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INTERCONNECTING HIGHWAYS*

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Abstract. We present the problem of constructing roads of minimum total length to interconnect n highways under the constraint that the roads can intersect each highway only at one point in a designated interval which is a line segment. We present a set of optimality conditions for the problem and show how to construct a solution to meet this set of optimality conditions.

Key words. interconnecting networks, optimality conditions, Steiner trees

AMS subject classifications. 05C05, 90C27

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1. Introduction.

We present the problem of constructing roads of minimum total length to interconnect n existing highways H_1, H_2, ..., H_n under the constraint that the roads can intersect H_i only at one point, called an *exit* of H_i, in a designated interval I_i. To avoid unnecessary complexity, we assume that all I_i's are disjoint. In this paper, we consider the case where I_i is a line segment, including the two extreme cases where I_i is a point or a line. The case where I_i is a point for all $i = 1, 2, \dots, n$ is the Steiner minimum tree problem which is NP-hard [5]. Thus, the current problem is also NP-hard. Some special cases for $n = 3$ have been studied by Chen [3] and Weng [15]. More applications and the relation to facility allocation problems can be found in [3, 8, 18].

We will first establish a set of optimality conditions and then show how to construct a solution to meet this set of conditions by generalizing Melzak's construction for Steiner trees. Finally, we will use those results to determine global optimal solutions for $n = 2$ and $n = 3$.

2. Optimality conditions.

Let us call each intersection of roads, which is not an exit, a *Steiner point*. Consider an optimal solution for the problem of interconnecting highways. Clearly, this solution must be the Steiner minimum tree for the n exits at the current positions. Thus, it must have properties for Steiner minimum trees as stated in the following [6, 8].

LEMMA 2.1. *An optimal solution for the problem of interconnecting highways must satisfy the following conditions:*

(a) *Every Steiner point has degree three (Figure* 2.1(a)*).*

(b) *Two roads meeting at a point form an angle of at least* $120°$ *(Figure* 2.1(b)*).*

Since each exit can move in the designated interval I_i, we have additional optimality conditions at exits.

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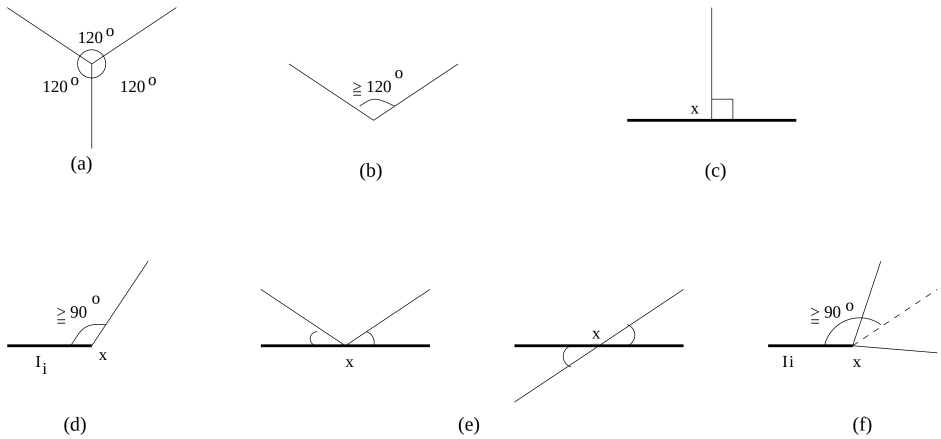

FIG. 2.1. *Optimality conditions.*

LEMMA 2.2. *Let x be an exit in interval I_i. An optimal solution for the problem of interconnecting highways must satisfy the following conditions:*

(c) *If exit x is connected to only one road and x is an interior point of I_i, then this road is perpendicular to I_i (Figure* 2.1(c)*).*

(d) *If exit x is connected to only one road and x is an endpoint of I_i, then this road together with I_i forms an angle of at least* $90°$ *(Figure* 2.1(d)*).*

(e) *If exit x is connected to exactly two roads and x is an interior point of I_i, then the angle formed by one road and a part of I_i equals the angle formed by the other road and the other part of I_i (Figure* 2.1(e)*).*

(f) *If exit x is connected to exactly two roads and x is an endpoint of I_i, then the bisector of the relative angle of the two roads and I_i form an angle of at least* $90°$ *(Figure* 2.1(f)*).*

Proof. Suppose $I_i = [A, B]$.

(c) Suppose exit x is connected to only one road (x, C). Since x is an interior point of $[A, B]$, if xC is not perpendicular to $[A, B]$, then either $\angle AxC < 90°$ or $\angle BxC < 90°$. In the former case, moving x in direction xA would decrease distance xC (Figure 2.2). In the latter case, moving x in direction xB would decrease xC. Thus, x is not at an optimal position.

(d) A similar argument to (c) could apply (Figure 2.2).

(e) Suppose that exit x is connected to two roads (x, C) and (x, D). If C and D lie on different sides of line AB, then C, x, and D must be on the same line. Otherwise, a little perturbation would make a shorter solution (Figure 2.3). If C and D are on the same side of line AB, then find the point C' which is symmetric to C with respect to line AB. Then, C', x, and D must be on the same line. Otherwise, a little perturbation would make a shorter solution (Figure 2.3). In both case, we have $\angle AxC = \angle BxD$.

(f) An argument similar to (e) could apply. ∎

We call a tree satisfying conditions (a)–(f) a *legitimate tree*. A legitimate tree is *full* if every exit is a leaf.

THEOREM 2.3. *The optimal solution of the highway interconnection problem must be a legitimate tree.*

Since a legitimate tree is a Steiner tree for a current position of exits, we may

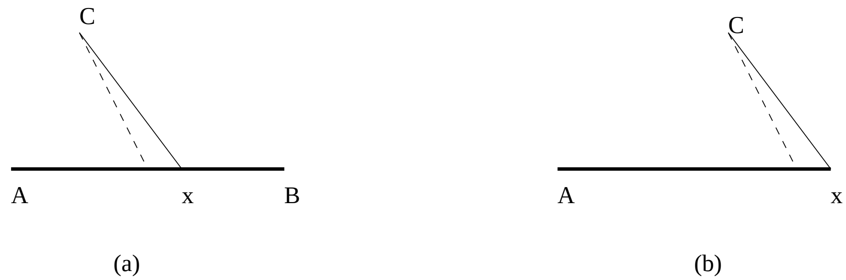

FIG. 2.2. *Moving x in direction xA decreases xC.*

FIG. 2.3. *A little perturbation would shorten the tree.*

extend some concepts about Steiner trees to legitimate trees as follows: A *topology* is the graph structure of a legitimate tree. The topology is full if the legitimate tree is full. A topology can be degenerated by shrinking an edge between a Steiner point and an exit. A topology is called a *degenerate* one of another topology if the former can be obtained from the latter by a sequence of degenerate operations.

THEOREM 2.4. *Among a full topology and its degenerate topologies, if there exists one with which the legitimate tree exists, then it is minimum among all trees with the full topology and its degenerate ones.*

Proof. Consider the problem of finding the shortest one among all trees under a full topology (including its degenerate topologies) interconnecting n exits each in a designated interval. This problem has a convex objective function with respect to coordinates of Steiner points and exits, which is a sum of Euclidean distances, and linear constraints on coordinates of exits. Therefore, it is a convex programming. Any local optimal solution is also a global optimal solution. In the following, we will show that if a legitimate tree with the full topology or its degenerate one exists, then it is a local minimum for the convex programming. To do so, we show that at the legitimate tree, every feasible direction is not descending. That is, the directional derivative of the objective function is nonnegative.

Let t be the full topology and $E(t)$ the edge set of t. Then, the objective function of the convex programming is

$$f(V(t)) = \sum_{(u,v)\in E(t)} \|u - v\|,$$

where $V(t)$ is the vertex set of t and all coordinates of vertices are variables of function f. A feasible direction ΔV of $V(t)$ consists of moving direction Δv for every vertex $v \in V(t)$. For a Steiner point, every direction can be feasible. For an exit, only direction along the exit's interval can be feasible. Suppose the feasible direction ΔV of $V(t)$ has unit length. Then, its directional derivative is

$$\lim_{\lambda \to 0} \frac{f(V(t) + \lambda \Delta V) - f(V(t))}{\lambda} = \sum_{(u,v) \in E(t)} \lim_{\lambda \to 0} \frac{\|(u + \lambda \Delta u) - (v + \lambda \Delta v)\| - \|u - v\|}{\lambda}.$$

We will first calculate this derivative and then show that it is nonnegative.

For $u \neq v$,

$$\lim_{\lambda \to 0} \frac{\|(u + \lambda \Delta u) - (v + \lambda \Delta v)\| - \|u - v\|}{\lambda} = \frac{(\Delta u - \Delta v)^T (u - v)}{\|u - v\|} = \frac{(\Delta u)^T (u - v) + (\Delta v)^T (v - u)}{\|u - v\|}.$$

For $u = v$,

$$\lim_{\lambda \to 0} \frac{\|(u + \lambda \Delta u) - (v - \lambda \Delta v)\| - \|u - v\|}{\lambda} = \lim_{\lambda \to 0} \|\Delta u - \Delta v\| = \|\Delta u - \Delta v\|.$$

Now, suppose $V(t)$ is the vertex set of a legitimate tree. If u is a Steiner point and its three edges in the legitimate tree are all of nonzero length, then there are three terms involving Δu in the directional derivative,

$$\frac{(\Delta u)^T (u - v_1)}{\|u - v_1\|} + \frac{(\Delta u)^T (u - v_2)}{\|u - v_2\|} + \frac{(\Delta u)^T (u - v_3)}{\|u - v_3\|} = 0,$$

since any two of three vectors $u - v_1$, $u - v_2$, and $u - v_3$ form an angle of $120°$.

If u is an exit and its only edge in the legitimate tree has nonzero length, then Δu is involved in only one term of the directional derivative,

$$\frac{(\Delta u)^T (u - v)}{\|u - v\|} \geq 0,$$

since Δu and $v - u$ form an angle of at least $90°$, that is, $(\Delta u)^T (v - u) \leq 0$.

Note that if degeneration occurs, it must occur around an exit v. If in the legitimate tree v is incident to two edges, this means that the edge (v, u) in t, where u is a Steiner point, has been shrunk to a point. If in the legitimate tree v is incident to three edges, then two edges (v, u) and (u, w), where u and w are Steiner points, have been shrunk to one point in the legitimate tree.

In the former case, i.e., edge (u, v), where u is a Steiner point and v is an exit in t, having length 0 in the legitimate tree, then the directional derivative has three terms involving Δu and Δv as follows:

$$\|\Delta u - \Delta v\| + \frac{(\Delta u)^T (u - v_2)}{\|u - v_2\|} + \frac{(\Delta u)^T (u - v_3)}{\|u - v_3\|}.$$

Denote

$$w = \frac{v_2 - u}{\|u - v_2\|} + \frac{v_3 - u}{\|u - v_3\|}.$$

Since $v_2 - u$ and $v_3 - u$ form an angle of at least 120°, we have

$$\|w\| \leq 1.$$

In addition, since w and Δv form an angle of at least 90°, we have

$$(\Delta v)^T w \leq 0.$$

Therefore,

$$\begin{aligned}&\|\Delta u - \Delta v\| - (\Delta u)^T w \\ &\quad \geq \|\Delta u - \Delta v\| + (\Delta v - \Delta u)^T w \\ &\quad \geq 0.\end{aligned}$$

Finally, we consider the case where, in the legitimate tree, v is incident to three edges, i.e., two edges (v, u) and (u, w) in t, where u and w are Steiner points, have been shrunk to one point in the legitimate tree. In this case, the directional derivative has five terms involving Δu, Δv, and Δw as follows:

$$\|\Delta v - \Delta u\| + \|\Delta u - \Delta w\| + \frac{(\Delta u)^T(u - v_1)}{\|u - v_1\|} + \frac{(\Delta w)^T(w - v_2)}{\|w - v_2\|} + \frac{(\Delta w)^T(w - v_3)}{\|w - v_3\|}.$$

Denote this summation by s. Note that in the legitimate tree, u and w are identical and any two of three vectors $u - v_1$, $w - v_2$, and $w - v_3$ form an angle of 120°. Thus,

$$\frac{(u - v_1)}{\|u - v_1\|} + \frac{(w - v_2)}{\|w - v_2\|} + \frac{(w - v_3)}{\|w - v_3\|} = 0.$$

It follows that

$$s = \|\Delta v - \Delta u\| + \|\Delta u - \Delta w\| + \frac{(\Delta u - \Delta w)^T(u - v_1)}{\|u - v_1\|} \geq 0.$$

By summarizing the above, we know that the directional derivative is nonnegative. This completes our proof. □

3. Generalized Melzak construction. In this section, we study the following question: If a legitimate tree exists, how do we construct it?

First, we show how to construct a legitimate tree with full topology if it exists. Let us start by recalling Melzak's construction [11].

Melzak's construction works for a Steiner tree with a full topology. In each step, it first finds a Steiner point adjacent to two exits (they are fixed in a Steiner tree problem). Then, it constructs an equilateral triangle with the two exits as its two vertices and replaces them by the third vertex (Figure 3.1), considered a new exit. After several steps, when only two exits exist, it connects them by a straight line segment and in reverse ordering, then finds all edges of the full Steiner tree.

Now, we also want to replace two exits (adjacent to the same Steiner point) by a new exit. However, a new situation is that each exit has a feasible region. (For each original exit, its feasible region is a line segment.) Thus, we also need to construct

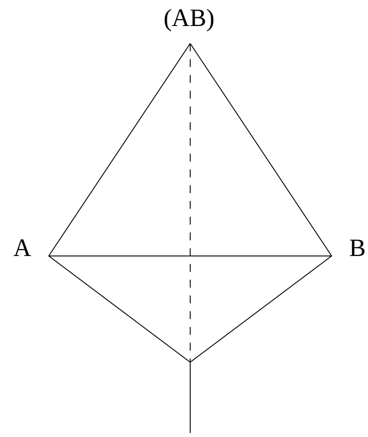

FIG. 3.1. *Melzak's construction.*

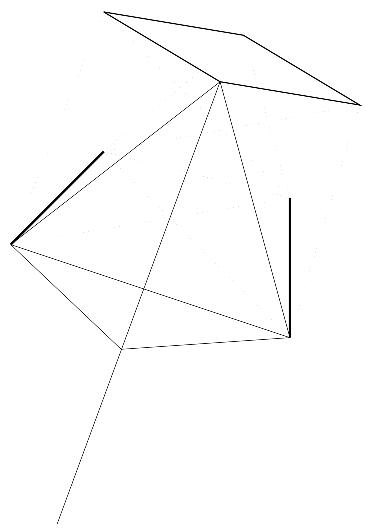

FIG. 3.2. *The feasible region of this new exit is a parallelogram.*

a feasible region for the new exit. Initially, a new exit is obtained from two original exits and the feasible region of this new exit is a parallelogram, as shown in Figure 3.2. In general, what is the feasible region for a new exit if it is obtained from k original exits through $k-1$ steps of Melzak's construction? An answer is given in the following.

Let us call a convex central symmetric $2k$-polygon a *parallel* $2k$-polygon if its $2k$ edges can be divided into k pairs of parallel edges with equal length (Figure 3.3). Note that every parallel $2k$-polygon can be covered in the following way: Choose an edge. Moving this edge along an adjacent edge will obtain a parallel 4-polygon (or a parallelogram). Moving the parallel 4-polygon along an adjacent edge will obtain a parallel 6-polygon. Continuing in this way, finally the parallel $2k$-polygon will be obtained (Figure 3.3) and all points in this parallel $2k$-polygon are covered by moving images.

THEOREM 3.1. *Let v be a new exit obtained from k original exits through $k-1$*

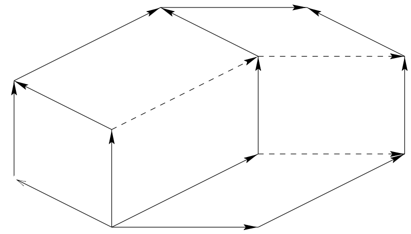

FIG. 3.3. *Parallel $2k$-polygon.*

steps of Melzak's construction. The feasible region of v is a parallel $2k$-polygon.

Proof. We prove it by induction on k. Suppose v is obtained from two exits u and w. Suppose u is obtained from i original exits and w is obtained from j original exits. Clearly, $i + j = k$ and $i < k$ and $j < k$. By the induction assumption, the feasible region of u is a parallel $2i$-polygon P and the feasible region of w is a parallel $2j$-polygon Q.

First, we fix u at a position in P and move w over region Q. It is easy to see that v will describe a region Q' isomorphic to Q. Actually, this region Q' can be obtained from Q by turning Q around center u in an angle of 60°.

Next, we move u along an edge e_1 of P. As u moves, Q' will move along a certain direction and all images will cover a parallel $2(j + 1)$-polygon Q''.

Now, we move edge e_1 along an adjacent edge e_2 of P. As all images of e_1 cover a parallel 4-polygon P', all images of Q'' cover a parallel $2(j + 2)$-polygon Q'''. Continue in this way. As all points in the parallel $2i$-polygon P are covered, a parallel $2(i + j)$-polygon will be covered by images of Q' (Figure 3.4). ∎

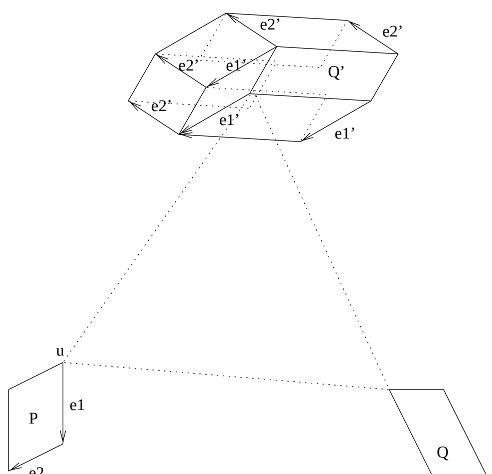

FIG. 3.4. *The proof of Theorem* 3.1.

For degenerate topology, Melzak's method for a Steiner tree is to decompose it into an edge-disjoint union of small full topologies. However, for the problem of interconnecting highways, such a decomposition does not exist since the position of an exit v connected to two roads (v, u) and (v, w) has to be determined by considering both roads. If the feasible region of u (or w) is known, then we may need to consider

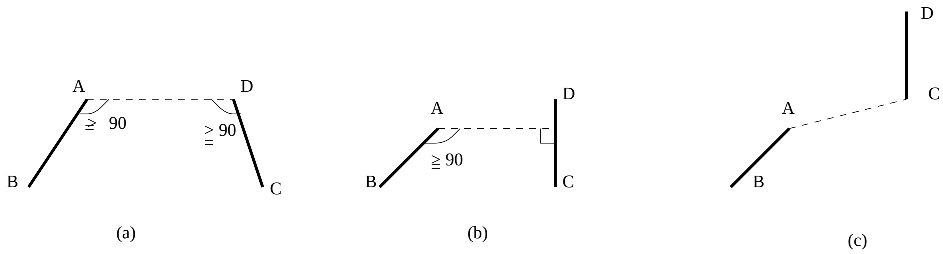

FIG. 4.1. *Two highways.*

two cases: v is in the interior of the designated interval I_i or v is at one of two endpoints of I_i. In the former case, we can replace I_i by one of its endpoints and then decompose the topology at v. In the latter case, we replace (v, u) and (v, w) by (w, u) and meanwhile replace the feasible region of u by its symmetric image with respect to I_i if u and w are in the same side of I_i.

From the above analysis, one may see that constructing the legitimate tree with a degenerate topology is much more complicated than the Steiner tree in a similar situation. It is indeed not a construction which can be finished in polynomial time with respect to the number of exits. However, Xue, Du, and Hwang [18] showed that there exists a fast way to construct a tree with length almost as short as the legitimate tree. This work draws from many previous contributions on shortest network under a given topology [7, 9, 13, 14, 16, 17].

4. Two or three highways. If our interest is only on highway interconnection, then $n = 2$ and $n = 3$ are the most practical cases. In these two cases, there is a unique full topology. Thus, a tree being legitimate is necessary and sufficient for optimality. In this section, we will apply the results from previous sections to determine the legitimate tree in these two cases.

For $n = 2$, suppose AB and CD are two line segments. Assume that BD and DA do not intersect, that is, $ABCD$ form a quadrilateral. Since $\angle A + \angle B + \angle C + \angle D = 360°$, either $\angle A + \angle D \geq 180°$ or $\angle B + \angle C \geq 180°$. Without loss of generality, assume the former occurs and $\angle A \geq \angle D$. Then $\angle A \geq 90°$. Now, we have three cases.

Case 1. $\angle D \geq 90°$. In this case, the line segment AD is the legitimate tree interconnecting AB and CD (Figure 4.1(a)).

Case 2. $\angle D < 90°$ and $\angle ACD < 90°$. In this case, draw line segment AE perpendicular to CD at E. Then AE is the legitimate tree (Figure 4.1(b)).

Case 3. $\angle D < 90°$ and $\angle ACD \geq 90°$. In this case, AC is the legitimate tree (Figure 4.1(c)).

For $n = 3$, there are four possible topologies for the legitimate tree. We first construct the one with full topology (Figure 4.2). If successful, then the work is done. If unsuccessful, then we construct the other three topologies in turn until a legitimate tree is found (Figure 4.3).

To construct the one with full topology, first replace two line segments I_1 and I_2 by a parallelogram with Melzak's construction. Then, find the shortest distance between the parallelogram and the third line segment I_3. Suppose this happens between point A in the parallelogram and point B in I_3. Note that A corresponds to two points C and D in I_1 and I_2, respectively. Draw the full Steiner tree for B, C, D. If it exists, the legitimate tree with the full topology is found; if not, then it means that the legitimate tree with the full topology does not exist.

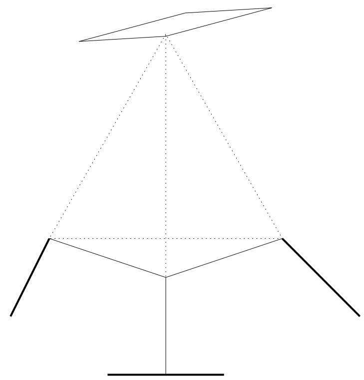

FIG. 4.2. *Three highways* (1).

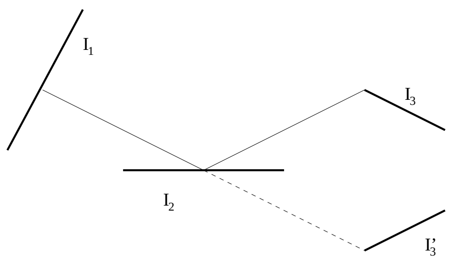

FIG. 4.3. *Three highways* (2).

To explain how to construct a legitimate tree with a degenerate topology, we may assume the topology consists of two edges between I_1 and I_2 and I_2 and I_3, respectively. Suppose I_3' is the mirror image of I_3 with respect to I_2. Find the shortest distance between I_1 and I_3 and the shortest distance between I_1 and I_3'. If a segment realizing either one of the two shortest distances intersects segment I_2, then the legitimate tree is found. If no such segment exists, then consider two endpoints of I_2. For each endpoint, find the shortest segments to connect it to I_1 and I_3, respectively. Check whether the two shortest segments form a legitimate tree. If no legitimate tree is found in this way, then it means that the legitimate tree with this degenerate topology does not exists and we should consider another degenerate topology. By Theorem 2.3, we would find a legitimate tree before all possible topologies are examined.

5. Discussion. A variation of the problem considered in this paper is to use a spanning tree instead of a Steiner tree, interconnecting points with each on a specified line segment, and to find the shortest one. So far, we do not know if such a spanning tree problem has a polynomial time solution. Therefore, approximation for the highway interconnection problem is also an open problem. No polynomial time approximation with constant performance ratio for this problem has been found, although many polynomial time approximations with good performance for Steiner minimum trees have been known [1, 2, 10].

When a new Steiner tree problem appears, one usually also considers the corresponding Steiner ratio problem. Note that a Steiner minimum tree interconnecting n line segments is also the Steiner minimum tree connecting the n exits. Let us fix these

n exits. Then, a minimum spanning tree for n line segments is not longer that the minimum spanning tree for the fixed n exits. Therefore, the ratio of lengths between the Steiner minimum tree and the minimum spanning tree for n line segments is at least $\sqrt{3}/2$. It follows that the Steiner ratio for the highway interconnection problem is the same as that for the Euclidean Steiner tree problem [4, 12].

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