The Consecutive-4 Digraphs are Hamiltonian

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Abstract: Du, Hsu, and Hwang conjectured that consecutive-d digraphs are Hamiltonian for $d = 3, 4$. Recently, we gave an infinite class of consecutive-3 digraphs, which are not Hamiltonian. In this article we prove the conjecture for $d = 4$. © 1999 John Wiley & Sons, Inc. J Graph Theory 31: 1–6, 1999

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1. INTRODUCTION

Define $G(d, n, q, r)$, also known as a *consecutive-d digraph*, to be a digraph whose n nodes are labeled by the residues modulo n, and a link $i \to j$ from node i to node j exists if and only if $j \in \{qi + k \pmod{n}\colon\ r \le k \le r + d - 1\}$, where $1 \le q \le n-1, 1 \le d \le n-1$ and $0 \le r \le n-1$ are given. Many computer networks and multiprocessor systems use consecutive-d digraphs for the topology of their interconnection networks. For example, $q = 1$ yields the *multiloop networks* [13], also known as *circulant digraphs* [14], with the skip set $\{r, r+1, \ldots, r+d-1\}$. $q = d$ and $r = 0$ yields the *generalized de Bruijn digraphs* [8, 12], and $q = r = n - d$ yields the *Imase–Itoh digraphs* [9].

In some applications, it is important to know whether a consecutive-d digraph embeds a Hamiltonian circuit. This issue was first raised by Pradhan [11]. Necessary and sufficient conditions for generalized de Bruijn digraphs and the Imase–Itoh digraphs to be Hamiltonian were given by Du, Hsu, Hwang, and Zhang [5]. For

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the case of $\gcd(n,q) \geq 2$, Du, Hsu, and Hwang [4] showed that $G(d,n,q,r)$ is Hamiltonian if and only if $d \geq \gcd(n,q)$. So, we may only consider the case when $\gcd(n,q) = 1$. Necessary and sufficient conditions for consecutive-d digraphs to be Hamiltonian were given by Hwang [7] for $d = 1$ and by Du and Hsu [3] (also see [2]) for $d = 2$. Furthermore, Du, Hsu, and Hwang [4] proved that consecutive-d digraphs are Hamiltonian for $d \geq 5$, and conjectured they are also for $d = 3, 4$. Du and Hsu [3] gave partial support to this conjecture by proving its validity under the condition $q \leq d$. Recently, we [1] gave an infinite class of examples that consecutive-3 digraphs are not necessarily Hamiltonian. In this article, we prove that consecutive-4 digraphs are Hamiltonian, and thus, completely settle the conjecture.

2. SOME GENERAL REMARKS

Throughout this article, we assume that $\gcd(n,q) = 1$. In this case, $G(d,n,q,r)$ is a regular digraph of indegree and outdegree both d. In particular, $G(1,n,q,r')$ is the disjoint union of cycles.

Let $G(4,n,q,r)$ denote the underlying consecutive-4 digraph. Consider the digraph $G(1,n,q,r+1)$. Suppose that $G(1,n,q,r+1)$ consists of c disjoint cycles $C_1, C_2, \ldots, C_c$. If $c = 1$, then $G(4,n,q,r)$ is Hamiltonian and we are done. Suppose that $c > 1$. A *link-interchange method* was introduced in [4] to merge two cycles. Since $0 < q < n$, there exists a cycle with more than one node. Furthermore, this cycle remains to contain more than one node throughout merges. Let i be a node on this cycle such that $i+1$ is not. Such an i always exists unless the cycle is Hamiltonian. Suppose that $i' \to i$ and $(i+1)' \to i+1$ are in $G(1,n,q,r+1)$, where $i' \neq i$ but $(i+1)'$ could be $i+1$. We replace these two links by the two links $i' \to i+1$ and $(i+1)' \to i$ and call this an $\{i, i+1\}$ *interchange*, which merges the two cycles i and $i+1$ are on into one. Note that the link $i' \to i+1$ is in $G(1,n,q,r+2)$ and the link $(i+1)' \to i$ is in $G(1,n,q,r)$.

Two interchanges $\{i, i+1\}$ and $\{j, j+1\}$ do not interfere with each other, if $\{i, i+1\} \cap \{j, j+1\} = \emptyset$. But if the intersection is not empty, say, $j+1 = i$, then doing the interchange $\{i-1, i\}$ after $\{i, i+1\}$ means replacing $(i+1)' \to i$ by $(i+1)' \to i-1$, which is in $G(1,n,q,r-1)$, but not in $G(4,n,q,r)$. However, we can do $\{i, i+1\}$ after $\{i-1, i\}$. This is because we are replacing $(i-1)' \to i$ and $(i+1)' \to i+1$ by $(i-1)' \to i+1$ and $(i+1)' \to i$, where $(i-1)' \to i+1$ is in $G(1,n,q,r+3)$ and $(i+1)' \to i$ is in $G(1,n,q,r)$. Therefore, we can do two consecutive interchanges, if we do it in the right order, namely, do the smaller pair first. Similarly, if we start with decomposing $G(1,n,q,r+2)$ into cycles, then we can do two consecutive interchanges, if we do the larger pair first.

We will now represent two consecutive interchanges $\{i-1, i\}$ and $\{i, i+1\}$ by the set $\{i-1, i, i+1\}$. In defining an interchange, the two nodes involved are assumed to be on different cycles, and these are cycles updated to previous merges. For example, when the interchange $\{i, i+1\}$ is performed after the interchange

$\{i-1, i\}$, then the three nodes $i-1, i, i+1$ are on different cycles originally (if $i-1$ and $i+1$ are on the same cycle, we have no reason to perform the second interchange). A *legitimate interchange set* (without three consecutive interchanges) can be represented by a set $S = \{S_1, S_2, \ldots, S_s\}$, where the S_i's are disjoint and each S_i is a subset of two or three consecutive nodes. Note that after one or more interchanges in S_i are performed, then all cycles intersecting S_i are connected.

Let X and Y be two sets of subsets of $\{1, 2, \ldots, m\}$. Define $B_m(X, Y)$ to be the bipartite graph with vertex set $X \cup Y$, and there exists an edge between $X_i \in X$ and $Y_j \in Y$ if and only if $X_i \cap Y_j \neq \emptyset$. Let C^{r+1} (respectively, C^{r+2}) denote the set of all disjoint cycles in $G(1, n, q, r+1)$ (respectively, $G(1, n, q, r+2)$). Then we have the following.

Lemma 1. *$G(4, n, q, r)$ is Hamiltonian if $\gcd(n, q) = 1$ and there exists a legitimate interchange set S such that either $B_n(S, C^{r+1})$ or $B_n(S, C^{r+2})$ is connected.*

Proof. Since $\gcd(n, q) = 1$, both $G(1, n, q, r+1)$ and $G(1, n, q, r+2)$ are disjoint unions of cycles. Applying the link-interchange method by using the legitimate interchange set S, we can merge the cycles into a Hamiltonian cycle of $G(4, n, q, r)$. Q.E.D.

3. ALGORITHM FOR CONSTRUCTING S

We are unable to find an explicit legitimate interchange set S such that $B_n(S, C^{r+1})$ or $B_n(S, C^{r+2})$ is connected for all n and q. However, for each given set (n, q, r), we give an algorithm to construct such S. In fact, our construction applies to a more general setting where $C_1, C_2, \ldots, C_c$ do not have to come from $G(1, n, q, r+1)$ or $G(1, n, q, r+2)$, but merely a disjoint partition of $\{1, 2, \ldots, n\}$.

Lemma 2. *Let $P = \{P_1, P_2, \ldots, P_p\}$ be a partition of $\{1, 2, \ldots, m\}$ such that all $|P_j| \geq 2$ except one part can be a singleton. Then there exists $S = \{S_1, S_2, \ldots, S_s\}$, where S_i's are disjoint consecutive subsets of $\{1, 2, \ldots, m\}$ with all $|S_i| = 2$ or 3 such that $B_m(S, P)$ is connected and the S_i containing m (if any) has $|S_i| = 2$.*

Proof. We shall prove the lemma by induction on m. It is trivially true for $m \leq 4$. Assume $m \in P_i$ and $m-1 \in P_j$.

If $|P_i| \geq 3$, then $|P_i - \{m\}| \geq 2$. Let P' be obtained from P by deleting m from P_i. By the induction hypothesis, there exists S such that $B_{m-1}(S, P')$ is connected. Clearly, $B_m(S, P)$ is also connected.

Now, suppose that $|P_i| \leq 2$. If $i \neq j$, let P' be obtained from P by replacing P_i and P_j by $P'_i = P_i \cup P_j - \{m-1, m\}$. Note that P'_i is nonempty, since P_i or P_j is not a singleton. Also, P'_i is a singleton only when P_i or P_j is. Thus, P' has at most one singleton. By the induction hypothesis, there exists S' such that $B_{m-2}(S', P')$ is connected. Then $B_m(S' \cup \{\{m-1, m\}\}, P)$ is connected.

If $i = j$, i.e., $P_i = \{m-1, m\}$, let $P' = P - \{P_i\}$. By the induction hypothesis, $B_{m-2}(S', P')$ is connected for some S'. Set $S = S' \cup \{\{m-2, m-1\}\}$, if $m-2$

is not in any S_k. Otherwise, assume $m-2 \in S_k$ (then $|S_k| = 2$). Let S be obtained from S' by adding $m-1$ to S_k. Then $B_m(S, P)$ is connected. Q.E.D.

Theorem 1. *Suppose that* $\gcd(n, q) = 1$. *Then* $G(4, n, q, r)$ *is Hamiltonian.*

Proof. We first note that a consecutive-1 digraph $G(1, n, q, r')$ has a loop $i \to i$ (i.e., $i \equiv qi + r'$ (mod n)) if and only if $\gcd(n, q-1)$ divides r'. In the affirmative case, the number of loops is $\gcd(n, q-1)$, see [7].

If $\gcd(n, q-1) > 1$, then either $G(1, n, q, r+1)$ or $G(1, n, q, r+2)$ has no loop, as $\gcd(n, q-1)$ cannot divide both $r+1$ and $r+2$. If $\gcd(n, q-1) = 1$, then both $G(1, n, q, r+1)$ and $G(1, n, q, r+2)$ have exactly one loop. In either case, since $\gcd(n, q) = 1$, either $G(1, n, q, r+1)$ or $G(1, n, q, r+2)$ partitions the node-set into a set C of disjoint cycles with at most one singleton-cycle. By Lemma 2, there exists a legitimate interchange set S such that $B_n(S, C)$ is connected. The theorem then follows from Lemma 1. Q.E.D.

Note that the inductive proof of Lemma 2 implies a linear-time algorithm to construct S.

4. EXPLICIT CONSTRUCTION OF S

When $\gcd(n, q) = 1$ and 3 divides n, we can give an explicit construction of S that works for all n and q. Throughout this section, $S = \{\{3i-2, 3i-1, 3i\} : i = 1, 2, \ldots, n/3\}$.

Theorem 2. *If* $\gcd(n, q) = 1$ *and* 3 *divides* n, *then either* $B_n(S, C^{r+1})$ *or* $B_n(S, C^{r+2})$ *is connected.*

Proof. It is now easier to consider S as a set E of links (a subset of size 3 corresponds to two consecutive links). To show $B_n(S, C^{r+1})$ or $B_n(S, C^{r+2})$ is connected, it suffices to show that $E \cup G(1, n, q, r+1)$ or $E \cup G(1, n, r+2)$ is connected. We first consider $E \cup G(1, n, q, r+1)$. Note that for $i \to i+1$ in E, both $i \to qi + r + 1$ and $i + 1 \to q(i+1) + r + 1$ are in $G(1, n, q, r+1)$. Hence, $qi + r + 1$ and $qi + r + 1 + q$ are connected in $E \cup G(1, n, q, r+1)$. Let $E \cup Q$ be obtained from $E \cup G(1, n, q, r+1)$ by replacing the two links $i \to qi + r + 1$ and $i + 1 \to q(i+1) + r + 1$ with the q-link $qi + r + 1 \to qi + r + 1 + q$ for every i such that $i \to i+1$ is in E. Then $E \cup G(1, n, q, r+1)$ is connected if $E \cup Q$ is. We now explore the connectivity of $E \cup Q$.

Partition the nodes into $n/3$ groups, where group i consists of nodes $3i-2, 3i-1, 3i$. We will refer to them as the first, second, and third node of the group. We show that the groups are interconnected through the q-links. A q-link (i, j) will be called an (x, y) q-link if i is the x^{th} node of a group and j the y^{th} node of a group. Since $\gcd(n, q) = 1$ and 3 divides n, we have that 3 does not divide q. Therefore, each group has two q-links going out and two q-links going in. The $2n/3$ q-links contain two patterns of size $n/3$ each: one pattern corresponds to the (x, y) pattern of the q-link generated by the link (1, 2), the other by the link (2, 3). As 3 does not divide q, we have $x \not\equiv y$ (mod 3). So there are six permissible combinations for these two patterns: (i) (1, 2), (2, 3); (ii) (1, 3), (3, 2); (iii) (2, 3), (3, 1); (iv) (2, 1), (1, 3); (v) (3, 1), (1, 2); (vi) (3, 2), (2, 1). The two q-links (i, j) and (i', j') going out from a group have different patterns (x, y) and (x', y'). Note that $i - j = i' - j'$. Since i and i' are in the same group, j and j' are either in the same group or in consecutive groups. Furthermore, it is easily seen that j and j' are in the same group if and only if $(x - y)(x' - y') > 0$. Thus, for the middle four combinations, the two q-links from a group go to two consecutive groups. This implies that every pair of consecutive groups is connected; hence $E \cup Q$ is.

For the first and last pattern, the two q-links from a group go to the same group. So $E \cup Q$ is not connected. However, let $E \cup Q'$ be obtained from $E \cup G(1, n, q, r+2)$ by replacing the two links $i \to qi + r + 2$ and $i + 1 \to q(i + 1) + r + 2$ with the q-link $qi + r + 2 \to qi + r + 2 + q$ for every i such that $i \to i + 1$ is in E. Then the combination of the two patterns of q-links is (2, 3), (3, 1) for case (i), and (1, 3), (3, 2) for case (vi). In either case, the two q-links from a group go to two different groups. So, $E \cup Q'$, consequently, $E \cup G(1, n, q, r + 2)$ is connected. Q.E.D.

Unfortunately, $E = \{(3i - 2 \to 3i - 1) \cup (3i - 1 \to 3i) : i = 1, 2, \ldots, \lfloor n/3 \rfloor\}$ does not work when 3 does not divide n. A counterexample $G(4, 25, 13, 7)$ was given by Xuding Zhu (group 1 and group 5 are not connected in $G(1, 25, 13, 8)$ and group 4 and group 8 not connected in $G(1, 25, 13, 9)$).

5. CONCLUSIONS

It is known that consecutive-d digraph is Hamiltonian for $d \geq 5$, but not necessarily so for $d \leq 3$. In this article, we prove the conjecture that consecutive-4 digraphs are Hamiltonian, and thus completely settle the issue. Of course, our result for $d = 4$ implies that for $d \geq 5$. Our result also implies that there exist at least $\lfloor d/4 \rfloor$ disjoint Hamiltonian circuits for a consecutive-d digraph.

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