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One-Loop Vilkovisky-Dewitt Effective Potential for Scale-Invariant Gravity

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For a pure gravity theory without matter, the scale symmetric phase represents an equivalent class of gravity theories, to which the Einstein gravity plus a cosmological constant belongs under a special gauge choice. The one-loop quantum correction of this scale-invariant theory is calculated by using Vilkovisky-Dewitt's method. It is shown that the resulting effective potential is gauge-independent as expected. Some discussions of the Vilkovisky-DeWitt method through this calculation are given.

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I. Introduction

As the effective potential is taken from the zeroth order of the momentum expansion of the effective action, the effective potential is off-shell except at the vacuum, where the field configuration satisfies the equation of motion. For this reason, it is not unexpected that there exist some ambiguities when the conventional Coleman-Weinberg formalism [1] is applied to field theories [2].

The first ambiguity is that the conventional effective action is not invariant under field reparameterizations. For two classically equivalent theories with different field parameterizations, their effective potentials given by conventional methods are not only different in form from the other but also cannot be transformed to each other by a field reparameterization.

Second, for gauge theories or theories with continuous internal symmetries, their conventional effective potentials are gauge-variant and gauge-dependent. In some cases, the gauge invariance of the effective action can be achieved technically by splitting each field into a quantum- and a background-part, and making an arrangement so that the gauge of the quantum-field-part is fixed by a gauge-invariant fixing term, while the effective action which is a function of background-field-part is gauge-invariant [3]. But it is not guaranteed that the same effective action can be obtained if one initially chooses another gauge condition or another field parameterization for quantum fields. Even if a gauge invariant effective potential is worked out, usually there still remains a dependence on the coupling factor of the gauge fixing term. This factor gives rise to another ambiguity, especially in direct applications of the effective potential in practical situations. Fortunately, the whole problem was resolved by Vilkovisky [4] and DeWitt [5].

Recalling Einstein ' s idea in formulating the theory of general relativity: the coordinate reparameterization dependence can be eliminated by simply choosing all variables and derivatives to be covariant, Vilkovisky and DeWitt pointed out that the space of the field configurations may not be trivially flat. Thus, in order to obtain an effective action independent of the field reparameterization, one should properly construct covariant variables and derivatives in the configuration space. Furthermore, it is possible to define a gauge-invariant metric in the configuration space, so that a gauge independent effective potential can be uniquely derived without any ambiguity.

Many Vilkovisky-Dewitt (VD) effecive potentials for varieties of gauge theories have been worked out [2, 6]. Besides, VD method can be applied as well to systems with continuous internal symmetries such as the reparameterization symmetry in general relativity [4, 7] and the scale symmetry. In this paper, we will use the VD method to calculate the one-loop correction of a gravity theory with scale symmetry.

II. Scale-invariant Gravity

The simplest gravity model with scale symmetry is [8]

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{12}\hat{\phi}^2 R + \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} + \frac{\lambda}{4!}\hat{\phi}^4\right], \tag{1}$$

where $\hat{\phi}$ is a scalar field, and λ is a coupling constant. The action S is invariant under the local scale transformations

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x), \tag{2}$$

$$\hat{\phi}(x) \rightarrow \hat{\phi}'(x) = \Omega^{-1}(x)\hat{\phi}(x). \tag{3}$$

(One would see below that $\hat{\phi}$ is in fact an auxiliary field in the spectrum. This is a manifestation of the scale symmetry.) To fix the degree of freedom corresponding to the scale symmetry, one can put a constraint, the Einstein *gauge*,

$$\hat{\phi} = v \tag{4}$$

with v being a non-zero constant, or equivalently,

$$\Omega^2(x) = v^{-2}\hat{\phi}^2, \tag{5}$$

so that the action S becomes

$$S_{EG} = \int d^4x\sqrt{-g}\left[-\frac{1}{16\pi G}R + 2\Lambda\right], \tag{6}$$

where the gravitational constant $G = 3/(4\pi v^2)$ and the cosmological constant $\Lambda = \lambda v^4/48$. We thus see that the scalar-tensor gravity (1) is no more than a generalized form of the Einstein gravity S_{EG}. In general, S represents a set of pure gravity theories with arbitrary space-time varying gravitational and cosmological "constants" interconnected by the local scale transformation.

As such, we will adopt the VD method to handle the scale symmetry. Instead of displaying the full VD calculation of the one-loop effective potential, we begin with the conventional Coleman-Weinberg formalism and modify it to the VD calculations when necessary. A brief account of the VD method and a detailed calculation of the effective' potential are given in Appendix A.

III. One-loop effective potential

Expanding the gravitational field and the scalar field about the ground-state background, one has

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{7}$$

$$\hat{\phi} = \phi + \sigma, \tag{8}$$

where $\eta_{\mu\nu}$ is a flat metric, ϕ is a constant field, and the quantum field $h_{\mu\nu}$ is graviton. *To* evaluate the one-loop effective potential, it suffices to expand the action (1) up to the second order in the quantum fields, namely,

$$\begin{aligned}-\sqrt{g}\mathcal{L} \approx \mathcal{L}_2 \equiv{}& V(\phi) + \frac{V(\phi)}{2\phi}h + V'(\phi)\sigma \\ &-\frac{\phi^2}{24}\left[h^{\mu\nu}{}_{,\mu}h_{,\nu} - \frac{1}{2}h_{,\mu}h^{,\mu} - h^{\mu\nu}{}_{,\nu}h_{\mu\rho,}{}^{\rho} + \frac{1}{2}h^{\mu\nu,\rho}h_{\mu\nu,\rho} - 4\sigma\phi^{-1}(h_{,\mu}{}^{,\mu} - h^{\mu\nu}{}_{,\mu\nu})\right] \\ &+\frac{1}{2}\sigma_{,\mu}\sigma^{,\mu} + \frac{1}{8}Vh^2 + \frac{1}{2}V'h\sigma + \frac{1}{2}V''\sigma^2 - \frac{1}{4}Vh^{\mu\nu}h_{\mu\nu},\end{aligned} \tag{9}$$

where $h \equiv h_{\mu\mu}$ and $V' \equiv \delta V/\delta\phi$. One may write the quadratic part of the Lagrangian in the following form,

$$\mathcal{L}_q = \frac{1}{2}\psi_a P^{ab}\psi_b, \tag{10}$$

where $a, b = 0, \cdots, 10$ and ψ_a represent the quantum fields σ and ten independent components of $h_{\mu\nu}$ respectively.

Suppose a transformation

$$\rho_{\mu\nu} = h_{\mu\nu} + 2\sigma\eta_{\mu\nu}\phi^{-1} \tag{11}$$

is performed, the quadratic Lagrangian becomes

$$
\begin{aligned}
\mathcal{L}_2 =& -\frac{\phi^2}{24}\left(\frac{1}{2}\rho\partial^2\rho + \rho_{\mu\nu}\partial^\nu\partial_\beta\rho^{\mu\beta} - \rho\partial_\mu\partial_\nu\rho^{\mu\nu} - \frac{1}{2}\rho_{\mu\nu}\partial^2\rho^{\mu\nu}\right)\\
&+\frac{V}{4}\left(\frac{1}{2}\rho^2 - \rho_{\mu\nu}\rho^{\mu\nu}\right) + \sigma A\rho + \sigma B\sigma,\\
A \equiv& \frac{V'}{2} - \frac{V}{\phi},\\
B \equiv& \frac{V''}{2} - \frac{4V'}{\phi} + \frac{4V}{\phi^2}.
\end{aligned} \tag{12}
$$

Here the kinetic term of σ field vanishes, in other words, σ is an auxiliary field. This gives a hint that there are some symmetries in this Lagrangian. Indeed, they originate from the scale transformations (2) and (3). When the Einstein gauge (5) is chosen, the gravitational field becomes

$$
g'_{\mu\nu} = v^{-2}\hat{\phi}^2 g_{\mu\nu}. \tag{13}
$$

Substituting the background field expansions (7) and (8) into the above transformation, one would obtain the infinitesimal version of the scale transformation,

$$
\rho_{\mu\nu} \equiv h'_{\mu\nu} = h_{\mu\nu} + 2\sigma\eta_{\mu\nu}\phi^{-1} + O(\psi^2), \tag{14}
$$

if $v=\phi$. This is exactly the transformation (11).

One can go further by letting

$$
\sigma' = \sigma + \frac{A}{2B}\rho = \left(1 + \frac{4A}{B}\right)\sigma + \frac{A}{2B}h, \tag{15}
$$

so that the Jacobian with respect to the field reparametrization $(h_{\mu\nu},\sigma) \to (\rho_{\mu\nu}, \sigma')$ equals unity. Then the last two terms in Eq. (12) turn into

$$
\sigma A\rho + \sigma B\sigma = \rho\left(\frac{A^2}{4B}\right)\rho + \sigma' B\sigma', \tag{16}
$$

where σ' is now decoupled. The field equation $\sigma' = 0$ corresponds to the Einstein gauge (4). We thus conclude that the only difference between the Einstein gravity and the Weyl gravity (1) is that the latter has the gravitational wave with a mass term proportional to ρ^2 as shown in the right-hand side of Eq. (16). Nevertheless, this difference dissappears by choosing a traceless gauge $\rho = 0$.

In fact, after the Lagrangian (10) being diagonized, there are other vanishing kinetic terms, which correspond to four degrees of freedom of the reparameterization symmetry. This property is not surprising in systems with internal symmetries. Consider a symmetry transformation operator U operating on a quantum field by $\psi_a \to \psi'_a = U_a{}^b\psi_b$. Assume the quadratic Lagrangian is invariant under such a symmetry transformation, then $P = U^T P U$ where U^T denotes the transverse of U. This would imply that det $P = 0$ if det $U \neq 1$. In our model (9), the determinants of the operators corresponding to the scale transformation and reparameterization are indeed not equal to unity. As such, the corresponding operator $P^{ab}(V=0)$ is not invertible, and hence its propagator cannot be defined. Similar situations occur in the theory of electromagnetism, where one can interpret P as a projection operator [9].

It is therefore preferable to set the gauge condition in a form of a first derivative with respect to the corresponding quantum field in this case. For example, in electromagnetism, one may choose the Lorentz gauge $\partial_\mu A^\mu = 0$, whose corresponding gauge fixing term reads

$$\frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \tag{17}$$

with an arbitrary factor α. Once this term is added into the Lagrangian of electromagnetism, the propagator of photon is then well-defined up to the factor cr.

In the present case, the reparameterization symmetry can be fixed by choosing

$$h_{\mu\nu},^{\nu} - \frac{1}{2}h,_{\mu} = 0, \tag{18}$$

while the gauge condition for the scale symmetry (4) is equivalent to

$$\partial_\mu\sigma = 0. \tag{19}$$

Hence, the gauge fixing term then reads

$$\mathcal{L}_{gf} = \frac{\phi^2}{2\alpha}\left(h_{\mu\rho,\mu} - \frac{1}{2}h_{,\rho}\right)\left(h^{\nu\rho}{}_{,\nu} - \frac{1}{2}h,^{\rho}\right) + \frac{1}{2\beta}\sigma_{,\mu}\sigma^{\mu}_{,}, \tag{20}$$

where α and β are arbitrary factors.

In the one-loop level, only quadratic terms are needed in path-integral calculation. Since the corresponding ghost Lagrangian,

$$\mathcal{L}_{ghost} = \bar{\eta}(-\partial^2)\eta^\mu, \tag{21}$$

is totally decoupled from the system in this level, the ghost field can be neglected here. So the relevant quadratic part of the Lagrangian considered in this approximation is

$$\begin{aligned}\mathcal{L}_q = {} & \frac{1}{4}h_{\mu\nu}\alpha_1 h^{\mu\nu} - \frac{1}{4}h\alpha_2 h + \frac{1}{2}h_{\mu\nu}\alpha_3\frac{\partial^\nu\partial_\rho}{\partial^2}h^{\mu\rho} - \frac{1}{2}h\alpha_4\frac{\partial_\mu\partial_\nu}{\partial^2}h^{\mu\nu} \\ & - \frac{1}{2}h_{\mu\nu}\alpha_5\frac{\partial^\mu\partial^\nu\partial_\rho a_\sigma}{\partial^2\partial^2}h^{\rho\sigma} + \frac{1}{2}\sigma\beta_1\sigma + \frac{1}{2}\sigma\beta_2 h + \frac{1}{2}\sigma\beta_3\frac{\partial_\mu\partial_\nu}{\partial^2}h^{\mu\nu},\end{aligned} \tag{22}$$

where

$$
\begin{aligned}
\alpha_1 &= -\frac{k^2\phi^2}{12} - \frac{\lambda\phi^4}{24}, \\
\alpha_2 &= -\left(\frac{1}{12} + \frac{1}{2\alpha}\right) k^2\phi^2 - \frac{\lambda\phi^4}{48}, \\
\alpha_3 &= \alpha_4 = \left(\frac{1}{12} + \frac{1}{\alpha}\right) k^2\phi^2, \\
\alpha_5 &= 0, \\
\beta_1 &= \left(1 + \frac{1}{\beta}\right) k^2 + \frac{\lambda\phi^2}{2}, \\
\beta_2 &= \frac{k^2\phi}{3} + \frac{\lambda\phi^3}{6}, \\
\beta_3 &= -\frac{k^2\phi}{3},
\end{aligned} \tag{23}
$$

with a replacement $-\partial^2 \to k^2$. Let us write $\mathcal{L}_q$ in the form of Eq. (10), the eigenvalues of P^{ab} are found to be

$$
\begin{aligned}
\lambda_1 = \lambda_2 = \lambda_3 &= -\frac{k^2\phi^2}{12} - \frac{\lambda\phi^4}{24}, \\
\lambda_4 = \lambda_5 = \lambda_6 &= \frac{k^2\phi^2}{\alpha} - \frac{\lambda\phi^4}{24}, \\
\lambda_7 = \lambda_8 &= -\frac{k^2\phi^2}{24} - \frac{\lambda\phi^4}{48}, \\
\lambda_0\lambda_9\lambda_{10} &= \phi^4 \left[\frac{k^6}{48\alpha\beta} + \lambda\phi^2 \left(\frac{1}{96\alpha\beta} - \frac{1}{48\alpha} - \frac{1}{1152\beta}\right) k^4 \right. \\
&\quad \left. + (\lambda\phi^2)^2 \left(\frac{1}{3456} - \frac{1}{64\alpha} - \frac{1}{2304\beta}\right) k^2 + \frac{5}{13824}(\lambda\phi^2)^3\right].
\end{aligned} \tag{24}
$$

Indeed, if $1/\alpha$ and $1/\beta$ are set to be zero such that $\mathcal{L}_{gf}$ vanishes, there would be five elements of the eigenvector which have no kinetic terms: they are λ_4, λ_5, λ_6 and two of the three eigenvalues λ_0, λ_9 and λ_{10}.

In terms of λ's, the unrenormalized one-loop effective potential can be written as

$$
V_1 = V - \frac{i}{2} \sum_{a=0}^{10} \mathrm{Tr}\ln\lambda_a . \tag{25}
$$

Obviously, the conventional effective potential obtained by substituting the eigenvalues (24) into Eq. (25) depends on arbitrary factors α and β. To eliminate this ambiguity, one should introduce the Vilkovisky-DeWitt effective potential.

From Appendix A, the Vilkovisky-DeWitt method changes the eigenvalues into

$$\begin{aligned}
\lambda_1 = \lambda_2 = \lambda_3 &= -\frac{k^2\phi^2}{12} - \frac{\lambda\phi^4}{24}, \\
\lambda_4 = \lambda_5 = \lambda_6 &= \frac{k^2\phi^2}{\alpha}, \\
\lambda_7 = \lambda_8 &= -\frac{k^2\phi^2}{24} - \frac{\lambda\phi^4}{48}, \\
\lambda_0\lambda_9\lambda_{10} &= \frac{k^4\phi^4}{48\alpha\beta}\left(k^2 + \frac{15}{46}\lambda\phi^2\right),
\end{aligned} \tag{26}$$

by combining the original quadratic Lagrangian (22) with the correction (A47). The one-loop VD effective potential then reads

$$V_1^{VD} = \frac{\lambda\phi^4}{4!} + \frac{5i}{2}\mathrm{Tr}\ln\left(k^2 + \frac{1}{2}\lambda\phi^2\right) + \frac{i}{2}\mathrm{Tr}\ln\left(k^2 + \frac{15}{46}\lambda\phi^2\right) + \text{constant}, \tag{27}$$

which does not depend on the gauge-fixing factors.

IV. Discussion

Note that $\mathcal{L}_2 + \mathcal{L}'$ (Eq. (9) and Eq. (A47)) is invariant under the infinitesimal transformations ($A19$) and (A20) as expected. This invariance guarantees that the one-loop VD effective potential is independent of gauge-choices, because different gauge-conditions of $\mathcal{L}_{gf}$ can be related to each other by a gauge-transformation while $\mathcal{L}_2 + \mathcal{L}'$ is unchanged after the transformation.

Also, gauge-invariance of the one-loop expansion implies that the determinant of $\delta^2 \int d^4x(\mathcal{L}_2 + \mathcal{L}')/\delta\psi_A\delta\psi_B$ vanishes, hence the gauge parameter in gauge fixing Lagrangian $\mathcal{L}_{gf}$ which is not gauge-invariant can be factored out in the determinant of $\delta^2 \int d^4x\mathcal{L}_q/\delta\psi_A \delta\psi_B$. This can be easily seen by noting that the only a-independent term of $\det(aA + B)$ is det B. In language of perturbation theory, this means that the vertices representing the interaction between the unphysical and physical fields are removed, that is to say, the unphysical fields are decoupled from the system.

The effect of VD method on the eigenvalues λ_4, λ_5 and λ_6 is to remove the vertex term $-\lambda\phi^2/24$ (see Eqs. (24) and (26)), so that the loops of ψ_4, ψ_5 and ψ_6 are decoupled from the system. The same decoupling can also be achieved by naively choosing the Landau gauge: $\alpha = \beta = 0$. It is, however, not the case in calculating $\lambda_0\lambda_9\lambda_{10}$. The complicated mixing between these quantum fields makes the VD residual non-zero vertex term different from that obtained from the Landau gauge. Thus in general, the equality between the naive and the VD effective potentials occurs only in some special cases by accident.

Recall that when a constrained or gauge system is quantized in the path integral formalism, the constraints or gauge conditions $\delta(F[\phi])$ are loosed into a distribution $e^{-F^2/2\alpha}$, where α can be understood as the width of this distribution. In other words, all of the off-shell field configurations, which are weighted by this distribution, are taken into account, and the constraint $F[\phi] = 0$ is true only when the system is on-shell. Therefore, to choose the Landau gauge, $\alpha \to 0$. is equivalent to narrowing the distribution to a delta-like function. However, if the configuration space is curved, the direction that $\alpha \to 0$ may not be orthogonal to the on-shell surface $F[\phi] = 0$ everywhere because α is not a covariant quantity in ***M***. Hence taking the Landau gauge naively without considering the curvature effect may give the wrong result.

For example, if we choose the Landau gauge in the conventional effective potential obtained from Eq. (24), the resulting effective potential would be identical to the conventional effective potential obtained from Einstein gravity (6) with a $\to$ 0. It should .be emphasized that the Einstein gravity is the consequence of choosing the gauge, $\sigma = 0$, in the scale invariant gravity (12) *before* quantization. However, neither of them is equal to the VD effective potential (27). Although the Einstein gravity and the scale invariant gravity are classically equivalent, the off-shell structure as well as the quantum theory of them are quite different.

Our final remark is that the field σ, which has no kinetic term, can be identified as an auxiliary field in the classical theory. One may substitute the equation of motion with respect to σ in the Lagrangian to eliminate the auxiliary field and get a new pure graviton theory. If we include only the graviton gauge-fixing a-term to compute the one-loop VD effective potential for this new theory, the results would depend on α. This is because we have ignored the scale symmetry hidden in the new theory. Therefore, if one finds that the obtained VD effective potential still depends on some arbitrary factor which corresponds to the known symmetry of a system, then the system would have to carry some extra hidden symmetry. In this case, one may apply the method developed by Dirac [10] to find all the constraints and then run the quantization process again.

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Appendix A: Vilkovisky-Dewitt method

In this appendix, would briefly introduce the Vilkovisky-Dewitt method [2, 4-7] and calculate the one-loop VD effective potential for the scale invariant gravity (1).

Let the naive metric of the space of the configuration of the quantum fields ***M*** be G_{ij}, where i and j are indices which runs over all the quantum fields at every point of the whole space-time. G_{ij} does not have to be gauge invariant.

In the calculation of the one-loop effective potential, one needs to know the second derivative (variation) of the action $S[\phi]$ with respect to the quantum fields. Since S is a scalar in ***M***, $S_{,i} \equiv \delta S/\delta\phi^i$ is a vector in ***M***. ***If*** G_{ij} describes a non-trivial curved space, one should take the covariant derivative of $S_{,i}$,

$$D_i S_{,j} = \frac{\delta}{\delta\phi^i} S_{,j} - \Gamma^k_{ij} S_{,k}, \tag{A1}$$

instead of $\delta^2 S/\delta\phi^i\delta\phi^j$ because the corresponding effective potential should not depend on the choice of special background field configurations. Here the connection Γ^k_{ij} can be written as

$$\Gamma^k_{ij} = \frac{1}{2} G^{kl}(G_{li,j} + G_{lj,i} - G_{ij,l}), \tag{A2}$$

which is the Christoffel symbol.

In gauge theories, it is possible to define a gauge independent metric from the naive one. Suppose that a general infinitesimal transformation of the fields is a pure gauge transformation, then

$$\delta\phi^i = Q^i_\alpha \epsilon^\alpha, \tag{A3}$$

where Q^i_α is the generator of the gauge symmetry, and ϵ^α is a parameter. In general, $\delta\phi^i$ should include the gauge transformation part and the physical transformation part. The line element of the general transformation $\delta\phi^i$ in *M can* be written as

$$\delta s^2 = G_{ij}\delta\phi^i\delta\phi^j . \tag{A4}$$

Define the projection operator as

$$\Pi^i_j \equiv \delta^i_j - Q^i_\alpha N^{\alpha\beta} Q^k_\beta G_{kj}, \tag{A5}$$

satisfying

$$\Pi^i_j Q^j_\alpha = 0, \tag{A6}$$

$$\Pi^i_j \Pi^j_k = \Pi^i_k, \tag{A7}$$

to project out the gauge transformation part of a general infinitesimal transformation of the field. Here $N^{\alpha\beta}$ is the inverse of

$$N_{\alpha\beta} = G_{ij} Q^i_\alpha Q^j_\beta . \tag{A8}$$

The component of $\delta\phi^i$ in the physical space can then be defined by

$$\delta_\perp \phi^i \equiv \Pi^i_j \delta\phi^j , \tag{A9}$$

and the line element of the physical transformation reads

$$\delta_\perp s^2 = G_{ij}\delta_\perp\phi^i\delta_\perp\phi^j \equiv \gamma_{ij}\delta\phi^i\delta\phi^j , \tag{A10}$$

where

$$\gamma_{ij} = G_{ik}\Pi^k_j \tag{A11}$$

is taken to be the gauge independent metric which measures the physical transformation part of a general transformation. Using γ_{ij}, the connection in the usual definition can be constructed in terms of G_{ij} and Q^i_α as

$$\Gamma^{(\gamma)k}_{ij} = \Gamma^k_{ij} + T^k_{ij}, \tag{A12}$$

where

$$T^k_{ij} = -2Q^k_{\alpha;(i}B^\alpha_{j)} + Q^l_\sigma B^\sigma_{(i}B^\rho_{j)}Q^k_{\rho;l} \tag{A13}$$

with

$$B^\alpha_i \equiv N^{\alpha\beta}Q^i_\beta G_{ij}\,, \tag{A14}$$

$$Q^k_{\alpha,i} \equiv \frac{\delta}{\delta\phi^i}Q^k_\alpha + \Gamma^k_{ij}Q^j_\alpha\,. \tag{A15}$$

Here the convention of symmetrization,

$$A_{(i}B_{j)} \equiv \frac{1}{2}(A_iB_j + A_jB_i), \tag{A16}$$

is understood. Now the covariant derivative of $S_{,i}$ with the connection $\Gamma^{(\gamma)k}_{ij}$ is gauge invariant so that the effective potential V_1^{VD} constructed from it is gauge independent.

Now we turn to our case of the scale invariant gravity (1). Following Vilkovisky's prescription [4], we take the naive metric,

$$G_{\hat\phi(x)\hat\phi(y)} = \sqrt{-g}\phi^{-2}\hat\phi^2\delta(x-y), \tag{A17}$$

$$G_{g_{\mu\nu}(x)g_{\rho\sigma}(y)} = \frac{1}{2}\sqrt{-g}\phi^{-2}\hat\phi^4(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma})\delta(x-y), \tag{A18}$$

and the infinitesimal transformations,

$$\delta\hat\phi = -\epsilon^\alpha\partial_\alpha\hat\phi - \omega\hat\phi \equiv Q^{\hat\phi}_{\epsilon^\alpha}\epsilon^\alpha + Q^{\hat\phi}_\omega\omega, \tag{A19}$$

$$\delta g_{\mu\nu} = -g_{\alpha\mu}\partial_\nu\epsilon^\alpha - g_{\alpha\nu}\partial_\mu\epsilon^\alpha - \epsilon^\alpha\partial_\alpha g_{\mu\nu} + 2\omega g_{\mu\nu} \equiv Q^{g_{\mu\nu}}_{\epsilon^\alpha}\epsilon^\alpha + Q^{g_{\mu\nu}}_\omega\omega, \tag{A20}$$

where ϵ^α and w are parameters corresponding to the reparameterization and scale transformations respectively. Then the generators of these transformations are

$$Q^{\hat\phi(z)}_{\epsilon^\alpha(x)} = -\partial_\alpha\hat\phi\delta(z-x), \tag{A21}$$

$$Q^{g_{\mu\nu}(z)}_{\epsilon^\alpha(x)} = -g_{\alpha\mu}\partial_\nu\delta(z-x) - g_{\alpha\nu}\partial_\mu\delta(z-x) - g(z-x)\partial_\alpha g_{\mu\nu}(z), \tag{A22}$$

$$Q^{\hat{\phi}(z)}_{\omega(x)} = -\hat{\phi}(z)\delta(z-x), \tag{A23}$$

$$Q^{g_{\mu\nu}(z)}_{\omega(x)} = 2g_{\mu\nu}(z)\delta(z-x). \tag{A24}$$

Here the derivative ∂_μ always acts on the first argument of the δ function. Note that the background value of $\sqrt{\det G}$ has a ϕ^{10} factor, which will cancel the same factor from the product of eigenvalues X ' s in path integral.

A straightforward calculation gives the quantities *(A8)* evaluated at the ground-state backgrourld values (7) and (8),

$$N_{\epsilon^\alpha(x)\epsilon^\beta(y)}|_{bg} = -2\phi^2\eta_{\alpha\beta}\partial^2\delta(x-y), \tag{A25}$$

$$N_{\omega(x)\omega(y)}|_{bg} = -15\phi^2\delta(x-y), \tag{A26}$$

$$N_{\epsilon^\alpha(x)\omega(y)}|_{bg} = -N_{\omega(y)\epsilon^\alpha(x)}|_{bg} = -4\phi^2\partial_\alpha\delta(x-y), \tag{A27}$$

whose inverses read

$$N^{\epsilon^\alpha(x)\epsilon^\beta(y)}|_{bg} = \phi^{-2}\left(-\frac{1}{2}\eta^{\alpha\beta}\frac{1}{\partial^2} + \frac{4}{23}\frac{\partial^\alpha\partial^\beta}{\partial^4}\right)\delta(x-y), \tag{A28}$$

$$N^{\omega(x)\omega(y)}|_{bg} = -\frac{1}{23}\phi^{-2}\delta(x-y), \tag{A29}$$

$$N^{\epsilon^\alpha(x)\omega(y)}|_{bg} = -N^{\omega(x)\epsilon^\alpha(y)}|_{bg} = \frac{2}{23}\phi^{-2}\frac{\partial^\alpha}{\partial^2}\delta(x-y), \tag{A30}$$

Hence, the background values of the quantities defined in Eq. *(A14)* are

$$B^{\epsilon^\alpha(u)}_{\hat{\phi}(z)}|_{bg} = -\frac{2}{23}\phi^{-1}\frac{\partial^\alpha}{\partial^2}\delta(u-z), \tag{A31}$$

$$B^{\omega(u)}_{\hat{\phi}(z)}|_{bg} = \frac{1}{23}\phi^{-1}\delta(u-z), \tag{A32}$$

$$B^{\epsilon^\alpha(u)}_{g_{\mu\nu}(z)}|_{bg} = \left(-\eta^{\alpha(\mu}\frac{\partial^{\nu)}}{\partial^2} + \frac{7}{46}\eta^{\mu\nu}\frac{\partial^\alpha}{\partial^2} + \frac{8}{23}\frac{\partial^\alpha\partial^\mu\partial^\nu}{\partial^2\partial^2}\right)\delta(u-z), \tag{A33}$$

$$B^{\omega(u)}_{g_{\mu\nu}(z)}|_{bg} = -\frac{4}{23}\left(\frac{\partial^\mu\partial^\nu}{\partial^2} - \eta^{\mu\nu}\right)\delta(u-z). \tag{A34}$$

It follows that T^k_{ij} in Eq. (A13) are given by

$$T^{\hat{\phi}(z)}_{\hat{\phi}(x)\hat{\phi}(y)}|_{bg} = \frac{4}{23}\phi^{-1}\delta(x-z)\delta(y-z), \tag{A35}$$

$$g^{\rho\sigma}(z)T^{g_{\rho\sigma}(z)}_{\hat{\phi}(x)\hat{\phi}(y)}|_{bg} = -\frac{47}{23}\phi^{-2}\delta(x-y), \tag{A36}$$

$$T^{\hat{\phi}(z)}_{g_{\mu\nu}(x)\hat{\phi}(y)}|_{bg} = \left(\frac{15}{46}\frac{\partial^\mu\partial^\nu}{\partial^2} - \frac{3}{46}\eta^{\mu\nu}\right)\delta(x-z)\delta(y-z), \tag{A37}$$

$$g^{\rho\sigma}(z)T^{g_{\rho\sigma}(z)}_{g_{\mu\nu}(x)\hat{\phi}(y)}|_{bg} = -\frac{48}{23}\phi^{-1}\eta^{\mu\nu}\delta(x-y), \tag{A38}$$

$$\begin{aligned}T^{\hat{\phi}(z)}_{g_{\alpha\beta}(x)g_{\mu\nu}(y)}|_{bg} = \phi\Bigg[&-\frac{32}{23}\frac{\partial^\alpha\partial^\beta\partial^\mu\partial^\nu}{\partial^2\partial^2} + 4\frac{\partial^{(\mu}\eta^{\nu)(\alpha}\partial^{\beta)}}{\partial^2} \\ &-\frac{15}{23}\left(\eta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\partial^2} + \eta^{\alpha\beta}\frac{\partial^\mu\partial^\nu}{\partial^2}\right) \\ &-\frac{7}{23}\eta^{\alpha\beta}\eta^{\mu\nu}\Bigg]\delta(x-z)\delta(y-z),\end{aligned} \tag{A39}$$

$$\begin{aligned}g^{\rho\sigma}(z)T^{g_{\rho\sigma}(z)}_{g_{\alpha\beta}(x)g_{\mu\nu}(y)}|_{bg} = \Bigg[&\frac{16}{23}\frac{\partial^\alpha\partial^\beta\partial^\mu\partial^\nu}{\partial^2\partial^2} - 2\frac{\partial^{(\mu}\eta^{\nu)(\alpha}\partial^{\beta)}}{\partial^2} \\ &+\frac{15}{23}\left(\eta^{\mu\nu}\frac{\partial^\alpha\partial^\beta}{\partial^2} + \eta^{\alpha\beta}\frac{\partial^\mu\partial^\nu}{\partial^2}\right) \\ &-\frac{1}{2}\eta^{\alpha\beta}\eta^{\mu\nu}\Bigg]\delta(x-z)\delta(y-z).\end{aligned} \tag{A40}$$

On the other hand, the connection of the naive metric $(A17)$ and $(A18)$ are

$$\Gamma^{\hat{\phi}(z)}_{\hat{\phi}(x)\hat{\phi}(y)} = \hat{\phi}^{-1}\delta(x-z)\delta(y-z), \tag{A41}$$

$$\Gamma^{\hat{\phi}(z)}_{g_{\mu\nu}(x)\hat{\phi}(y)} = \frac{1}{4}g^{\mu\nu}\delta(x-z)\delta(y-z), \tag{A42}$$

$$\Gamma_{g_{\mu\nu}(x)g_{\alpha\beta}(y)} = -\hat{\phi}(g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\mu\nu})\delta(x-z)\delta(y-z), \tag{A43}$$

$$g^{\mu\nu}(z)\Gamma^{g_{\mu\nu}(z)}_{\hat{\phi}(x)\hat{y}} = \hat{\phi}^{-2}\delta(x-y), \tag{A44}$$

$$g^{\alpha\beta}(z)\Gamma^{g_{\mu\nu}(z)}_{g_{\mu\nu}(x)\hat{\phi}(y)} = 2g^{\mu\nu}\hat{\phi}^{-1}\delta(x-y), \tag{A45}$$

$$g^{\rho\sigma}(z)\Gamma^{g_{\rho\sigma}(z)}_{g_{\mu\nu}(x)g_{\alpha\beta}(y)} = 0. \tag{A46}$$

Combining these with T^k_{ij}, the gauge independent connection $(A12)$ and hence the covariant derivative $(A1)$ can be worked out. This is equivalent to adding the correction terms,

$$\mathcal{L}' = \frac{1}{2}\sigma\left(-\frac{4}{23}\lambda\phi^2\right)\sigma + \sigma\left(-\frac{5}{92}\lambda\phi^3\right)\frac{\partial^\mu\partial^\nu}{\partial^2}h_{\mu\nu} + \sigma\left(-\frac{2}{69}\lambda\phi^3\right)h$$
$$+\frac{1}{2}h_{\mu\nu}\left(-\frac{5}{46}\lambda\phi^4\right)\frac{\partial^\mu\partial^\nu\partial^\alpha\partial^\beta}{\partial^2\partial^2}h_{\alpha\beta} + \frac{1}{2}h_{\mu\alpha}\left(-\frac{5}{16}\lambda\phi^4\right)\frac{\partial^\alpha\partial_\beta}{\partial^2}h^{\mu\beta} \tag{A47}$$
$$+h\left(\frac{35}{368}\lambda\phi^4\right)\frac{\partial^\mu\partial^\nu}{\partial^2}h_{\mu\nu} + \frac{1}{2}h\left(-\frac{233}{2208}\lambda\phi^4\right)h + h_{\mu\nu}\left(\frac{1}{6}\lambda\phi^4\right)h^{\mu\nu},$$

to the original quadratic Lagrangian (22).

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