chain of F', then $F = \cap_{t=1}^{k} F_{I_t} \subseteq \cap_{t=1}^{k'} F_{I'_t} = F'$.

To establish (b), let F be a nonempty face of $P^{(n_1,\cdots,n_p)}$ with representing chain $I_1, \dots, I_k$. Now, if v is a vertex of $P^{(n_1,\cdots,n_p)}$ which is in F, then $F' \equiv \{v\} \subseteq F$ is a face of $P^{(n_1,\cdots,n_p)}$ and part (a) implies the existence of a reperesenting chain of F' which is a superchain of $I_1, \dots, I_k$, say $I'_1, \dots, I'_{k'}$. By Lemma 5, there exists a consecutive partition π which is consistent with $I'_1, \dots, I'_{k'}$ and has $\theta_\pi \in \cap_{t=1}^{k} F_{I_t} = \{v\}$, that is, $\theta_\pi = v$; of course, π is also consistent with the subchain $I_1, \dots, I_k$ of $I'_1, \dots, I'_{k'}$. Alternatively, assume that v is a vertex of $P^{(n_1,\dots,n_p)}$ with representation $v = \theta_\pi$ where π is consistent with $I_1, \dots, I_k$. Then there exists a chain of length $p-1$ corresponding to π which is a superchain of $I_1, \dots, I_k$. By Lemma 4, this chain is a representing chain of $\{v\} = \{\theta_\pi\}$, and the implication (iii) $\Rightarrow$ (i) in (a) assures that $\{\theta_\pi\} \subseteq F$, that is, $\theta_\pi \in F$.

□

The next lemma characterizes the sources for non-minimality of chains.

Lemma 8. *Let $I_1, \dots, I_k$ be a chain and let $s \in \{1, \dots, k\}$. Then the following are equivalent:*

(a) $\cap_{t=1}^{k} F_{I_t} = \cap_{t=1, t\neq s}^{k} F_{I_t}$.
(b) $F_{I_{s-1}} \cap F_{I_{s+1}} \subseteq F_{I_s}$*, and*
(c) θ^i *is constant for* $n_{I_{s-1}} < i \leq n_{I_{s+1}}$*,*

Proof. (b) $\Rightarrow$ (a): This implication is trite.

(c) $\Rightarrow$ (b): Suppose $\theta^i = K$ for $n_{I_{s-1}} < i \leq n_{I_{s+1}}$. Then $\theta_{(I_s)} - \theta_{(I_{s-1})} = (n_{I_s} - n_{I_{s-1}})K$ and (as $n_{I_{s-1}} < n_{I_{s+1} \setminus (I_{s+1} \setminus I_s)} \leq n_{I_{s+1}}$)$\theta_{(I_{s+1})} - \theta_{(I_{s+1} \setminus (I_s \setminus I_{s-1}))} = (n_{I_s} - n_{I_{s-1}})K$. To prove that $F_{I_{s-1}} \cap F_{I_{s+1}} \subseteq F_{I_s}$, let $x \in F_{I_{s-1}} \cap F_{I_{s+1}}$ and we will show that $x \in F_{I_s}$. Indeed, we have that

$$\sum_{j \in I_s \setminus I_{s-1}} x_j = \sum_{j \in I_s} x_j - \sum_{j \in I_{s-1}} x_j \geq \theta_{(I_s)} - \theta_{(I_{s-1})} = \sum_{i=n_{I_{s-1}}+1}^{n_{I_s}} \theta^i = (n_{I_s} - n_{I_{s-1}})K \text{ and}$$

$$\sum_{j \in I_s \setminus I_{s-1}} x_j = \sum_{j \in I_{s+1}} x_j - \sum_{j \in I_{s+1} \setminus (I_s \setminus I_{s-1})} x_j \leq \theta_{(I_{s+1})} - \theta_{(I_{s+1} \setminus (I_s \setminus I_{s-1}))} = (n_{I_s} - n_{I_{s-1}})K\ ;$$

It follows that the inequality in the first string holds as equality; implying that $x \in F_{I_s}$.

(a) $\Rightarrow$ (c): Suppose $\cap_{t=1}^{k} F_{I_t} = \cap_{t=1, t\neq s}^{k} F_{I_t}$. By Lemma 5, there exists a consecutive partition π which is consistent with the chain $I_1, \dots, I_{s-1}, J \equiv I_{s-1} \cup (I_{s+1} \setminus I_s), I_{s+1}, \dots, I_k$; for this π, $\theta_\pi \in (\cap_{t=1, t\neq s}^{k} F_{I_t}) \cap F_J \subseteq \cap_{t=1, t\neq s}^{k} F_{I_t} = \cap_{t=1}^{k} F_{I_t}$. So, $\theta_\pi \in F_{I_s} \cap F_J$. As I_s and $J = I_{s-1} \cup (I_{s+1} \setminus I_s)$ are not ordered by set inclusion, as $I_s \cap J = I_{s-1}$ and as $I_s \cup J = I_{s+1}$, Lemma 3 implies that θ^i is constant for $n_{I_{s-1}} < i \leq n_{I_{s+1}}$.

□

The next lemma is needed for exploring minimal chains.

Lemma 9. *Let I be a subset of $\{1, \dots, p\}$ and let $x \in F_I$. Then:*

(a) if $t \in I$*, then* $x_t \leq n_t \theta^{n_I}$*, and*
(b) if $t \in \{1, \dots, p\} \setminus I$*, then* $x_t \geq n_t \theta^{n_{I}+1}$*.*

Proof. As $x \in F_I$, $\sum_{j\in I} x_j = \theta_{(I)}$. Also, $\sum_{j\in J} x_j \geq \theta_{(J)}$ for all $J \subseteq \{1, \ldots, p\}$. Using (3),

$$x_t = \sum_{j\in I} x_j - \sum_{j\in I\setminus\{t\}} x_j \leq \theta_{(I)} - \theta_{(I\setminus\{t\})} = \sum_{i=n_I-n_t+1}^{n_I} \theta^i \leq n_t\theta^{n_I} \text{ for } t \in I, \text{ and}$$

$$x_t = \sum_{j\in I\cup\{t\}} x_j - \sum_{j\in I} x_j \geq \theta_{(I\cup\{t\})} - \theta_{(I)} = \sum_{i=n_I+1}^{n_I+n_t} \theta^i \geq n_t\theta^{n_I+1} \text{ for } t \in \{1, \ldots, p)\setminus I.$$

□

The next lemma proves uniqueness of minimal representing chains of faces.

Lemma 10. *If $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ are minimal chains with $\cap_{t=1}^{k} F_{I_t} = \cap_{t=1}^{k'} F_{I'_t} \neq \emptyset$, then $k = k'$ and $I_t = I'_t$ for $t = 0, 1, \ldots, k = k'$.*

Proof. Let $F \equiv \cap_{t=1}^{k} F_{I_t} = \cap_{t=1}^{k'} I'_t \neq \emptyset$. We proceed with an inductive argument and prove that for each positive integer $s \leq \min\{k, k'\} + 1$, $I_s = I'_s$, in particular, if $s = \min\{k, k'\} + 1$, then $I_s = I'_s = \{1, \ldots, p\}$ and $s = k + 1 = k' + 1$. As $I_0 = I'_0$, the assertion is trite for $s = 0$. Assume the assertion holds for integer $s < k + 1$ and we will establish the assertion with $s + 1$ replacing s.

We first argue that I_{s+1} and I'_{s+1} are ordered by set-inclusion. Aiming to establish a contradiction we assume that this conclusion is false. With $J \equiv I_{s+1} \cap I'_{s+1}$, we have that $I_s \subseteq J \subset I_{s+1}$. Also, Lemma 3 and the fact that $F_{I_{s+1}} \cap F_{I'_{s+1}} \supseteq F \neq \emptyset$ imply that $F_J = F_{I_{s+1}\cap I'_{s+1}} \supseteq F_{I_{s+1}} \cap F_{I'_{s+1}} \supseteq F$ and that θ^i is a constant for $n_J < i \leq n_{I_{s+1}\cup I'_{s+1}}$. As $I_s \subseteq J \subset I_{s+1}$ and $F_J \supseteq F$, the insertion of J into the chain $I_1, \ldots, I_k$ between I_s and I_{s+1} yields a superchain which is another representation of $F = \cap_{t=1}^{k} F_{I_t}$; by Lemma 8, it follows that if $I_s \subset J$, then θ^i is constant for $n_{I_s} < i \leq n_{I_{s+1}}$. As θ^i was shown to be constant for $n_J = n_{I_{s+1}\cap I'_{s+1}} < i \leq n_{I_{s+1}\cup I'_{s+1}}$ and as $n_{I_{s+1}} > n_{I_{s+1}} \cap I'_{s+1} = n_J$ (because I_{s+1} and I'_{s+1} are not ordered by set-inclusion), we conclude that when either $I_s = J$ or $I_s \subset J$, θ^i is a constant for $n_{I_s} < i \leq n_{I_{s+1}\cup I'_{s+1}}$.

As I_{s+1} and I'_{s+1} are not ordered by set-inclusion, neither I_{s+1} nor I'_{s+1} equals $\{1, \ldots, p\}$ assuring that $s + 1 < \min\{k, k'\} + 1$. Also, Lemma 8, the minimality of $I_1, \ldots, I_k$ and the conclusion that θ^i is a constant for $n_{I_s} < i \leq n_{I_{s+1}\cup I'_{s+1}}$ imply that $n_{I_{s+1}\cup I'_{s+1}} < n_{I_{s+2}}$. Our next step is to argue that $I'_{s+1} \subseteq I_{s+2}$. Indeed, if there exists an index u in $I'_{s+1} \setminus I_{s+2}$, then Lemma 9 and (3) imply that

$$\frac{x_u}{n_u} \leq \theta^{n_{I'_{s+1}}} \leq \theta^{n_{I_{s+2}}+1} \leq \frac{x_u}{n_u} \tag{21}$$

and equality must hold throughout (21), in particular, θ^i must be constant for $n_{I_{s+1}} \leq i \leq n_{I_{s+2}} + 1$. As we already established that θ^i is a constant for $n_{I_s} < i \leq n_{I_{s+1}\cup I'_{s+1}}$ and $n_{I_{s+1}\cup I'_{s+1}} > n_{I_{s+1}}$, θ^i must then be constant for $n_{I_s} < i \leq n_{I_{s+2}} + 1$. From Lemma 8 this conclusion is false due to the minimality of the chain $I_1, \ldots, I_k$. So, indeed, $I'_{s+1} \subseteq I_{s+2}$. It follows that $I_{s+1} \subseteq I_{s+1} \cup I'_{s+1} \subseteq I_{s+2}$, further, as I_{s+1} and I'_{s+1} are not ordered and as $I_{s+1} \cup I'_{s+1} \neq I_{s+2}$ (because $n_{I_{s+1}\cup I'_{s+1}} < n_{I_{s+2}}$), these inclusions are strict. So, with $H \equiv I_{s+1} \cup I'_{s+1}$, we have that $I_{s+1} \subset H \subset I_{s+2}$. Also, Lemma 3 implies that $F_H = F_{I_{s+1} \cup I'_{s+1}} \supseteq F_{I_{s+1}} \cap F_{I'_{s+1}} \supseteq F \neq \emptyset$. We conclude that the insertion of H into the chain $I_1, \ldots, I_k$ between I_{s+1} and I_{s+2} yields a proper superchain which is another representation of $F = \cap_{t=1}^{k} F_{I_t}$; Lemma 8 then implies that θ^i is constant for $n_{I_{s+1}} < i \leq n_{I_{s+2}}$. As we already established that θ^i is a constant for $n_{I_s} < i \leq n_{I_{s+1} \cup I'_{s+1}}$ and $n_{I_{s+1} \cup I'_{s+1}} > n_{I_{s+1}}$, we conclude that θ^i is a constant for $n_{I_s} < i \leq n_{I_{s+2}}$. This conclusion combines with Lemma 8 to contradict the minimality of $I_1, \ldots, I_k$. This contradiction proves that the assertion that I_{s+1} and I'_{s+1} are not ordered by set-inclusion is false.

We have established that I_{s+1} and I'_{s+1} are ordered by set-inclusion. Without loss of generality, we henceforth assume that $I'_{s+1} \subseteq I_{s+1}$. Assume that $I'_{s+1} \neq I_{s+1}$, that is, $I'_{s+1} \subset I_{s+1}$, and we will establish a contradiction. As $I_s = I'_s \subset I'_{s+1} \subset I_{s+1}$ and $F \subseteq F_{I'_{s+1}} = F_J$, we conclude that the insertion of I'_{s+1} into the chain $I_1, \ldots, I_k$ between I_s and I_{s+1} yields a proper superchain which is another representation of $F = (\cap_{t=1}^{k} F_{I_t})$, hence, Lemma 8 implies that θ^i is constant for $n_{I'_s} = n_{I_s} < i \leq n_{I_{s+1}}$.

We next argue that $I_{s+1} \subseteq I'_{s+2}$. Indeed, if there exists an index u in $I_{s+1} \setminus I'_{s+2}$, Lemma 9 and (3) imply that

$$\frac{x_u}{n_u} \leq \theta^{n_{I_{s+1}}} \leq \theta^{n_{I'_{s+2}}+1} \leq \frac{x_u}{n_u} \tag{22}$$

and equality must hold throughout (22), in particular, θ^i must be constant for $n_{I_{s+1}} \leq i \leq n_{I'_{s+2}} + 1$. As we already established that θ^i is a constant for $n_{I'_s} = n_{I_s} < i \leq n_{I_{s+1}}$, θ^i must then be constant for $n_{I'_s} = n_{I_s} < i \leq n_{I'_{s+2}} + 1$. From Lemma 8 this conclusion is false due to the minimality of the chain $I'_1, \ldots, I'_{k'}$. So, indeed, $I_{s+1} \subseteq I'_{s+2}$. Lemma 8, the minimality of $I'_1, \ldots, I'_{k'}$ and the conclusion that θ^i is a constant for $n_{I_s} < i \leq n_{I_{s+1}}$ imply that $I_{s+1} \neq I'_{s+2}$. So, $I'_{s+1} \subset I_{s+1} \subset I'_{s+2}$. Also, from Lemma 3, $F_{I_{s+1}} \supseteq F \neq \emptyset$. It follows that the insertion of I_{s+1} into the chain $I'_1, \ldots, I'_{k'}$ between I'_{s+1} and I'_{s+2} yields a proper superchain which is another representation of $F = \cap_{t=1}^{k} F_{I_t}$; Lemma 8 then implies that θ^i is constant for $n_{I'_{s+1}} < i \leq n_{I'_{s+2}}$. As we already established that θ^i is a constant for $n_{I_s} < i \leq n_{I_{s+1}}$ and $n_{I'_{s+1}} < n_{I_{s+1}}$, we conclude that θ^i is a constant for $n_{I'_s} = n_{I_s} < i \leq n_{I'_{s+2}}$. This conclusion combines with Lemma 8 to contradict the minimality of $I_1, \ldots, I_k$. This contradiction proves the assertion that $I_{s+1} = I'_{s+1}$.

□

Proof of Theorem 3. By Theorem 2, a subset F of R^p is a nonempty face of $P^{(n_1,\ldots,n_p)}$ if and only if there is a chain $I_1, \ldots, I_k$ with $F = \cap_{t=1}^{k} F_{I_t}$; each such chain has a minimal subchain $I'_1, \ldots, I'_{k'}$ with $\cap_{t=1}^{k'} F_{I'_t} = \cap_{t=1}^{k} F_{I_t} = F$. We conclude that a set $F \subseteq R^p$ is a nonempty face of $P^{(n_1,\ldots,n_p)}$ if and only if it has a representing chain which is minimal. By Lemma 10, a minimal chain representing a given face is unique; also, trivially, a minimal chain uniquely defines the corresponding face. So, the correspondence of nonempty faces of $P^{(n_1,\ldots,n_p)}$ onto minimal chains is one-to-one.

Next, let F be a nonempty face of $P^{(n_1,\ldots,n_p)}$ with representing minimal chain $I_1, \ldots, I_k$.

(a): If $I'_1, \ldots, I'_{k'}$ is a representing chain of F, then it has a minimal subchain $I''_1, \ldots, I''_{k''}$ with $\cap_{t=1}^{k''} F_{I''_t} = \cap_{t=1}^{k'} F_{I'_t} = F = \cap_{t=1}^{k} F_{I_t}$. As $\cap_{t=1}^{k''} F_{I''_t} = \cap_{t=1}^{k} F_{I_t}$, Lemma 10 assures that the minimal chains $I_1, \ldots, I_k$ and $I''_1, \ldots, I''_{k''}$ coincide, thus $I_1, \ldots, I_k$ is a subchain of $I'_1, \ldots, I'_{k'}$.

(b): Suppose F' is a nonempty face of $P^{(n_1,\cdots,n_p)}$. The implication (i) $\Rightarrow$ (ii) follows from part (a) of Theorem 2. To see that (ii) $\Rightarrow$ (iii) assume that F' has a representing chain $I''_1, \ldots, I''_{k''}$ which is a superchain of $I_1, \ldots, I_k$. It then follows from the established part (a) that the minimal chain representing F', say $I'_1, \ldots, I'_{k'}$, is a subchain of $I''_1, \ldots, I''_{k''}$. As minimal chains $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ are both subchains of $I''_1, \ldots, I''_{k''}$ and $\cap_{t=1}^{k''} F_{I''_t} = F' = \cap_{t=1}^{k'} F_{I'_t}$, we have that $I'_1, \ldots, I'_{k'}$ refines $I_1, \ldots, I_k$. Finally, to see that (iii) $\Rightarrow$ (i) assume the minimal chain representing F', say $I'_1, \ldots, I'_{k'}$, refines $I_1, \ldots, I_k$. Then there exists a chain $I''_1, \ldots, I''_{k''}$ which is a superchain of both $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ with $\cap_{t=1}^{k''} F_{I''_t} = \cap_{t=1}^{k'} F_{I'_t}$; in particular, $F' = \cap_{t=1}^{k''} F_{I''_t} \subseteq \cap_{t=1}^{k} F_{I_t} = F$.

(c): The characterization of the vertices of F via corresponding consecutive partitions follows from part (b) of Theorem 2 (where it is established for all representing chains, not just the minimal ones). Also, from part (a) of Proposition 1, F (as a polytope) is the convex hull of its vertices, and from part (e) of Proposition 1 the vertices of F are the vertices of $P^{(n_1,\cdots,n_p)}$ that are in F. As Theorem 1 shows that the vertices of $P^{(n_1,\cdots,n_p)}$ are precisely the θ_π's where π ranges over the consecutive partitions, the second part of (c) is immediate from the first part

(d): The minimality of the chain $I_1, \ldots, I_k$ implies that the sets $F_0 \equiv P^{(n_1,\cdots,n_p)}$ and $F_s \equiv \cap_{t=1}^{s} F_{I_t}$ for $s = 1, \ldots, k$ are distinct. As these sets are faces of $P^{(n_1,\cdots,n_p)}$ and $F_0 = P^{(n_1,\cdots,n_p)} \supset F_1 \supset \ldots \supset F_{k-1} \supset F_k = F$, part (f) of Proposition 1 implies that $\dim P^{(n_1,\cdots,n_p)} = \dim F_0 > \dim F_1 > \ldots > \dim F_{k-1} > \dim F_k = \dim F$; it follows that $\dim F \le \dim P^{(n_1,\cdots,n_p)} - k$.

□

Proof of Theorem 4. The existence of superchains of representing chains of faces of $P^{(n_1,\cdots,n_p)}$ that are maximal is immediate by iteratively taking superchains. Next, let $I_1, \ldots, I_k$ be a representing chain of face F and we will establish the equivalence of the stated four conditions.

(a) $\Leftrightarrow$ (b): By Lemma 8, a proper superchain $I'_1, \ldots, I'_{k'}$ of $I_1, \ldots, I_k$ has $\cap_{t=1}^{k'} F_{I'_t} = \cap_{t=1}^{k} F_{I_t}$ if and only if it is obtained from $I_1, \ldots, I_k$ by inserting sets between pairs I_t and I_{t+1} for $t \in T(I_1, \ldots, I_k)$; no such insertion is possible if and only if $T(I_1, \ldots, I_k) = \emptyset$.

(a) $\Rightarrow$ (c): Theorem 3 assures that F has a unique minimal representing chain, say $I'_1, \ldots, I'_{k'}$. By Lemma 8, a proper superchain of $I'_1, \ldots, I'_{k'}$ is a representing chain of F if and only if it is obtained from $I'_1, \ldots, I'_{k'}$ by inserting nested sets between pairs I'_t and I'_{t+1} for $t \in T(I'_1, \ldots, I'_{k'})$; as the number of possible distinct insertions between I'_t and I'_{t+1} for $t \in T(I'_1, \ldots, I'_{k'})$ is precisely $|I'_{t+1} \setminus I'_t| - 1$, the length of each superchain of $I'_1, \ldots, I'_{k'}$ which is a maximal representing chain of F is $c(F) \equiv k' + \sum_{t \in T(I'_1,\cdots,I'_{k'})} (|I'_{t+1} \setminus I'_t| - 1)$. By Theorem 3, all maximal representing chains of F are superchains of $I'_1, \ldots, I'_{k'}$. Thus, $c(F)$ is the common length of all maximal representing chains of F. We prove that $c(F) = p - 1 - (\dim F)$ by induction on $\dim F$.

Suppose $\dim F = 0$. Then F is a vertex and Theorem 1 implies that $F = \{\theta_\pi\}$ for some consecutive partition π. By Lemma 4, the chain of length $p - 1$ corresponding to π is a representing chain of $\{\theta_\pi\} = F$. As a chain of length $p - 1$ has no superchains, that chain is maximal and $c(F) = p - 1 = p - 1 - (\dim F)$, verifying the induction hypothesis when $\dim F = 0$.

Assume that $d \equiv \dim F > 0$, and $c(F') = p - 1 - (\dim F')$ for every face F' of $P^{(n_1,\dots,n_p)}$ with $\dim F' < d$. Let $k \equiv c(F)$ and let $I_1, \dots, I_k$ be a maximal representing chain of F. From parts (c) and (d) of Proposition 1, F has a facet F' which is a face of $P^{(n_1,\dots,n_p)}$ of dimension $d - 1 \geq 0$. Also, parts (c) and (g) of Proposition 1 imply that F has a representation which uses (10) and (9) except that some of the inequalities of (9) are tightened to equalities; consequently, part (g) of Proposition 1 applied to polytope F shows that $F' = F \cap F_I$ for some nonempty proper subset I of $\{1, \dots, p\}$. So, $\cap_{t=1}^k F_{I_t} = F \supset F' = F \cap F_I = (\cap_{t=1}^k F_{I_t}) \cap F_I$. As $\emptyset \neq F' \subset F$, Lemma 6 implies that F' has a representing chain which is a <u>proper</u> superchain of $I_1, \dots, I_k$; in particular, its length is $k + 1$ or more. As we already observed that each chain has a maximal superchain, we conclude that $c(F') \geq k + 1$. As $\dim F' = d - 1 < d$, the induction assumption implies that $c(F') = p - 1 - (\dim F') = p - 1 - (d - 1)$, implying that $c(F) = k \leq c(F') - 1 = p - d - 1$.

Next, as $\dim F = d > 0$, the established inequality $k \leq p - 1 - d$ assures that $k < p - 1$, implying that $|I_{s+1} \setminus I_s| \geq 2$ for some $s = 1, \dots, k$. We will construct a set $I_s \subset I' \subset I_{s+1}$ such that the chain $I_1, \dots, I_s, I', I_{s+1}, \dots, I_k$ is maximal. As $I_1, \dots, I_k$ is a maximal representing chain of F, the established implication (a) $\Rightarrow$ (b) implies that $T(I_1, \dots, I_k) = \emptyset$, and from the established implication (b) $\Rightarrow$ (a), the maximality of the new chain holds if:

(i) either $|I' \setminus I_s| = 1$ or θ^i is not constant for $n_{I_s} < i \leq n_{I'}$, and
(ii) either $|I_{s+1} \setminus I'| = 1$ or θ^i is constant for $n_{I'} < i \leq n_{I_{s+1}}$.

We consider two cases.

Case 1. $|I_{s+1} \setminus I_s| = 2$. Let j' be any one of the (two) elements of $I_{s+1} \setminus I_s$ and let $I' \equiv I_s \cup \{j'\}$. Then $|I' \setminus Is| = |I_{s+1} \setminus I'| = 1$ and conditions (i) and (ii) are clearly satisfied.

Case 2. $|I_{s+1} \setminus I_s| \geq 3$. Let j' be a minimizer of n_j when j ranges over $I_{s+1} \setminus I_s$, let $I_+ \equiv I_s \cup \{j'\}$ and let $I_- \equiv I_{s+1} \setminus \{j'\}$. Then $n_{I_+} = n_{I_s} + n_{j'}$, $n_{I_-} = n_{I_{s+1}} - n_{j'}$ and the selection of j' assures that $n_{j'} < (n_{I_{s+1}} - n_{I_s})/2$; thus, $n_{I_+} < n_{I_-}$ implying that $\{i : n_{I_s} < i \leq n_{I_-}\} \cap \{i : n_{I_+} < i \leq n_{I_{s+1}}\} \neq \emptyset$. As the maximality of $I_1, \dots, I_k$ assures that $T(I_1, \dots, I_k) = \emptyset$, we have that θ^i is not a constant for $n_{I_s} < i \leq n_{I_{s+1}}$. We conclude that θ^i is not constant over both $n_{I_s} < i \leq n_{I_-}$ and $n_{I_+} < i \leq n_{I_{s+1}}$. Conditions (i) and (ii) are satisfied by $I' \equiv I_+$ if θ^i is not constant over $n_{I_+} < i \leq n_{I_{s+1}}$ and by $I' \equiv I_-$ if θ^i is not constant over $n_{I_s} < i \leq n_{I_-}$.

Evidently, $I_1, \dots, I_s, I', I_{s+1}, \dots, I_k$ is a chain which is a representing chain of $F' \equiv F \cap F_{I'} = (\cap_{t=1}^k F_{I_t}) \cap F_{I'} F$. As this chain is a proper superchain of $I_1, \dots, I_k$, Theorem 2 implies that F' is a nonempty face of $P^{(n_1,\dots,n_p)}$ which is included in F; further the maximality of F assures that $F' \subset F$, implying that $\dim F' < \dim F = d$ (see part (f) of Proposition 1). As $I_1, \dots, I_s, I', I_{s+1}, \dots, I_k$ is a maximal representing chain of F', the induction assumption assures that $k + 1 = p - 1 - (\dim F')$. So, $c(F) = k = p - 2 - (\dim F') \geq p - 2 - (d - 1) = p - 1 - d$. Thus, the proof that $c(F) = p - 1 - d$ is completed.

(c) $\Rightarrow$ (a): If $I_1, \dots, I_k$ is not a maximal representing chain of F it has a proper superchain $I'_1, \dots, I'_{k'}$ which is a maximal representing chain of F; in particular, $k' > k$. By the established implication (a) $\Rightarrow$ (c), $p - 1 - (\dim F) = k' > k$.

(c) $\Leftrightarrow$ (d): As $F = \cap_{t=1}^{k} F_{I_t}$, the definition of the F_I's in (11) assures that for $x, y \in F$ and $t = 1, \dots, k+1$, $\sum_{j \in I_t} (x - y)_j = \theta_{(It)} - \theta_{(It)} = 0$, implying that tng $F \subseteq L \equiv \cap_{t=1}^{k+1} \{z \in R^p : \sum_{j \in I_t \setminus I_{t-1}} z_j = 0\}$. As L and tng F are linear subspaces, we conclude that tng $F = L$ if and only if dim(tng F) $= \dim L$. From standard arguments $\dim L = p - (k+1)$; thus, tng $F = L$ if and only if $\dim F = p - 1 - k$.

□

Proof of Corollary 1. The empty chain is a representing chain of $P^{(n_1,\dots,n_p)}$ and is the only chain of length 0. Hence by the equivalence (c) $\Leftrightarrow$ (a) of Theorem 4, $\dim P^{(n_1,\dots,n_p)} = p - 1$ if and only if the empty chain is maximal. Observing that $T(\emptyset) = \emptyset$ if and only if either $p = |I_1 \setminus I_0| = 1$ or not all θ^i's coincide, the equivalence (a) $\Leftrightarrow$ (b) of Theorem 4, implies that the empty chain is maximal if and only if either $p = 1$ or not all θ^i's coincide.

□

Proof of Theorem 5. Let $I_1, \dots, I_k$ be a chain and let c be a vector in $N(I_1, \dots, I_k)$, as defined in (12). Then c has a representation $\sum_{t=1}^{k+1} \beta_t \, \mathrm{e}^{I_t}$ where $\beta_1, \dots, \beta_k$ are negative and b_{k+1} is unrestricted. For $x \in P^{(n_1,\dots,n_p)}$, $c^{\mathrm{T}} x = \sum_{t=1}^{k+1} \beta_t (\mathrm{e}^{I_t})^{\mathrm{T}} x = \sum_{t=1}^{k+1} \beta_t (\sum_{j \in I_t} x_j) \leq \sum_{t=1}^{k+1} \beta_t \theta_{(I_t)}$ and equality holds if and only if $\sum_{j \in I_t} x_j = \theta_{(I_t)}$ for $t = 1, \dots, k$; so, $\mathrm{argmax}_{x \in P^{(n_1,\dots,n_p)}} c^{\mathrm{T}} x = \cap_{t=1}^{k} F_{I_t}$. In particular, we have that for each face F having representing chain $I_1, \dots, I_k$, $N(I_1, \dots, I_k) \subseteq N_F$.

Suppose F is a face of $P^{(n_1,\dots,n_p)}$ with representing chain $I_1, \dots, I_k$ and with $N_F = N(I_1, \dots, I_k)$. For $s \in \{1, \dots, k\}$ let $c^s \equiv \sum_{t=1, t \neq s}^{k+1} (-\mathrm{e}^{I_t})$ and $F_s \equiv \cap_{t=1, t \neq s}^{k} F_{I_t}$. Then $c^s \in N(I_1, \dots, I_{s-1}, I_{s+1}, \dots, I_k) \subseteq N_{F^s}$ (the inclusion following from the first paragraph) and $c^s \notin N(I_1, \dots, I_k) = N_F$. We conclude that $\cap_{t=1, t \neq s}^{k} F_{I_t} = F_s \neq F$ for each $s = 1, \dots, k$, assuring that $I_1, \dots, I_k$ is minimal. Next, by standard results from linear algebra tng $N(I_1, \dots, I_k) = \{\sum_{t=1}^{k+1} \beta_t \, \mathrm{e}^{I_t} : \beta_1, \dots, \beta_{k+1} \in R(\text{unrestricted})\}$; in particular, $\dim N(I_1, \dots, I_k) = k+1$. The assertion $N_F = N(I_1, \dots, I_k)$ next combines with Proposition 2 to show that $\dim F = p - \dim N_F = p - \dim N(I_1, \dots, I_k) = p - (k+1) = p - 1 - k$, and Theorem 4 implies that $I_1, \dots, I_k$ is maximal.

Next assume that F is a face of $P^{(n_1,\dots,n_p)}$ with representing chain $I_1, \dots, I_k$ which is both minimal and maximal. The first paragraph of our proof shows that $N(I_1, \dots, I_k) \subseteq N_F$. To see the reverse inclusion let $c \in N_F$ and we will show that $c \in N(I_1, \dots, I_k)$. From Proposition 2 we then have that $c \in N_F \subseteq \mathrm{tng}\, N_F = (\mathrm{tng}\, F)^{\perp}$. By Theorem 4, the maximality of $I_1, \dots, I_k$ implies that tng $F = \{z \in$

$R^p : \sum_{j\in I_t\setminus I_{t-1}} z_j = 0$ for $t = 1, \dots, k+1\}$ and therefore (by standard arguments) $(\text{tng}\, F)^{\perp}$ is the linear span of $\{e^{I_1\setminus I_0}, e^{I_2\setminus I_1}, \dots, e^{I_{k+1}\setminus I_k}\}$, or equivalently of $\{e^{I_1}, e^{I_2}, \dots, e^{I_{k+1}}\}$ (as $e^{I_j} = \sum_{u=1}^{j} e^{I_u\setminus I_{u-1}}$ for $j = 1, \dots, k+1$). Thus, the conclusion $c \in (\text{tng}\, F)^{\perp}$ implies that c has a representation $c = \sum_{t=1}^{k+1} \beta_t e^{I_t}$ and it remains to show that $\beta_1, \dots, \beta_k$ are negative. Fix $s \in \{1, \dots, k\}$. Let $F^s \equiv (\cap_{t=1,t\neq s}^{k} F_{I_t})$; the minimality of $I_1, \dots, I_k$ then assures that $F = \cap_{t=1}^{k} F_{I_t} \subset F_s$, so there is a vector x^s in $F^s \setminus F$. As $c \in N_F$, $F = \text{argmax}_{x\in P^{(n_1,\dots,n_p)}} c^{\mathrm{T}}x$; further, for all $x \in F$, $c^{\mathrm{T}}x = \sum_{t=1}^{k+1} \beta_t (e^{I_t})^{\mathrm{T}} x = \sum_{t=1}^{k+1} \beta_t (\sum_{j\in I_t} x_j) = \sum_{t=1}^{k+1} \beta_t \theta_{(I_t)}$. As $x^s \notin F$ and $x^s \in F^s = \cap_{t=1,t\neq s}^{k} F_{I_t}$, we have $\sum_{t=1}^{k+1} \beta_t \theta_{(I_t)} > c^{\mathrm{T}} x^s = \sum_{t=1,t\neq s}^{k+1} \beta_t (\sum_{j\in I_t} x_j^s) + \beta_s (\sum_{j\in I_t} x_j^s) = \sum_{t=1,t\neq s}^{k+1} \beta_t \theta_{(I_t)} + \beta_s (\sum_{j\in I_t} x_j^s)$, and therefore $\beta_s \theta_{(Is)} > \beta_s (\sum_{j\in I_t} x_j^s)$. As $\sum_{j\in I_t} x_j^s \geq \theta_{(Is)}$ it follows that $\beta_s < 0$.

□

Let F be a face of $P^{(n_1,\dots,n_p)}$. Theorem 5 provides a representation of N_F in terms of a representing chain of F which is both minimal and maximal. Further, our arguments show that if the definition of $N(I_1, \dots, I_k)$ is extended through (12) to subsets $I_1, \dots, I_k$ of $\{1, \dots, p\}$ which do not form a chain, then $N_F = N(I_1, \dots, I_k)$ provided:

(a) $F = \cap_{t=1}^{k} F_{I_t}$,
(b) for $s = 1, \dots, k$, $F \neq \cap_{t=1,t\neq s}^{k} F_{I_t}$, and
(c) $e^{I_1}, e^{I_2}, \dots, e^{I_{k+1}}$ is a basis of $(\text{tng}\, F)^{\perp}$ (with $I_{k+1} = \{1, \dots, p\}$).

(If $I_1, \dots, I_k$ is a chain, (a) implies that it is a representing chain of F, (b) implies that it is minimal, and (c) implies that $\dim F = p - \dim(\text{tng}\, F)^{\perp} = p - 1 - k$ and, in view of Theorem 4, the chain is maximal). Now, suppose $I_1, \dots, I_q$ is a minimal representing chain of F and $T \equiv T(I_1, \dots, I_q)$. It is easy to verify that $\{I_t : t = 1, \dots, q+1, t-1 \notin T\} \cup \{I_t \cup \{j\} : t \in T, j \in I_{t+1} \setminus I_t\}$ is a family of subset of $\{1, \dots, p\}$ that satisfy (a)–(c), hence, they yield a representation to N_F. If $I_1, \dots, I_q$ is maximal (on top of being minimal), then the constructed family is $\{I_1, \dots, I_q\}$ and the derived representation of N_F reduces to the one asserted in Theorem 5.

Proof of Theorem 6. (a) The implications (i) ⇒ (iii), (iii) ⇒ (ii), (ii) ⇒ (i) follow, respectively, from Theorem 4, Lemma 4 and Theorem 1.
(b) Let v and v' be two distinct vertices of $P^{(n_1,\dots,n_p)}$ and let $E \equiv \text{conv}\{v, v'\}$.

(i) ⇒ (ii): Assume E is an edge of $P^{(n_1,\dots,n_p)}$, that is, a face of dimension 1. Then v and v' are the only vertices of $P^{(n_1,\dots,n_p)}$ in E. By Theorem 4, E has a maximal representing chain of length $p-2$, say $I_1, \dots, I_{p-2}$. Evidently, $\{I_t \setminus I_{t-1} : t = 1, \dots, p-1\}$ has $p-2$ singletons and one set containing two elements, say $I_s \setminus I_{s-1} = \{j, k\}$. Thus, the chain $I_1, \dots, I_{p-2}$ has exactly two proper superchains of length $p-1$ obtained by inserting either $I \equiv I_{s-1} \cup \{j\}$ or $I' \equiv I_{s-1} \cup \{k\}$ between I_{s-1} and I_s. Let π and π' be the consecutive partitions corresponding to these two superchains, respectively, then π and π' coincide on all but parts j and k and $\pi_j \cup \pi_k = \pi'_j \cup \pi'_k = \{n_{I_{s-1}} + 1, \dots, n_{I_s}\}$. By Lemma 4, $\theta_\pi \in \cap_{t=1}^{p-2} F_{I_t} = E$, $\theta_{\pi'} \in \cap_{t=1}^{p-2} F_{I_t} = E$ and θ_π and $\theta_{\pi'}$ are vertices of $P^{(n_1,\dots,n_p)}$, thus $\{\theta_\pi, \theta_{\pi'}\} \subseteq \{v, v'\}$. We next demonstrate that $\theta_\pi \neq \theta_{\pi'}$ which will imply the equality $\{\theta_\pi, \theta_{\pi'}\} = \{v, v'\}$. Indeed, if $\theta_\pi = \theta\pi'$ then $F_I \cap F_{I'} \neq \emptyset$ and Lemma 3 implies that $F_I \cap F_{I'} = F_{I \cup I'} \cap F_{I' \cap I'} = F_{I_s} \cap F_{I_{s-1}}$; consequently, $F_I \subseteq \cap_{t=1}^{p-2} F_{I_t}$ and $\{\theta_\pi\} = \cap_{t=1}^{p-2} F_{I_t} \cap F_I = \cap_{t=1}^{p-2} F_{I_t} = E$, a contradiction to the assertion that $\dim E = 1$.

(ii) $\Rightarrow$ (iii): Assume that (ii) holds. Without loss of generality we assume that $v = \theta_\pi$ and $v' = \theta\pi'$. Consider the two chains of length $p - 1$ corresponding to π and π', say $I_1, \ldots, I_{p-1}$ and $I'_1, \ldots, I'_{p-1}$, respectively. By Lemma 4, $\{v\} = \{\theta_\pi\} = \cap_{t=1}^{p-1} F_{I_t}$, $\{v'\} = \{\theta_{\pi'}\} = \cap_{t=1}^{p-1} F_{I'_t}$. Also, the asserted properties of π and π' and the description of chains of length $p-1$ corresponding to consecutive partitions, imply that $I_1, \ldots, I_{p-1}$ and $I'_1, \ldots, I'_{p-1}$ coincide, except for a single element, that is, they have a common subchain of length $p - 2$.

(iii) $\Rightarrow$ (i): Assume that (iii) holds and let $J_1, \ldots, J_{p-2}$ be the common subchain of length $p - 2$ of $I_1, \ldots, I_{p-1}$ and $I'_1, \ldots, I'_{p-1}$ and let $F \equiv \cap_{t=1}^{p-1} F_{I_t}$. Without loss of generality assume that $\{v\} = \cap_{t=1}^{p-1} F_{I_t}$ and $\{v'\} = \cap_{t=1}^{p-1} F_{I'_t}$. We observed above that a chain of length $p-2$ has exactly two proper superchains. As $\cap_{t=1}^{p-1} F_{I_t} = \{v\} \neq \{v'\} = \cap_{t=1}^{p-1} F_{I'_t}$, it follows that $J_1, \ldots, J_{p-2}$ is maximal (for otherwise either $F = \{v\}$ or $F = \{v'\}$). Now, by part (c) of Theorem 4, $\dim F = (p-1)-(p-2) = 1$. So, F is an edge of $P^{(n_1,\ldots,n_p)}$ that contains vertices v and v' and standard results show that $F = \text{conv}\{v, v'\} = E$. Thus, indeed, E is an edge of $P^{(n_1,\ldots,n_p)}$.

Finally, if j and k are as in (ii), trivially, $v - v'$ is a scalar multiple of $(e^j - e^k)$. □

The next lemma considers the case where the θ^i's are distinct.

Lemma 11. *Suppose the θ^i's are distinct. Then:*

(a) the function $\theta_{(.)}$ mapping $I \subseteq \{1, \ldots, p\}$ into $\theta_{(I)}$ given by (7)–(8) is strictly supermodular,

(b) if $I_1, \ldots, I_k$ are distinct subsets of $\{1, \ldots, p\}$ with $\cap_{t=1}^{k} F_{I_t} \neq \emptyset$, then $I_1, \ldots, I_k$ are well-ordered under set inclusion,

(c) every chain is both minimal and maximal,

(d) if $I_1, \ldots, I_k, I'_1, \ldots, I'_{k'}$ are subsets of N with $\cap_{t=1}^{k} F_{I_t} = \cap_{t=1}^{k'} F_{I'_t} \neq \emptyset$, then $k = k'$ and $\{I_1, \ldots, I_k\} = \{I'_1, \ldots, I'_{k'}\}$, and

(e) the function mapping a partition π into θ_π is one-to-one on the set of consecutive partitions.

Proof. (a) Let I and J be subsets of $\{1, \ldots, p\}$ with $I \not\subseteq J$ and $J \not\subseteq I$. Then $|I \cup J| - |I \cap J| \geq 2$ and the assumption that the θ^i's are distinct and Lemma 2 imply that $\theta_{(I)} + \theta_{(J)} \neq \theta_{(I \cap J)} + \theta_{(I \cup J)}$.

(b) Let $I_1, \ldots, I_k$ be distinct subsets of $\{1, \ldots, p\}$ with $F \equiv \cap_{t=1}^{k} F_{I_t} \neq \emptyset$ and let $r, s \in \{1, \ldots, k\}$. If $F_{I_r} \not\subseteq F_{I_s}$ and $F_{I_s} \not\subseteq F_{I_r}$, then $n_{I_r \cup I_s} > \max\{n_{I_r}, n_{I_s}\} \geq \min\{n_{I_r}, n_{I_s}\} > n_{I_r \cap I_s}$ implying that $n_{I_r \cup I_s} - n_{I_r \cap I_s} \geq 2$. As $F_{I_r} \cap F_{I_s} \supseteq F \neq \emptyset$, Lemma 3 yields a contradiction to the assertion that the θ^i's are distinct.

(c) As the θ^i's are distinct, Lemma 8 implies that every chain is minimal. Also, trivially, for every chain $I_1, \ldots, I_k$, $T(I_1, \ldots, I_k) = \emptyset$ and therefore it must be maximal by Theorem 4.

(d) Suppose $I_1, \ldots, I_k, I'_1, \ldots, I'_{k'}$ are subsets of N with $F \equiv \cap_{t=1}^{k} F_{I_t} = \cap_{t=1}^{k'} F_{I'_t} \neq \emptyset$. By part (b), $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ are, respectively, well ordered by set inclusion, hence, by possibly permuting the sets in each group we may assume that $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ are chains. Consider an enumeration of the distinct sets in $\{I_1, \ldots, I_k, I'_1, \ldots, I'_{k'}\}$, say $I''_1, \ldots, I''_{k''}$. Then $\cap_{t=1}^{k''} F_{I''_t} = F \neq \emptyset$ and, again by part (b), after possible permutation we may assume that $I''_1, \ldots, I''_{k''}$ is a chain, further, by part (c), this chain is minimal. As $I_1, \ldots, I_k$ and $I'_1, \ldots, I'_{k'}$ are subchains of $I''_1, \ldots, I''_{k''}$ with $\cap_{t=1}^{k''} F_{I''_t} = \cap_{t=1}^{k} F_{I_t} = \cap_{t=1}^{k'} F_{I'_t} = F \neq \emptyset$, the minimality of $I''_1, \ldots, I''_{k''}$ implies that $\{I''_1, \ldots, I''_{k''}\} = \{I_1, \ldots, I_k\} = \{I'_1, \ldots, I'_{k'}\}$.

(e) Let π and π' be consecutive partitions with $\theta_\pi = \theta_{\pi'}$ and let $I_1, \ldots, I_{p-1}$ and $I'_1, \ldots, I'_{p-1}$ be chains of length $p-1$ corresponding to π and π', respectively. Then $\cap_{t=1}^{p-1} F_{I_t} = \{\theta_\pi\} = \{\theta_{\pi'}\} = \cap_{t=1}^{p-1} F_{I'_t}$ and part (d) assures that $\{I_1, \ldots, I_{p-1}\} = \{I'_1, \ldots, I'_{p-1}\}$, implying that $\pi = \pi'$.

□

The next example shows that the last conclusion of Lemma 11 is false for nonconsecutive partitions, even when the θ^i's are distinct.

Example 5. Suppose $\theta^i = i$ for $i \in N = \{1, 2, 3, 4, 5\}$, $p = n_1 = 2$ and $n_2 = 3$. Then $\pi_1 = (\{1, 4\}, \{2, 3, 5\})$ and $\pi_2 = (\{2, 3\}, \{1, 4, 5\})$. Then $\theta_{\pi_1} = \theta_{\pi_2} = (5, 10) \in R^2$.

□

Proof of Theorem 7. Proposition 1 established the representation of nonempty faces of $P^{(n_1,\ldots,n_p)}$ as intersections $\cap_{t=1}^{k} F_{I_t}$, in fact, with the sets $I_1, \ldots, I_k$ well-ordered by set inclusion. The uniqueness of these representations follows from part (d) of Lemma 11.

□

Proof of Corollary 2.

(a): The first conclusion is immediate from Corollary 1.

(b): Lemma 7 shows the characterization of nonempty faces via the existence of representing chains and Lemma 11 gives the uniqueness of representing chains when the θ^i's are distinct. Theorem 3 and Theorem 4 establish the existence of minimal and maximal representing chains, thus, the unique representing chain of a nonempty face must be both minimal and maximal.

(c): Recall the notation $N(P)$ for the normal fan of a polytope P. Let F be a face of $P^{(n_1,\ldots,n_p)}$. By part (b), F has a unique representing chain, say $I_1, \ldots, I_k$, and part (c) of Lemma 11 assures that this chain is both minimal and maximal; thus, by Theorem 5, $N(I_1, \ldots, I_k)$ defined through (12) equals N_F. Thus $N(P^{(n_1,\ldots,n_p)}) \subseteq \mathcal{N} \equiv \{N(I_1, \ldots, I_k) : I_1, \ldots, I_k$ is a chain on $\{1, \ldots, p\}\}$. To see that this inclusion holds as equality observe that if $I_1, \ldots, I_k$ is a chain on $\{1, \ldots, p\}$, the above arguments show that $N(I_1, \ldots, I_k) = N_F$ for $F = \cap_{t=1}^{k} F_{I_t}$.

The family of cones $\mathcal{N}$ depends only on p and not on the values of the n_j's or the θ^i's; and we proved that when the θ^i's are distinct $\mathcal{N} = N(P^{(n_1,\ldots,n_p)})$. Thus, with p fixed, the normal fans of all partition polytopes coincide, as long as the data has the θ^i's distince; in particular, they all coincide with the normal fan of the standard permutahedron.

(d): We have seen that the unique representing chain of a face is both minimal and maximal. Listed properties (i)–(ii) of the chain corresponding to a face F follow from Theorem 3 and the minimality of the chain, whereas (iii)–(iv) follow from Theorem 4 and its maximality.

(e) and (f): These conclusions are immediate from parts (a) and (b) of Theorem 6 and the established uniqueness of the representing chain of a face.

□

Appendix

Proof of Proposition 2.

(a): Let F be a nonempty face of P. The definition of a face assures that F is the set of maximizers over P of a linear function; that is, $N_F \neq \emptyset$. Also, it is easy to verify that if $c, d \in N_F$ and $\alpha, \beta > 0$ then $\alpha c + \beta d \in N_F$; that is, N_F is a cone in R^p.

(b): Standard results show that for each $c \in R^p$ the function mapping $x \in P$ into $c^T x$ attains a maximum over (the bounded set) P and the definition of faces assures that the corresponding set of maximizers is a face; so, each $c \in R^p$ defines a nonempty face F with $c \in N_F$. Further, as $\text{argmax}_{x \in P}\, c^T x$ is well defined, the N_F's are necessarily disjoint.

(c): Suppose F and G are two faces of P with $\text{cl}\, N_G \subseteq \text{cl}\, N_F$ and let $x \in F$. Let $c \in N_G$ (such a vector exists by part (a)). Then $c \in N_G \subseteq \text{cl}\, N_G \subseteq \text{cl}\, N_F$, implying that there exists a sequence of vectors $\{c_n\}_{n=1,2,\ldots}$ in N_F which converges to c as $n \to \infty$. In particular, for each $y \in P$, $(c_n)^T x \geq (c_n)^T y$ for $n = 1, 2, \ldots,$ implying that $c^T x \geq c^T y$. Thus, $x \in \text{argmax}_{x \in P}\, c^T x = G$. Thus we proved that $F \subseteq G$.

Next assume that $F \subseteq G$ and let $c \in N_G$. Then $\text{argmax}_{x \in P}\, c^T x = G \supseteq F$, that is, $c^T x \geq c^T y$ for every $x \in F$ and $y \in P$. Now, let c^* be a vector in N_F (such a vector exists by part (a)). Then $\text{argmax}_{x \in P} (c^*)^T x = F$, that is, $(c^*)^T x \geq (c^*)^T y$ for every $x \in F$ and $y \in P$ with strict inequality holding for $y \in P \setminus F$. For each $\varepsilon > 0$, $(c + \varepsilon c^*)^T x \geq (c + \varepsilon c^*)^T y$ for every $x \in F$ and $y \in P$ with strict inequality holding for $y \in P \setminus F$, implying that $c + \varepsilon c^* \in N_F$. As $c = \lim_{\varepsilon \downarrow 0}(c + \varepsilon c^*)$ and each $c + \varepsilon c^*$ is in N_F, it follows that $c \in \text{cl}\, N_F$. Thus $N_G \subseteq \text{cl}\, N_F$, implying that $\text{cl}\, N_G \subseteq \text{cl}\, N_F$.

(d): Suppose F and G are two faces with $N_F = N_G$. Then $\text{cl}\, N_F = \text{cl}\, N_G$, and two applications of part (c) imply that $F = G$.

(e): Let F be a nonempty face of P. If $u \in N_F$, then $u^T x = u^T y$ for every $x, y \in F$ implying that $u^T[\alpha(x-y)] = 0$ for every $\alpha \in R$, that is, $u^T z = 0$ for every $z \in \text{tng}\, F$; so, $N_F \subseteq (\text{tng}\, F)^\perp$ implying that $\text{tng}\, N_F \subseteq (\text{tng}\, F)^\perp$. In order to prove that this inclusion holds as equality it suffices to show that $\dim(\text{tng}\, N_F) = \dim[(\text{tng}\, F)^\perp]$. By part (a) there exists a vector in N_F, say c, and with $\gamma \equiv \max_{x \in P} c^T x$ and V as the set of vertices of P, we have that

$$c^T v \begin{cases} = \gamma & \text{if } v \in V \cap F, \text{ and} \\ < \gamma & \text{if } v \in V \setminus F\,. \end{cases} \qquad \text{(A.1)}$$

Let $\nu \equiv \dim(\operatorname{tng} F)^{\perp}$. Then there exist ν linearly independent vectors in $(\operatorname{tng} F)^{\perp}$, say $u^1, \ldots, u^{\nu}$. Fix $j \in \{1, \ldots, \nu\}$. As $u^j \in (\operatorname{tng} F)^{\perp}$, $(u^j)^{\mathrm{T}}(v - v') = 0$ for each pair of vectors $v, v' \in F$, implying that $(u^j)^{\mathrm{T}} v$ is constant over F; let β_j be the common value of $(u^j)^{\mathrm{T}} v$ when v ranges over F. As (A.1) implies that $c^{\mathrm{T}} v > c^{\mathrm{T}} w$ for all $v \in V \cap F$ and $w \in V \setminus F$, we have that for sufficiently small positive ε,

$$\gamma + \varepsilon\beta_j = c^{\mathrm{T}} v + \varepsilon(u^j)^{\mathrm{T}} v < c^{\mathrm{T}} w + \varepsilon(u^j)^{\mathrm{T}} w \text{ for all } v \in V \cap F \text{ and } w \in V \setminus F\,;$$

as $F = \operatorname{conv} V \cap F$ and $P = \operatorname{conv} V$, it follows that $\gamma + \varepsilon\beta_j = c^{\mathrm{T}} v + \varepsilon(u^j)^{\mathrm{T}} v$ for each $v \in F$ and $\gamma + \varepsilon\beta_j < c^{\mathrm{T}} w + \varepsilon(u^j)^{\mathrm{T}} w$ for each $w \in P \setminus F$, that is, $\operatorname{argmax}_{x \in P}(c + \varepsilon u^j)^{\mathrm{T}} x = F$. Thus, for sufficiently small positive ε, $c + \varepsilon u^j \in N_F$ and therefore (as $c \in N_F$) $u^j \in \operatorname{tng} N_F$. As $u^1, \ldots, u^{\nu}$ are linearly independent, we conclude that $\dim(\operatorname{tng} N_F) \geq \nu = \dim(\operatorname{tng} F)^{\perp}$ completing the proof that $\operatorname{tng} N_F = (\operatorname{tng} F)^{\perp}$; in particular, $\dim(\operatorname{tng} N_F) = (\operatorname{tng} F)^{\perp} = p - \dim F$.

□

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