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# Classical Distance-Regular Graphs of Negative Type

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We prove the following theorem.

THEOREM. *Let* $\Gamma = (X, R)$ *denote a distance-regular graph with classical parameters* $(d, b, \alpha, \beta)$ *and* $d \geqslant 4$. *Suppose* $b < -1$, *and suppose the intersection numbers* $a_1 \neq 0$, $c_2 > 1$. *Then precisely one of the following* (i)–(iii) *holds.*

(i) $\Gamma$ *is the dual polar graph* ${}^2A_{2d-1}(-b)$.

(ii) $\Gamma$ *is the Hermitian forms graph* $Her_{-b}(d)$.

(iii) $\alpha = (b-1)/2$, $\beta = -(1+b^d)/2$, *and* $-b$ *is a power of an odd prime.*



## 1. INTRODUCTION

Brouwer, Cohen, and Neumaier found that the intersection numbers of most known families of distance-regular graphs could be described in terms of four parameters $(d, b, \alpha, \beta)$ [2, pp. ix, 193]. They invented the term *classical* to describe those graphs for which this could be done. All classical distance-regular graphs with $b = 1$ are classified by Y. Egawa, A. Neumaier, and P. Terwilliger in a sequence of papers (see [2, p. 195] for a detailed description). In the present paper, we focus on the classical distance-regular graphs with $b < -1$. The following is our main result.

MAIN THEOREM. *Let* $\Gamma = (X, R)$ *denote a distance-regular graph with classical parameters* $(d, b, \alpha, \beta)$ *and* $d \geqslant 4$. *Suppose* $b < -1$, *and suppose the intersection numbers* $a_1 \neq 0$, $c_2 > 1$. *Then precisely one of the following* (i)–(iii) *holds.*

(i) $\Gamma$ *is the dual polar graph* ${}^2A_{2d-1}(-b)$.

(ii) $\Gamma$ *is the Hermitian forms graph* $Her_{-b}(d)$.

(iii) $\alpha = (b-1)/2$, $\beta = -(1+b^d)/2$, *and* $-b$ *is a power of an odd prime.*



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In addition to the above result, we believe Corollary 2.3, Lemma 3.2, Lemma 10.1, and Lemma 10.2 are of independent interest.

For the rest of this section, we review some definitions and basic concepts. See the books of Bannai and Ito [1], and Brouwer, Cohen, and Neumaier [2] for more background information.

Let $\Gamma=(X, R)$ denote a finite undirected graph without loops or multiple edges, with vertex set $X$ and edge set $R$. We say vertices $x, y \in X$ are *adjacent* if $\{x, y\} \in R$. Pick any nonnegative integer $i$ and vertices $x, y \in X$. By a *path of length i* from $x$ to $y$, we mean a sequence $x=x_0, x_1, ..., x_i=y$ of vertices from $X$ such that $x_j, x_{j+1}$ are adjacent for all $j$ $(0 \leqslant j \leqslant i-1)$. $\Gamma$ is said to be *connected* whenever for each pair of vertices $x, y \in X$, there exists a path in $\Gamma$ from $x$ to $y$. Assume $\Gamma$ is connected, and pick any vertices $x, y \in X$. By the *distance* $\delta(x, y)$, we mean the length of the shortest path from $x$ to $y$. By the *diameter* of $\Gamma$, we mean the scalar

$$d := \max\{\delta(x, y) \mid x, y \in X\}.$$

By a *subgraph* of $\Gamma$, we mean a graph $(\Omega, S)$, where $\Omega$ is a nonempty subset of $X$ and

$$S=\{\{x, y\} \mid x, y \in \Omega, \{x, y\} \in R\}.$$

By abuse of notation, we refer to the subgraph $(\Omega, S)$ as $\Omega$. Assume $\Gamma$ is connected, and let $\Omega$ denote a subgraph of $\Gamma$. Then $\Omega$ is said to be *geodesic* whenever for all vertices $x, y \in \Omega$, all vertices on all shortest paths in $\Gamma$ from $x$ to $y$ are contained in $\Omega$. One can see that if $\Omega$ is a geodesic subgraph of $\Gamma$ then the distances as measured in $\Omega$ are the same as the distances as measured in $\Gamma$.

Assume $\Gamma$ is connected with diameter $d$. For all $x \in X$ and for all integers $i$ $(0 \leqslant i \leqslant d)$, set

$$\Gamma_i(x) := \{y \mid y \in X, \delta(x, y)=i\}.$$

$\Gamma$ is said to be *distance-regular* whenever for all integers $h, i, j$ $(0 \leqslant h, i, j \leqslant d)$ and for all $x, y \in X$ with $\delta(x, y)=h$, the number

$$p^h_{ij} := |\Gamma_i(x) \cap \Gamma_j(y)| \tag{1.1}$$

is independent of $x, y$. The constants $p^h_{ij}$ $(0 \leqslant h, i, j \leqslant d)$ are known as the *intersection numbers* of $\Gamma$. For convenience, set $c_i := p^i_{1\,i-1}$ $(1 \leqslant i \leqslant d)$, $a_i := p^i_{1i}$ $(0 \leqslant i \leqslant d)$, $b_i := p^i_{1\,i+1}$ $(0 \leqslant i \leqslant d-1)$, $k_i := p^0_{ii}$ $(0 \leqslant i \leqslant d)$, and put $c_0 := 0$, $b_d := 0$, $k := k_1$. Note that $c_1=1$, $a_0=0$, and

$$k = c_i + a_i + b_i \qquad (0 \leqslant i \leqslant d), \tag{1.2}$$

$$k_i = \frac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i} \qquad (0 \leqslant i \leqslant d), \tag{1.3}$$

$$|X| = 1 + k_1 + \cdots + k_d. \tag{1.4}$$

From now on, we assume $\Gamma$ is distance-regular with diameter $d$. For each integer $i$ ($0 \leqslant i \leqslant d$), the $i$th *distance matrix* $A_i$ of $\Gamma$ has rows and columns indexed by $X$, and $x$, $y$ entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \delta(x, y) = i \\ 0, & \text{if } \delta(x, y) \neq i \end{cases} \qquad (x, y \in X).$$

Then

$$A_0 = I, \tag{1.5}$$

$$A_0 + \cdots + A_d = J \qquad (J = \text{all 1's matrix}), \tag{1.6}$$

$$A_i^t = A_i \qquad (0 \leqslant i \leqslant d), \tag{1.7}$$

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h \qquad (0 \leqslant i, j \leqslant d). \tag{1.8}$$

Let $\mathbb{R}$ denote the real field. By (1.5)–(1.8), the matrices $A_0, ..., A_d$ form a basis for a commutative semi-simple $\mathbb{R}$-algebra $M$, known as the *Bose–Mesner algebra*. By [1, pp. 59, 64], $M$ has a second basis $E_0, ..., E_d$ such that

$$E_0 = |X|^{-1} J, \tag{1.9}$$

$$E_i E_j = \delta_{ij} E_i \qquad (0 \leqslant i, j \leqslant d), \tag{1.10}$$

$$E_0 + E_1 + \cdots + E_d = I, \tag{1.11}$$

$$E_i^t = E_i \qquad (0 \leqslant i \leqslant d). \tag{1.12}$$

The $E_0, ..., E_d$ are known as the *primitive idempotents* of $\Gamma$, and $E_0$ is called the *trivial* primitive idempotent.

Since $A_0, ..., A_d \in M$,

$$A_i = \sum_{j=0}^{d} p_i(j)\, E_j \qquad (0 \leqslant i \leqslant d), \tag{1.13}$$

for some real scalars $p_i(j)$ $(0 \leqslant i, j \leqslant d)$. The scalar $p_i(j)$ is known as the *eigenvalue* of $A_i$ associated with $E_j$ $(0 \leqslant i, j \leqslant d)$. We often abbreviate

$$\theta_i := p_1(i) \qquad (0 \leqslant i \leqslant d).$$

Since $E_0, ..., E_d \in M$,

$$E_i = |X|^{-1} \sum_{j=0}^{d} q_i(j)\, A_j \qquad (0 \leqslant i \leqslant d), \tag{1.14}$$

for some real scalars $q_i(j)$ $(0 \leqslant i, j \leqslant d)$.

We set

$$m_i := q_i(0) \qquad (0 \leqslant i \leqslant d), \tag{1.15}$$

and note $m_i$ equals the rank of $E_i$ $(0 \leqslant i \leqslant d)$. In particular

$$m_i \neq 0 \qquad (0 \leqslant i \leqslant d). \tag{1.16}$$

We refer to $m_i$ as the *multiplicity* of $\Gamma$ associated with $E_i$. By [1, p. 63], we have

$$\frac{q_j(i)}{m_j} = \frac{p_i(j)}{k_i} \qquad (0 \leqslant i, j \leqslant d). \tag{1.17}$$

Set $V = \mathbb{R}^{|X|}$ (column vectors), and view the coordinates of $V$ as being indexed by $X$. For each vertex $x \in X$, set

$$\hat{x} = (0, 0, ..., 1, 0, ..., 0)^t, \tag{1.18}$$

where 1 is in coordinate $x$. We define the inner product

$$\langle u, v \rangle = u^t v \qquad (u, v \in V). \tag{1.19}$$

From (1.10), (1.12), (1.14), (1.18)–(1.19), we find that for all integers $i, j$ $(0 \leqslant i, j \leqslant d)$, and for all $x, y \in X$ such that $\delta(x, y) = j$,

$$\langle E_i \hat{x}, E_i \hat{y} \rangle = |X|^{-1}\, q_i(j). \tag{1.20}$$

A distance-regular graph $\Gamma$ is said to have *classical parameters* $(d, b, \alpha, \beta)$ whenever the diameter of $\Gamma$ is $d$, and the intersection numbers of $\Gamma$ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \qquad (0 \leqslant i \leqslant d), \tag{1.21}$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \qquad (0 \leqslant i \leqslant d), \tag{1.22}$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{j-1}. \tag{1.23}$$

Suppose $\Gamma$ has classical parameters $(d, b, \alpha, \beta)$. Combining (1.2), (1.21)–(1.23),

$$a_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left( \beta - 1 + \alpha \left( \begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \right) \qquad (0 \leqslant i \leqslant d). \tag{1.24}$$

Suppose $\Gamma$ has classical parameters $(d, b, \alpha, \beta)$ and $d \geqslant 3$. Then it is known $b$ is an integer, and that $b \neq 0$, $b \neq -1$ [2, p. 195].

## 2. THE $Q$-POLYNOMIAL PROPERTY

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d$. Observe the distance matrices of $\Gamma$ satisfy

$$A_i \circ A_j = \delta_{ij} A_i \qquad (0 \leqslant i, j \leqslant d),$$

where $\circ$ denotes entry-wise multiplication, so the Bose–Mesner algebra $M$ is closed under $\circ$. Thus there exist $q^h_{ij} \in \mathbb{R}$ $(0 \leqslant h, i, j \leqslant d)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{d} q^h_{ij} E_h.$$

The $q^h_{ij}$ are known as the *Krein parameters* of $\Gamma$. $\Gamma$ is said to be *$Q$-polynomial* (with respect to the given ordering $E_0, ..., E_d$ of the primitive idempotents) whenever for all integers $h, i, j$ $(0 \leqslant h, i, j \leqslant d)$, $q^h_{ij} = 0$ (resp. $q^h_{ij} \neq 0$) whenever one of $h, i, j$ is greater than (resp. equal to) the sum of the other two. Let $E$ denote a nontrivial primitive idempotent of $\Gamma$. Then $\Gamma$ is said to be *$Q$-polynomial with respect to $E$* whenever there exists an ordering $E_0, E_1 = E, ..., E_d$ of the primitive idempotents with respect to which $\Gamma$ is $Q$-polynomial. Suppose $\Gamma$ is $Q$-polynomial with respect to the ordering $E_0, ..., E_d$ of the primitive idempotents. Then we often abbreviate

$$\theta^*_i := q_1(i) \qquad (0 \leqslant i \leqslant d).$$

LEMMA 2.1. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 3$, and assume $\Gamma$ is $Q$-polynomial with respect to the primitive idempotent*

$$E = |X|^{-1} \sum_{i=0}^{d} \theta^*_i A_i. \tag{2.1}$$

*Then the following* (i)–(iii) *hold.*

(i) [9, Theorem 3.3(vii)] *For all integers* $i$ $(1 \leqslant i \leqslant d)$, *and for all* $x, y \in X$ *at distance* $\delta(x, y) = i$,

$$\sum_{\substack{w \in X \\ \delta(w, x) = 1 \\ \delta(w, y) = i-1}} E\hat{w} - \sum_{\substack{v \in X \\ \delta(v, x) = i-1 \\ \delta(v, y) = 1}} E\hat{v} = \lambda_i (E\hat{x} - E\hat{y}), \tag{2.2}$$

*where* $\lambda_i$ *is an appropriate real scalar that depends only on* $i$.

(ii) $c_i(\theta^*_{i+j-1} - \theta^*_{j+1}) = \lambda_i(\theta^*_{i+j} - \theta^*_j)$ $(1 \leqslant i \leqslant d)$, $(0 \leqslant j \leqslant d-i)$.

(iii) ([8, p. 384]) *The scalars* $\theta^*_0, ..., \theta^*_d$ *are distinct.*

*Proof.* (ii) Fix $x, y \in X$ at distance $\delta(x, y) = i$. Let $u$ denote a vertex at distances $\delta(u, x) = i + j$, $\delta(u, y) = j$. Taking the inner products of $E\hat{u}$ with both sides of (2.2), and evaluating the result using (1.20), we obtain the result.

Proposition 2.2. *Let* $\Gamma = (X, R)$ *denote a Q-polynomial distance-regular graph with diameter* $d \geqslant 4$. *For all integers* $i$ $(3 \leqslant i \leqslant d-1)$ *and for all* $x, y, z \in X$ *at distances* $\delta(x, y) = i$, $\delta(x, z) = i$, $\delta(y, z) = 1$,

$$\Gamma_1(x) \cap \Gamma_{i-1}(y) \cap \Gamma_{i-1}(z) \neq \emptyset. \tag{2.3}$$

*Proof.* Let $i, x, y, z$ be given, and suppose (2.3) does not hold. Then

$$\Gamma_j(x) \cap \Gamma_{i-j}(y) \subseteq \Gamma_{i-j+1}(z) \qquad (1 \leqslant j \leqslant i). \tag{2.4}$$

Taking the inner product of $E\hat{z}$ with both sides of (2.2) and evaluating the result using (1.20) and (2.4), we find

$$c_i(\theta^*_i - \theta^*_2) = \lambda_i(\theta^*_i - \theta^*_1). \tag{2.5}$$

Setting $j = 1$ in Lemma 2.1(ii),

$$c_i(\theta^*_i - \theta^*_2) = \lambda_i(\theta^*_{i+1} - \theta^*_1). \tag{2.6}$$

Combining (2.5), (2.6), and observing $\lambda_i \neq 0$ since $i \neq 2$, we find $\theta^*_i = \theta^*_{i+1}$, contradicting Lemma 2.1(iii).

Corollary 2.3. *Let* $\Gamma = (X, R)$ *denote a Q-polynomial distance-regular graph with diameter* $d \geqslant 3$. *For all integers* $i$ $(2 \leqslant i \leqslant d-1)$ *and for all* $x, y, z \in X$ *at distances* $\delta(x, y) = i$, $\delta(x, z) = i$, $\delta(y, z) = 1$,

$$\Gamma_{i-2}(x) \cap \Gamma_2(y) \cap \Gamma_2(z) \neq \emptyset. \tag{2.7}$$

*Proof.* The proof is by induction on $i$. The case $i=2$ is trivial, so assume $i \geqslant 3$. Recall $i \leqslant d-1$, so $d \geqslant 4$. By Proposition 2.2, there exists $w \in \Gamma_1(x) \cap \Gamma_{i-1}(y) \cap \Gamma_{i-1}(z)$. By the construction and induction,

$$\Gamma_{i-2}(x) \cap \Gamma_2(y) \cap \Gamma_2(z) \supseteq \Gamma_{i-3}(w) \cap \Gamma_2(y) \cap \Gamma_2(z) \neq \emptyset,$$

as desired. This proves the corollary.

## 3. SOME EIGENVALUES

LEMMA 3.1 [2, p. 250]. *Let $\Gamma=(X, R)$ denote a distance-regular graph with classical parameters $(d, b, \alpha, \beta)$ and $d \geqslant 3$. Then there exists an ordering $E_0, E_1, ..., E_d$ of the primitive idempotents of $\Gamma$ such that the following* (i)–(iii) *hold.*

$$\text{(i)}\quad p_1(i)=b^{-i}b_i-\begin{bmatrix} i \\ 1 \end{bmatrix} \qquad (0 \leqslant i \leqslant d),$$

*where* $\begin{bmatrix} i \\ 1 \end{bmatrix}$ *is defined in* (1.23). (3.1)

$$\text{(ii)}\quad \theta_d=-\begin{bmatrix} d \\ 1 \end{bmatrix}.$$

(iii) *$\Gamma$ is $Q$-polynomial with respect to $E_0, E_1, ..., E_d$.*

LEMMA 3.2. *Let $\Gamma=(X, R)$ denote a distance-regular graph with classical parameters $(d, b, \alpha, \beta)$, and assume $d \geqslant 3$, $b \neq 1$. Then with respect to the ordering of the primitive idempotents in Lemma* 3.1,

$$p_i(d)=(-1)^i \frac{(b^d-1)(b^d-b)\cdots(b^d-b^{i-1})}{(b-1)(b^2-1)\cdots(b^i-1)} \qquad (0 \leqslant i \leqslant d). \tag{3.2}$$

*Proof.* Let $\sigma_i$ denote the expression on the right in (3.2), for $0 \leqslant i \leqslant d$. By (3.1) and (1.21)–(1.24), we obtain

$$\sigma_0=1, \qquad \sigma_1=\theta_d,$$

$$b_{i-1}\sigma_{i-1}+a_i\sigma_i+c_{i+1}\sigma_{i+1}=\theta_d\sigma_i \qquad (1 \leqslant i \leqslant d-1).$$

By [1, p. 191], the sequence $p_0(d), p_1(d), ..., p_d(d)$ satisfies the same recursion and has the same initial conditions, so $\sigma_i=p_i(d)$ $(0 \leqslant i \leqslant d)$. This proves the lemma.

## 4. THE SUBSPACES

DEFINITION 4.1. Let $\Gamma=(X, R)$ denote a distance-regular graph.

(i) A subset $\Omega \subseteq X$ is said to be *weak-geodetically closed* whenever $\Omega \neq \emptyset$, and for all $x, y \in \Omega$,

$$\delta(x, z)+\delta(z, y) \leqslant \delta(x, y)+1 \rightarrow z \in \Omega \qquad (\forall z \in X). \tag{4.1}$$

(ii) By a *subspace* of $\Gamma$ we mean a regular subgraph which is induced on a weak-geodetically closed subset. We observe the subspaces of diameter 0 are just the vertices of $\Gamma$. We refer to the subspaces of diameter 1 (resp. diameter 2) as the *lines* (resp. *planes*) of $\Gamma$.

Properties of weak-geodetically closed subgraphs are discussed first by H. Suzuki in [7], where the term *strongly closed* is used for weak-geodetically closed. In this paper, we quote some of their properties from [11].

LEMMA 4.2 [11]. *Let $\Gamma=(X, R)$ denote a distance-regular graph. Then the following* (i)–(ii) *hold.*

(i) *The intersection of two weak-geodetically closed subsets is either empty, or weak-geodetically closed.*

(ii) *Let $\Omega$ denote a subspace of $\Gamma$. Then $\Omega$ is distance-regular with intersection numbers*

$$c_i(\Omega)=c_i(\Gamma) \qquad (0 \leqslant i \leqslant t), \tag{4.2}$$

$$a_i(\Omega)=a_i(\Gamma) \qquad (0 \leqslant i \leqslant t), \tag{4.3}$$

$$b_i(\Omega)=b_i(\Gamma)-b_t(\Gamma) \qquad (0 \leqslant i \leqslant t), \tag{4.4}$$

*where $t$ denotes the diameter of $\Omega$.*

LEMMA 4.3. *Let $\Gamma=(X, R)$ denote a distance-regular graph with diameter $d \geqslant 1$, and let $\Omega$ denote a subspace of $\Gamma$ containing at least two vertices. Then the intersection numbers of $\Gamma$ satisfy*

$$b_{t-1}>b_t, \tag{4.5}$$

*where $t$ denotes the diameter of $\Omega$.*

*Proof.* Set $i=t-1$ in line (4.4), and observe $b_{t-1}(\Omega) \neq 0$.

DEFINITION 4.4. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d$. Then $\Gamma$ is said to be *$d$-bounded* whenever the following (i)–(ii) hold.

(i) The subgraph induced on each weak-geodetically closed subset is regular.

(ii) For all $x, y \in X$, $x, y$ are contained in a subspace of diameter $\delta(x, y)$.

LEMMA 4.5. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d$, and suppose $\Gamma$ is $d$-bounded. Then the following* (i)–(iv) *hold.*

(i) *The intersection of two subspaces is either a subspace or the empty set.*

(ii) *Let $\Omega$ and $\Omega'$ denote subspaces of $\Gamma$ such that $\Omega$ is properly contained in $\Omega'$. Then the diameters*

$$\operatorname{diam}(\Omega) < \operatorname{diam}(\Omega').$$

(iii) *For any $x, y \in X$, the subspace of diameter $\delta(x, y)$ containing $x, y$ is unique.*

(iv) *Suppose $\Omega$ is a subspace of $\Gamma$. Then $\Omega$ is $t$-bounded, where $t$ is the diameter of $\Omega$.*

*Proof.* (i) This follows from Definition 4.1(ii), Lemma 4.2(i), and Definition 4.4(i).

(ii) Suppose $\Omega$ and $\Omega'$ have the same diameter. Then they have the same cardinality by (1.3)–(1.4), (4.2), (4.4). Thus $\Omega = \Omega'$, a contradiction.

(iii) Suppose subspaces $\Omega$ and $\Omega'$ contain $x, y$ and have diameter $\delta(x, y)$. Then $\Omega \cap \Omega'$ contains $x, y$ and has diameter $\delta(x, y)$. By (i), (ii) above,

$$\Omega = \Omega \cap \Omega' = \Omega'.$$

(iv) This follows from Definition in 4.1, since the distances as measured in $\Omega$ are the same as the distances as measured in $\Gamma$.

LEMMA 4.6. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 2$. Suppose $\Gamma$ is $d$-bounded, let $\Omega$ denote a plane of $\Gamma$, let $x$ denote a vertex of $\Gamma$, and set*

$$j := \max\{\delta(x, y) \mid y \in \Omega\}.$$

*Then there exists a unique subspace of $\Gamma$ that has diameter $j$ and contains both $x$ and $\Omega$.*

*Proof.* Pick a vertex $y \in \Gamma_j(x) \cap \Omega$. Let $\Delta$ denote the subspace of $\Gamma$ that has diameter $j$ and contains $x$, $y$. Observe $\Omega \cap \Delta$ is a subspace, and clearly $\Omega \cap \Delta \subseteq \Omega$, so $\Omega \cap \Delta$ has diameter at most 2. Observe by (4.1) that $\Omega \cap \Delta$ contains all vertices in $\Omega$ adjacent $y$. Now $\Omega \cap \Delta$ must contain a pair of non-adjacent vertices, so $\Omega \cap \Delta$ is a plane. Applying Lemma 4.5(ii), we find $\Omega \cap \Delta = \Omega$, so $\Omega \subseteq \Delta$. The uniqueness of $\Delta$ follows from Lemma 4.5(iii).

DEFINITION 4.7. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 2$. Let $\Omega$ denote a plane of $\Gamma$. Pick any $x \in X$, and set

$$S = \{i \mid 0 \leqslant i \leqslant d, \Gamma_i(x) \cap \Omega \neq \emptyset\}.$$

We refer to $S$ as the *shape of $\Omega$ with respect to $x$.*

*Note* 4.8. With reference to Definition 4.7, $S$ is a set of consecutive integers with cardinality at most 3, since $\Omega$ is connected and has diameter 2.

LEMMA 4.9. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 3$, and suppose $\Gamma$ is d-bounded. Pick an integer $i$ $(2 \leqslant i \leqslant d)$, pick a vertex $x \in X$, and let $\Omega$ denote a plane of shape $\{i-2, i-1, i\}$ with respect to $x$. Then $\Gamma_{i-2}(x) \cap \Omega$ consists of a single vertex. Denoting this vertex by $y$,*

$$\delta(x, z) = i - 2 + \delta(y, z) \qquad (\forall z \in \Omega). \tag{4.6}$$

*Proof.* By assumption, there exists a vertex $y \in \Gamma_{i-2}(x) \cap \Omega$. We show $y$ satisfies (4.6). To do this, we first show

$$\Gamma_2(y) \cap \Omega \subseteq \Gamma_i(x). \tag{4.7}$$

Pick any vertex $u \in \Gamma_2(y) \cap \Omega$. To obtain a contradiction, suppose $\delta(x, u) < i$. Pick a vertex $w$ adjacent to $u$ and $y$. Observe that $\delta(x, w) < i$, and $w \in \Omega$ by (4.1). Pick $v \in \{u, w\}$ with $\delta(x, v) = \max\{\delta(x, u), \delta(x, w)\}$. Note that $\delta(x, v) \in \{i-2, i-1\}$. Let $\Delta$ denote the subspace of diameter $\delta(x, v)$ containing $x$ and $v$. Observe that $y, u \in \Delta$ by (4.1), so $y, u \in \Omega \cap \Delta$. Apparently $\Omega \cap \Delta$ has diameter 2, so $\Omega \cap \Delta = \Omega$ by Lemma 4.5(ii), forcing $\Omega \subseteq \Delta$. But this is impossible, since the diameter of $\Delta$ is at most $i-1$, and since $\Omega$ has shape $\{i-2, i-1, i\}$ with respect to $x$. We now have a contradiction, and (4.7) follows. We now show

$$\Gamma_1(y) \cap \Omega \subseteq \Gamma_{i-1}(x).$$

This is immediate from (4.7), since $\Omega$ is distance-regular with diameter 2 and for each $w \in \Gamma_1(y) \cap \Omega$, $w$ is adjacent to some vertex in $\Gamma_2(y) \cap \Omega$. This proves (4.6). The uniqueness of $y$ is clear from (4.6).

LEMMA 4.10. *Let $\Gamma = (X, R)$ denote a distance-regular graph with classical parameters $(d, b, \alpha, \beta)$. Let $\Omega$ denote a subspace of $\Gamma$. Then $\Omega$ is distance-regular with classical parameters $(t, b, \alpha, \beta')$, where $t$ denotes the diameter of $\Omega$, and where*

$$\beta' = \beta + \alpha \left( \begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} t \\ 1 \end{bmatrix} \right). \tag{4.8}$$

*Proof.* Recall $\Omega$ is distance-regular by Lemma 4.2(ii). By (4.2), (1.21),

$$\begin{aligned} c_i(\Omega) &= c_i(\Gamma) \\ &= \begin{bmatrix} i \\ 1 \end{bmatrix} \left( 1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \qquad (0 \leqslant i \leqslant t), \end{aligned}$$

and by (4.4), (1.22),

$$\begin{aligned} b_i(\Omega) &= b_i(\Gamma) - b_t(\Gamma) \\ &= \left( \begin{bmatrix} t \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left( \beta + \alpha \left( \begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} t \\ 1 \end{bmatrix} \right) - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \qquad (0 \leqslant i \leqslant t). \end{aligned}$$

This proves the lemma.

## 5. DISTANCE-REGULAR GRAPHS WITH GEOMETRIC PARAMETERS

DEFINITION 5.1. A distance-regular graph $\Gamma$ is said to have *geometric parameters* $(d, b, \alpha)$ whenever it has classical parameters $(d, b, \alpha, \beta)$, where $b \neq 1$ and

$$\beta = \alpha \frac{1 + b^d}{1 - b}. \tag{5.1}$$

The following two classes of distance-regular graphs have geometric parameters.

THEOREM 5.2 [3]. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 3$, and let $b$ denote a complex number. Then the following* (i)–(ii) *are equivalent.*

(i) *$-b$ is a power of a prime, and $\Gamma$ is the dual polar graph ${}^2A_{2d-1}(-b)$.*

(ii) *$\Gamma$ has geometric parameters $(d, b, \alpha)$, where $\alpha = b(b-1)/(b+1)$.*

THEOREM 5.3 [4, 5, 10]. *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geqslant 3$, and let $b$ denote a complex number. Then the following* (i)–(ii) *are equivalent.*

(i) *$-b$ is a power of a prime, and $\Gamma$ is the Hermitian forms graph $Her_{-b}(d)$.*

(ii) *$\Gamma$ has geometric parameters $(d, b, \alpha)$, where $\alpha = b - 1$.*

We now consider the properties of distance-regular graphs with geometric parameters.

LEMMA 5.4. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$. Suppose that $\Omega$ is a subspace of $\Gamma$. Then $\Omega$ has geometric parameters $(t, b, \alpha)$, where $t$ is the diameter of $\Omega$.*

*Proof.* By Lemma 4.10, $\Omega$ has classical parameters $(t, b, \alpha, \beta')$, where

$$\begin{aligned}\beta' &= \beta + \alpha\left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} t \\ 1 \end{bmatrix}\right)\\ &= \alpha\frac{1 + b^d}{1 - b} + \alpha\left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} t \\ 1 \end{bmatrix}\right)\\ &= \alpha\frac{1 + b^t}{1 - b}\end{aligned}$$

by (5.1). Hence $\Omega$ has geometric parameters $(t, b, \alpha)$.

We are mainly concerned with distance-regular graphs that have geometric parameters and are $d$-bounded. We now mention a few results concerning these graphs.

LEMMA 5.5. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$ and $d \geqslant 3$. Suppose $\Gamma$ is $d$-bounded. Then $b < -1$.*

*Proof.* Recall that $b \neq -1$. Suppose $b > -1$. Then by [11, Lemma 3.6], we have $c_2 = b + 1$. Hence $\alpha = 0$ by (1.21), and $\beta = 0$ by (5.1). This implies $b_0 = 0$ by (1.22), a contradiction.

LEMMA 5.6. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$ and $d \geqslant 3$. Suppose $\Gamma$ is $d$-bounded. Let $\Omega, \Omega'$ denote subspaces of $\Gamma$ such that $\Omega \cap \Omega' \neq \emptyset$. Then the minimum subspace $\Delta$ containing $\Omega$ and $\Omega'$ satisfies*

$$\operatorname{diam}(\Delta) = \operatorname{diam}(\Omega) + \operatorname{diam}(\Omega') - \operatorname{diam}(\Omega \cap \Omega'), \tag{5.2}$$

*where* diam *denotes the diameter.*

*Proof.* Fix a vertex $x \in \Omega \cap \Omega'$, and let $I$ denote the poset consisting of all subspaces of $\Gamma$ that contain $x$, with partial order by reverse inclusion. We show $I$ is a modular atomic lattice with rank function $d$-diam. To see this, we use some results from [12]. First of all, we observe $I$ is an interval in the poset $P(\Gamma)$ defined in [12, Definition 2.3]. Applying [12, Corollary 2.8], we find $I$ is atomic, and applying [12, Lemma 2.4(i)] we find $I$ is ranked, with rank function $d$-diam. To see that $I$ is modular, observe equality holds for all $i$ in [12, Lemma 4.1(iii)]. It follows the scalars $f_1, f_2, ..., f_{d-1}$ from that lemma are equal, so $I$ is modular by [12, Proposition 2.10(i), (iii)]. We have now shown $I$ is a modular atomic lattice with rank function $d$-diam. Line (5.2) follows in view of [6, p. 104].

We now consider how to tell if a given distance-regular graph is $d$-bounded and has geometric parameters.

THEOREM 5.7. *Let $\Gamma = (X, R)$ denote a distance-regular graph with classical parameters $(d, b, \alpha, \beta)$ and $d \geqslant 3$. Suppose $b < -1$, and suppose the intersection numbers $a_1 \neq 0$, $c_2 > 1$. Then $\Gamma$ is d-bounded.*

*Proof.* Referring to Definition 4.4, condition (i) holds by [11, Lemma 5.2]. Condition (ii) is immediate from [11, Theorem 7.2(v), (vii)].

THEOREM 5.8 [12, Theorem 4.2]. *Let $\Gamma = (X, R)$ denote a distance-regular graph with classical parameters $(d, b, \alpha, \beta)$ and $b < -1$. Suppose $\Gamma$ is d-bounded and $d \geqslant 4$. Then*

(i) *$\Gamma$ has geometric parameters $(d, b, \alpha)$.*

(ii) *$-b$ is a power of a prime.*

## 6. SOME PROPERTIES ON THE SUBSPACES

LEMMA 6.1. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$ and $d \geqslant 3$. Suppose $\Gamma$ is d-bounded. Then the following* (i)–(ii) *hold.*

(i) *Each line has cardinality*

$$a_1 + 2 = \frac{b - 1 - \alpha(b+1)}{b-1}. \tag{6.1}$$

(ii) *Each plane has cardinality*

$$\frac{(1 - b + \alpha(1 + b^2))(1 - b + \alpha(1 + b^3))}{(b-1)^2\,(1+\alpha)}. \tag{6.2}$$

*Proof.* (i) Each line $L$ has cardinality

$$1 + b_0(L) = 1 + b_0(\Gamma) - b_1(\Gamma)$$
$$= a_1 + 2 \tag{6.3}$$

by (1.4), (4.4), (1.2). Now (6.1) follows by evaluating (6.3) using (1.24), (5.1).

(ii) Each plane $\Omega$ has cardinality

$$1 + b_0(\Omega) + \frac{b_0(\Omega)\, b_1(\Omega)}{c_2(\Omega)}$$
$$= 1 + b_0(\Gamma) - b_2(\Gamma) + \frac{(b_0(\Gamma) - b_2(\Gamma))(b_1(\Gamma) - b_2(\Gamma))}{c_2(\Gamma)} \tag{6.4}$$

by (1.3)–(1.4), (4.2), (4.4). Now (6.2) follows by evaluating (6.4) using (1.21)–(1.22), (5.1). This proves the lemma.

LEMMA 6.2. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$ and $d \geqslant 3$. Suppose $\Gamma$ is d-bounded. Pick $x \in X$, pick an integer $i$ $(3 \leqslant i \leqslant d)$, and suppose there exists a plane $\Omega$ of shape $\{i-1, i\}$ with respect to $x$. Then the following* (i)–(ii) *hold.*

(i) *$\Gamma_{i-1}(x) \cap \Omega$ is a disjoint union of lines.*

(ii) *Each $z \in \Gamma_{i-1}(x) \cap \Omega$ is adjacent to exactly*

$$b_1 - b_2 = \frac{\alpha b^2(b+1)}{1-b} \tag{6.5}$$

*vertices in $\Gamma_i(x) \cap \Omega$.*

*Proof.* (i) Let $C$ denote a maximal connected subset in $\Gamma_{i-1}(x) \cap \Omega$. We show $C$ is a line. Pick any vertex $w \in C$, let $\Delta$ denote the subspace of diameter $i-1$ containing $x$ and $w$, and observe $\Delta \cap \Omega \subseteq \Gamma_{i-1}(x) \cap \Omega$. Observe $\Delta \cap \Omega$ is a subspace, so it is connected, forcing

$$\Delta \cap \Omega \subseteq C. \tag{6.6}$$

By (4.1),

$$C \subseteq \Delta \cap \Omega, \tag{6.7}$$

and combining (6.6)–(6.7),

$$C = \Delta \cap \Omega. \tag{6.8}$$

By Lemma 4.6, there exists a unique subspace $\Delta'$ of $\Gamma$ that has diameter $i$, and contains both $x$ and $\Omega$. Observe $\Delta'$ equals the minimal subspace of $\Gamma$ containing $\Delta$ and $\Omega$. Applying (5.2),

$$\begin{aligned}\operatorname{diam}(\Delta \cap \Omega) &= \operatorname{diam}(\Delta) + \operatorname{diam}(\Omega) - \operatorname{diam}(\Delta') \\ &= 1, \end{aligned} \tag{6.9}$$

so $C$ is a line by (6.8), (6.9). This proves (i).

(ii) Recall that $\Omega$ is regular with valency $b_0 - b_2$ by (4.4), and $z$ is adjacent to $a_1 + 1$ vertices in $\Gamma_{i-1}(x) \cap \Omega$ by (i) and (6.1). Hence $z$ is adjacent to

$$\begin{aligned} b_0 - b_2 - (a_1 + 1) &= b_1 - b_2 \\ &= \frac{\alpha b^2 (b+1)}{1-b} \end{aligned}$$

vertices in $\Gamma_i(x) \cap \Omega$ by (1.22) and (5.1). This proves (ii).

## 7. THE CASE $D = 3$

In this paper, we are mainly concerned with distance-regular graphs that have geometric parameters and are $d$-bounded, where $d$ denotes the diameter. In this section, we consider the special case $d = 3$.

LEMMA 7.1. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(3, b, \alpha)$. Suppose $\Gamma$ is 3-bounded, and let $\Omega$ denote a plane of $\Gamma$. Then with respect to the ordering of the primitive idempotents from Lemma 3.1,*

$$\sum_{y \in \Omega} E_3 \hat{y} = 0. \tag{7.1}$$

*Proof.* With reference to (1.19), it suffices to show the inner product

$$\left\langle \sum_{y \in \Omega} E_3 \hat{y}, \sum_{y \in \Omega} E_3 \hat{y} \right\rangle = 0. \tag{7.2}$$

Observe by (1.20), (1.15)–(1.17),

$$\begin{aligned}\left\langle \sum_{y \in \Omega} E_3 \hat{y}, \sum_{y \in \Omega} E_3 \hat{y} \right\rangle &= \frac{|\Omega|}{|X|} (q_3(0) + k_1(\Omega)\, q_3(1) + k_2(\Omega)\, q_3(2)) \\ &= \frac{|\Omega|\, m_3}{|X|} \left( 1 + \frac{k_1(\Omega)}{k_1(\Gamma)} p_1(3) + \frac{k_2(\Omega)}{k_2(\Gamma)} p_2(3) \right). \end{aligned} \tag{7.3}$$

To show (7.3) is zero, observe by (5.1), (1.3), (1.21)–(1.22), (4.2), (4.4), (3.2),

$$\frac{k_1(\Omega)}{k_1(\Gamma)} = \frac{b^4-1}{b^6-1}, \tag{7.4}$$

$$\frac{k_2(\Omega)}{k_2(\Gamma)} = \frac{(b^4-1)(b^4-b^2)}{(b^6-1)(b^6-b^2)}, \tag{7.5}$$

$$p_1(3) = -\frac{b^3-1}{b-1}, \tag{7.6}$$

$$p_2(3) = \frac{(b^3-1)(b^3-b)}{(b-1)(b^2-1)}. \tag{7.7}$$

Evaluating (7.3) using (7.4)–(7.7), we find it equals zero. Now (7.2) holds, and the lemma is proved.

LEMMA 7.2. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(3, b, \alpha)$. Suppose $\Gamma$ is 3-bounded. Pick $x \in X$, and suppose there exists a plane $\Omega$ of shape $\{2, 3\}$ with respect to $x$. Then*

$$\text{(i)}\quad |\Gamma_2(x) \cap \Omega| = \frac{(1+\alpha(1+b))(1-b+\alpha(1+b^2))}{(1+\alpha)(1-b)}, \tag{7.8}$$

$$\text{(ii)}\quad |\Gamma_3(x) \cap \Omega| = \frac{\alpha b^2(1+b)(1-b+\alpha(1+b^2))}{(b-1)^2\,(1+\alpha)}. \tag{7.9}$$

*Proof.* Of course

$$|\Gamma_2(x) \cap \Omega| + |\Gamma_3(x) \cap \Omega| = |\Omega|. \tag{7.10}$$

Taking the inner product of $E_3\hat{x}$ with both sides of (7.1), and evaluating the result using (1.20), (1.17),

$$\begin{aligned} 0 &= |X|^{-1}\,(|\Gamma_2(x) \cap \Omega|\; q_3(2) + |\Gamma_3(x) \cap \Omega|\; q_3(3)) \\ &= \frac{m_3}{|X|}\left(|\Gamma_2(x) \cap \Omega|\,\frac{p_2(3)}{k_2(\Gamma)} + |\Gamma_3(x) \cap \Omega|\,\frac{p_3(3)}{k_3(\Gamma)}\right). \end{aligned} \tag{7.11}$$

Solving (7.10), (7.11) for $|\Gamma_2(x) \cap \Omega|$, $|\Gamma_3(x) \cap \Omega|$, and simplifying the result using (3.2), (1.3), (1.21)–(1.22), (6.2), (5.1), we obtain (7.8)–(7.9).

LEMMA 7.3. *Let $\Gamma=(X, R)$ denote a distance-regular graph with geometric parameters $(3, b, \alpha)$. Suppose $\Gamma$ is 3-bounded. Pick $x\in X$, and suppose there exists a plane $\Omega$ of shape $\{2, 3\}$ with respect to $x$. Then each vertex in $\Gamma_3(x)\cap\Omega$ is adjacent to exactly*

$$c_2-b=1+\alpha(b+1) \tag{7.12}$$

*vertices in $\Gamma_2(x)\cap\Omega$. Moreover, the scalar on either side of (7.12) is positive.*

*Proof.* For all $y\in\Gamma_3(x)\cap\Omega$, let $f_y$ denote the number of vertices in $\Gamma_2(x)\cap\Omega$ adjacent $y$. Observe

$$\sum_{y\in\Gamma_3(x)\cap\Omega}(f_y-c_2+b)^2=\sum_{y\in\Gamma_3(x)\cap\Omega}f_y^2-2(c_2-b)\sum_{y\in\Gamma_3(x)\cap\Omega}f_y +|\Gamma_3(x)\cap\Omega|\,(c_2-b)^2. \tag{7.13}$$

Applying (6.5),

$$\sum_{y\in\Gamma_3(x)\cap\Omega}f_y=|\Gamma_2(x)\cap\Omega|\,(b_1-b_2). \tag{7.14}$$

Counting the number of paths $z, y, w\in\Omega$ such that

$$z, w\in\Gamma_2(x)\cap\Omega,\qquad z\neq w,\quad y\in\Gamma_3(x)\cap\Omega,$$

in two ways, one in the sequence $y, z, w$, the other in the sequence $z, w, y$, and observing $\delta(z, w)=2$ by Lemma 6.2(i), we obtain

$$\sum_{y\in\Gamma_3(x)\cap\Omega}f_y(f_y-1)=|\Gamma_2(x)\cap\Omega|\,(|\Gamma_2(x)\cap\Omega|-a_1-2)\,c_2. \tag{7.15}$$

Of course

$$\sum_{y\in\Gamma_3(x)\cap\Omega}f_y^2=\sum_{y\in\Gamma_3(x)\cap\Omega}f_y(f_y-1)+\sum_{y\in\Gamma_3(x)\cap\Omega}f_y. \tag{7.16}$$

Eliminating $\sum f_y, \sum f_y^2$ in (7.13) using (7.14)–(7.16), and eliminating the result using (1.21)–(1.24), (5.1), (7.8)–(7.9), (6.5), we find the right hand side of (7.13) equals zero. This forces

$$f_y=c_2-b\qquad(\forall y\in\Gamma_3(x)\cap\Omega).$$

By (1.21), we find

$$c_2-b=1+\alpha(b+1),$$

and this scalar is positive since $\Omega$ is connected. This proves the lemma.

## 8. THE GENERAL CASE

LEMMA 8.1. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$ and $d \geqslant 3$. Suppose $\Gamma$ is $d$-bounded. Fix a vertex $x \in X$ and fix an integer $i$ $(3 \leqslant i \leqslant d)$. Suppose there exists a plane $\Omega$ of shape $\{i-1, i\}$ with respect to $x$. Then there exists a vertex $w \in \Gamma_{i-3}(x)$ such that the following* (i)–(iii) *hold.*

(i) *$w$ and $\Omega$ are contained in a common subspace of diameter* 3.

(ii) $\Gamma_{i-1}(x) \cap \Omega = \Gamma_2(w) \cap \Omega$.

(iii) $\Gamma_i(x) \cap \Omega = \Gamma_3(w) \cap \Omega$.

*Proof.* By Lemma 6.2(i), $\Gamma_{i-1}(x) \cap \Omega$ is a disjoint union of lines. In particular, there exist adjacent vertices $y, z \in \Gamma_{i-1}(x) \cap \Omega$. Recall $\Gamma$ is $Q$-polynomial by Lemma 3.1, so by (2.7),

$$\Gamma_{i-3}(x) \cap \Gamma_2(y) \cap \Gamma_2(z) \neq \emptyset.$$

Pick

$$w \in \Gamma_{i-3}(x) \cap \Gamma_2(y) \cap \Gamma_2(z).$$

Observe that $\Omega$ has shape $\{2, 3\}$ or shape $\{2, 3, 4\}$ with respect to $w$. In view of Lemma 4.9, $\Omega$ has shape $\{2, 3\}$ with respect to $w$. Applying Lemma 4.6, we find $w$ and $\Omega$ are contained in a common subspace $\Delta$ of diameter 3, so we have (i). To obtain (ii), first observe $\Delta$ has geometric parameters $(3, b, \alpha)$ by Lemma 5.4, and is 3-bounded by Lemma 4.5(iv), so the results of Section 7 apply to $\Delta$. Observe that

$$\Gamma_2(w) \cap \Omega \subseteq \Gamma_{i-1}(x) \cap \Omega. \tag{8.1}$$

To show equality in the above line, suppose there exists a vertex $u$ that is contained in $\Gamma_{i-1}(x) \cap \Omega$ but not contained in $\Gamma_2(w) \cap \Omega$. Then $u \in \Gamma_3(w) \cap \Omega$. Applying Lemma 7.3 to $\Delta$, we find $u$ is adjacent to some vertex in $\Gamma_2(w) \cap \Omega$. There is no such edge inside $\Gamma_{i-1}(x) \cap \Omega$ since both $\Gamma_{i-1}(x) \cap \Omega$ and $\Gamma_2(w) \cap \Omega$ are a disjoint union of lines by Lemma 6.2(i). We now have a contradiction, so equality holds in (8.1), and we have (ii). Condition (iii) is immediate from (ii), and since $\Omega$ has shape $\{2, 3\}$ with respect to $w$.

LEMMA 8.2. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$ and $d \geqslant 3$. Suppose $\Gamma$ is $d$-bounded. Pick any $x \in X$, pick an integer $i$ $(3 \leqslant i \leqslant d)$, and suppose there exists a plane $\Omega$ of shape $\{i-1, i\}$ with respect to $x$. Then the following* (i)–(iii) *hold.*

(i) $$|\Gamma_{i-1}(x) \cap \Omega| = \frac{(1+\alpha(1+b))(1-b+\alpha(1+b^2))}{(1+\alpha)(1-b)}. \tag{8.2}$$

(ii) $$|\Gamma_i(x) \cap \Omega| = \frac{\alpha b^2(1+b)(1-b+\alpha(1+b^2))}{(b-1)^2(1+\alpha)}. \tag{8.3}$$

(iii) *Each vertex in $\Gamma_i(x) \cap \Omega$ is adjacent to exactly*

$$c_2 - b = 1 + \alpha(b+1) \tag{8.4}$$

*vertices in $\Gamma_{i-1}(x) \cap \Omega$. Moreover, the expression on either side of* (8.4) *is positive.*

*Proof.* Pick $w \in \Gamma_{i-3}(x)$ satisfying (i)–(iii) of Lemma 8.1. From Lemma 8.1(i), there exists a subspace $\Delta$ of diameter 3 containing $w$ and $\Omega$. Note that $\Delta$ has geometric parameters $(3, b, \alpha)$ by Lemma 5.4, and $\Delta$ is 3-bounded by Lemma 4.5(iv). Now (i)–(iii) are immediate from Lemma 8.1(ii)–(iii) and Lemmas 7.2, 7.3 (with $\Gamma = \Delta$).

Corollary 8.3. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$ and $d \geqslant 3$. Suppose $\Gamma$ is d-bounded. Pick any $x \in X$, pick an integer $i$ $(3 \leqslant i \leqslant d)$, and suppose there exists a plane $\Omega$ of shape $\{i-1, i\}$ with respect to x. Then $\Gamma_{i-1}(x) \cap \Omega$ is a disjoint union of*

$$\sigma := \frac{(1+\alpha(1+b))(1-b+\alpha(1+b^2))}{(1+\alpha)(1-b+\alpha(1+b))} \tag{8.5}$$

*lines.*

*Proof.* By Lemma 6.2(i), $\Gamma_{i-1}(x) \cap \Omega$ is a disjoint union of lines. Dividing (8.2) by the line size (6.1), we obtain (8.5).

## 9. COUNTING ARGUMENTS

Proposition 9.1. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$ and $d \geqslant 3$. Suppose $\Gamma$ is d-bounded. Fix an integer $i$ $(3 \leqslant i \leqslant d)$, and pick vertices $x, y \in X$ at distance $\delta(x, y) = i$. Then the following* (i)–(iii) *hold.*

(i) *y is contained in*

$$\frac{c_i c_{i-1}}{c_2} \tag{9.1}$$

*planes of shape $\{i-2, i-1, i\}$ with respect to x.*

(ii) *y is contained in*

$$\frac{c_i(a_{i-1}-a_1c_{i-1})}{(c_2-b)(a_1+1)}=\frac{c_i(b^i-b)(b^i-b^2)(b^2-b-\alpha(b+1))}{(c_2-b)\,b^3(b+1)(b-1)^2} \tag{9.2}$$

*planes of shape* $\{i-1, i\}$ *with respect to* $x$.

(iii) *y is contained in*

$$\frac{(b^i-1)(b^i-b)(b^i-b^2)(b^i-b^3)(b-2\alpha-1)(b^2-b-\alpha(b+1))}{b^3(1+\alpha(b+1))(b+1)^2\,(b^2+1)(b-1)^4\,(1+\alpha)} \tag{9.3}$$

*planes of shape* $\{i\}$ *with respect to* $x$.

*Proof.* Let $\sharp_1$ (respectively $\sharp_2, \sharp_3$) denote the number of planes containing $y$ that have shape $\{i-2, i-1, i\}$ (respectively $\{i-1, i\}, \{i\}$) with respect to $x$.

To obtain $\sharp_1$, we count in two ways the number of ordered pairs $(z, w)$, where $z, w$ are vertices in $X$ such that

$$\delta(x, w)=i-2, \qquad \delta(y, z)=1, \qquad \delta(z, w)=1.$$

Observe that for such a pair $(z, w)$ we have $\delta(y, w)=2$, so there is a unique plane containing $y, w$; it contains $z$ and has shape $\{i-2, i-1, i\}$ with respect to $x$. On the one hand, we count in the sequence $z, w$ in $X$, and on the other hand, we count in the sequence $w, z$ in each of the planes containing $y$ of shape $\{i-2, i-1, i\}$ with respect to $x$. By Lemma 4.9, we obtain

$$c_ic_{i-1}=\sharp_1c_2, \tag{9.4}$$

and (9.1) follows by solving (9.4) for $\sharp_1$.

To obtain $\sharp_2$, we count in two ways the number of ordered pairs $(z, w)$, where $z, w$ are vertices in $X$ such that

$$\delta(x, z)=i-1, \qquad \delta(x, w)=i-1, \qquad \delta(y, z)=1, \qquad \delta(z, w)=1.$$

Note that $\delta(y, w)=2$, otherwise by (4.1), $y$ is contained in the subspace of diameter $\delta(x, z)$ containing $x$ and $z$, a contradiction. Observe from Lemma 4.9 that $y, w$ determine a plane containing $y, z, w$ of shape $\{i-2, i-1, i\}$ or of shape $\{i-1, i\}$ with respect to $x$.

On the one hand, we count such $z, w$ in $X$, and on the other hand, we count such $z, w$ in each of the planes containing $y$ of shape $\{i-2, i-1, i\}$ or $\{i-1, i\}$ with respect to $x$. We obtain

$$c_ia_{i-1}=\sharp_1c_2a_1+\sharp_2(c_2-b)(a_1+1) \tag{9.5}$$

by Lemma 4.9, Lemma 8.2(iii), Lemma 6.2(i), (6.1). Solving (9.4)–(9.5) for $\sharp_2$ and simplifying the result using (1.21)–(1.24), (5.1), we obtain (9.2).

To obtain $\sharp_3$, we count in two ways the number of ordered pairs $(z, w)$, where $z, w$ are vertices in $X$ such that

$$\delta(x, z) = i, \qquad \delta(x, w) = i, \qquad \delta(y, z) = 1, \qquad \delta(z, w) = 1, \qquad \delta(y, w) = 2.$$

On the one hand, we count such $z, w$ in $X$, and on the other hand, we count such $z, w$ in each of the planes containing $y$ of shape $\{i-2, i-1, i\}$, $\{i-1, i\}$, or $\{i\}$ with respect to $x$. Using an argument similar to the proof of (9.5),

$$\begin{aligned} a_i(a_i - a_1 - 1) + c_i a_1 \\ = \sharp_1(a_2(a_2 - a_1 - 1) + c_2 a_1) \\ + \sharp_2((b_0 - b_2 - (c_2 - b))(b_0 - b_2 - (c_2 - b) - a_1 - 1) + (c_2 - b)\, a_1) \\ + \sharp_3(b_0 - b_2)(b_1 - b_2). \end{aligned} \tag{9.6}$$

Solving (9.4)–(9.6) for $\sharp_3$ and simplifying the result using (1.21)–(1.24), (5.1), (9.1)–(9.2), we obtain (9.3).

COROLLARY 9.2. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$ and $d \geqslant 3$. Suppose $\Gamma$ is d-bounded. Furthermore, suppose $\Gamma$ is not the dual polar graph ${}^2A_{2d-1}(-b)$. Then for all integers $i$ $(3 \leqslant i \leqslant d)$, and for all vertices $x, y \in X$ at distance $\delta(x, y) = i$, there exists a plane containing $x, y$ that has shape $\{i-1, i\}$ with respect to $x$.*

*Proof.* It suffices to show (9.2) is not zero. Recall that $b$ is an integer, and that $b \neq 1$, $b \neq 0$, $b \neq -1$. In particular $b^i - b \neq 0$ and $b^i - b^2 \neq 0$. Note that $b^2 - b - \alpha(b+1) \neq 0$ by Theorem 5.2. We have now shown (9.2) is not zero, and the result follows. This proves the corollary.

## 10. THE MAIN THEOREM

We prove the main theorem in this section.

LEMMA 10.1. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$ and $d \geqslant 3$. Suppose $\Gamma$ is d-bounded. Suppose $\Gamma$ is not the dual polar graph ${}^2A_{2d-1}(-b)$, and $\Gamma$ is not the Hermitian forms graph $Her_{-b}(d)$. Then*

$$\alpha \geqslant (b-1)/2. \tag{10.1}$$

*Proof.* Observe by Corollary 9.2 that for each $x \in X$ and for each integer $i$ $(3 \leqslant i \leqslant d)$, there exists a plane of shape $\{i-1, i\}$ with respect to $x$. It follows the scalar $\sigma$ from (8.5) is an integer. Observe

$$\sigma - b^2 = \frac{(b+1)(b-1-\alpha)^2}{(1+\alpha)(1-b+\alpha(1+b))} \tag{10.2}$$

and

$$\sigma - b^2 - 1 = \frac{b^2(b-2\alpha-1)}{(1+\alpha)(1-b+\alpha(1+b))}. \tag{10.3}$$

Note that $\alpha \neq b-1$ by Theorem 5.3, and recall $b \neq -1$, so $\sigma - b^2 \neq 0$. Dividing (10.3) by (10.2),

$$\frac{b^2(b-2\alpha-1)}{(b+1)(b-1-\alpha)^2} = \frac{\sigma - b^2 - 1}{\sigma - b^2}$$
$$\geqslant 0,$$

since the ratio of two consecutive integers is nonnegative. Note that $b+1 < 0$ by Lemma 5.5, so $b - 2\alpha - 1 \leqslant 0$, and (10.1) follows. This proves the lemma.

LEMMA 10.2. *Let $\Gamma = (X, R)$ denote a distance-regular graph with geometric parameters $(d, b, \alpha)$ and $d \geqslant 4$. Suppose $\Gamma$ is d-bounded. Suppose $\Gamma$ is not the dual polar graph ${}^2A_{2d-1}(-b)$, and $\Gamma$ is not the Hermitian forms graph $Her_{-b}(d)$. Then* (i), (ii) *hold below.*

(i) $\alpha = (b-1)/2$.

(ii) $-b$ *is a power of an odd prime.*

*Proof.* (i) Set $i = 4$ in (9.2), and observe the resulting integer is positive by Corollary 9.2. Set $i = 4$ in (9.3), and observe the resulting integer is nonnegative. Dividing the second integer above by the first,

$$\frac{b^3(b^4-1)(b-2\alpha-1)}{c_2 c_4 (b^2+1)(b-1)} \geqslant 0. \tag{10.4}$$

Recall $b < -1$ by Lemma 5.5, so $b^3 < 0$, $b-1 < 0$, $b^4 - 1 > 0$. Hence

$$b - 2\alpha - 1 \geqslant 0. \tag{10.5}$$

Combining (10.1) and (10.5), we find $\alpha = (b-1)/2$.

(ii) Applying Lemma 5.5, we find $b < -1$. By Theorem 5.8(ii), we see $-b$ is a power of a prime. To show the prime is odd, we show $b$ is odd. Setting $i = 2$ in (1.21), and evaluating the result using (i) above,

$$c_2 = (1+\alpha)(1+b)$$
$$= \frac{(1+b)^2}{2}$$

is an integer, so $b$ is odd. We now have (ii), and the lemma is proved.

We now come to the main theorem of this paper.

THEOREM 10.3. *Let $\Gamma = (X, R)$ denote a distance-regular graph with classical parameters $(d, b, \alpha, \beta)$ and $d \geqslant 4$. Suppose $b < -1$, and suppose the intersection numbers $a_1 \neq 0$, $c_2 > 1$. Then precisely one of the following* (i)–(iii) *holds.*

(i) *$\Gamma$ is the dual polar graph ${}^2A_{2d-1}(-b)$.*

(ii) *$\Gamma$ is the Hermitian forms graph $Her_{-b}(d)$.*

(iii) *$\alpha = (b-1)/2$, $\beta = -(1+b^d)/2$, and $-b$ is a power of an odd prime.*

*Proof.* We assume (i), (ii) fail, and show (iii). Note that by Theorem 5.7, $\Gamma$ is $d$-bounded, and by Theorem 5.8(i), $\Gamma$ has geometric parameters $(d, b, \alpha)$, so conditions (i), (ii) hold in Lemma 10.2. Eliminating $\alpha$ in (5.1) using Lemma 10.2(i),

$$\beta = -\frac{1+b^d}{2},$$

and (iii) follows. This proves the theorem.

*Remark* 10.4. With the notation and assumptions of Lemma 10.2, pick $x \in X$, pick an integer $i$ $(3 \leqslant i \leqslant d)$, and let $\Omega$ denote a plane in $\Gamma$. Then the following (i)–(ii) hold.

(i) $\Omega$ does not have shape $\{i\}$ with respect to $x$.

(ii) Suppose $\Omega$ has shape $\{i-1, i\}$ with respect to $x$. Then $\Gamma_{i-1}(x) \cap \Omega$ is a disjoint union of $b^2+1$ lines. Moreover, each vertex in $\Gamma_i(x) \cap \Omega$ is adjacent to exactly

$$\frac{b^2+1}{2}$$

vertices in $\Gamma_{i-1}(x) \cap \Omega$.

*Proof.* We obtain the results by setting $\alpha = (b-1)/2$ in (9.3), (8.5), (8.4).

*Remark* 10.5. In [12], we proved that there is no distance-regular graph with classical parameters $(d, b, \alpha, \beta)$ that satisfies $d \geqslant 4$, $c_2 = 1$, and $a_2 > a_1 > 1$ (without assuming $b < -1$).

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