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Distance Graphs and T-Coloring

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We discuss relationships among T-colorings of graphs and chromatic numbers, fractional chromatic numbers, and circular chromatic numbers of distance graphs. We first prove that for any finite integral set T that contains 0, the asymptotic T-coloring ratio $R(T)$ is equal to the fractional chromatic number of the distance graph $G(Z, D)$, where $D = T - \{0\}$. This fact is then used to study the distance graphs with distance sets of the form $D_{m,k} = \{1, 2, ..., m\} - \{k\}$. The chromatic numbers and the fractional chromatic numbers of $G(Z, D_{m,k})$ are determined for all values of m and k. Furthermore, circular chromatic numbers of $G(Z, D_{m,k})$ for some special values of m and k are obtained.

1. INTRODUCTION

The T-coloring problem was formulated by Hale [18] as a model for the channel assignment problem, in which an integer broadcast channel is

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assigned to each of several locations so that interference among nearby locations is avoided. Interference is modeled by a non-negative integral set T containing 0 (called a *T-set*) as forbidden channel separations. One can construct a graph $G=(V, E)$ such that each vertex represents a location; two vertices are adjacent if their corresponding locations are nearby. Thereafter a valid channel assignment or *T-coloring* is a mapping f from the vertex set of V of G to the set of non-negative integers $\{0, 1, 2, ...\}$ such that $|f(x)-f(y)| \notin T$ whenever $xy \in E$. The *span* of a T-coloring f is the difference between the largest and the smallest numbers in $f(V)$, i.e., $\max\{|f(u)-f(v)|: u, v \in V\}$. Given T and G, the *T-span* of G, denoted by $\mathrm{sp}_T(G)$, is the minimum span among all T-colorings of G.

T-coloring has been extensively studied in the literature (see [4, 5, 16, 23–26, 30–32, 35]). Let σ_n denote $\mathrm{sp}_T(K_n)$, where K_n is a complete graph with n vertices. Griggs and Liu [16] proved that the *difference optimum sequence*, $\Delta\sigma=(\sigma_{n+1}-\sigma_n)_{n=1}^{\infty}$, is eventually periodic. This implies that for any T-set, the *asymptotic T-coloring ratio*

$$R(T) := \lim_{n \to \infty} \frac{\sigma_n}{n}$$

exists and is a rational number. This result was also proven by Rabinowitz and Proulx [30] and Cantor and Gordon [1] by different approaches.

The notion of distance graphs originated with the plane-coloring problem: What is the smallest number of colors needed to color all points of a Euclidean plane such that points at unit distances are colored with different colors. It is well known that four colors are necessary [28] and seven colors are sufficient [17]. However, the exact number of colors needed remains unknown (see [6]). Motivated by this problem, Eggleton [10] made the following generalization. Suppose S is a subset of a metric space $\mathcal{M}$ with metic d, and D is a set of positive real numbers. The *distance graph* $G(S, D)$ with *distance set* D is the graph with vertex set S and edge set $\{xy: d(x, y) \in D\}$. The objective is to determine $\chi(S, D)$, the chromatic number of $G(S, D)$. Note that the plane-coloring problem introduced above is equivalent to finding $\chi(R^2, \{1\})$.

Let D be a set of positive integers (called a D-set). The distance graph to be studied in this article is $G(Z, D)$, which has Z as the vertex set and $\{uv: |u-v| \in D\}$ as the edge set. The problem of finding $\chi(Z, D)$ for different D-sets has been studied extensively (see [2, 7, 9, 11–14, 21, 37–41]).

A *fractional coloring* of a graph G is a mapping c from $\mathcal{I}(G)$, the set of all independent sets of G, to the interval $[0, 1]$ such that $\sum_{x \in I \in \mathcal{I}(G)} c(I) \geqslant 1$ for all vertices x in G. The *fractional chromatic number* $\chi_f(G)$ of G is the infimum of the value $\sum_{I \in \mathcal{I}(G)} c(I)$ of a fractional coloring c of G ([15, 22, 33, 34]).

For a given T-set, letting $D = T - \{0\}$, Liu [26] proved that the asymptotic T-coloring ratio $R(T)$ is a lower bound of $\chi(Z, D)$. Hence, T-colorings and distance graphs are closely related. We shall explore further relationships between these two concepts. In Section 2, we prove that for any T-set, $R(T)$ is equal to the fractional chromatic number of the distance graph $G(Z, D)$, if $D = T - \{0\}$. This relationship provides new insights concerning the parameter $R(T)$, and can be used to simplify the proofs of some known results regarding $R(T)$.

Section 3 focuses on the family of distance graphs with D-sets of the form $D_{m,k} = \{1, 2, ..., m\} - \{k\}$. The chromatic numbers of such distance graphs, denoted as $\chi(Z, D_{m,k})$, have been investigated in the following articles. In [11], Eggleton, Erdős and Skilton obtained the solution for $k=1$, and partial solutions for $k=2$: $\chi(Z, D_{m,1}) = \lfloor (m+3)/2 \rfloor$ for any $m \geqslant 2$, $\chi(Z, D_{m,2}) = \lfloor (m+4)/2 \rfloor$ when $m \not\equiv 3 \pmod 4$, and $\lfloor (m+3)/2 \rfloor \leqslant \chi(Z, D_{m,2}) \leqslant \lfloor (m+5)/2 \rfloor$ for any $m \geqslant 4$ with $m \equiv 3 \pmod 4$. For $3 \leqslant k < m$, the same authors provided the bounds

$$\max\left\{k, \left\lfloor \frac{1}{2}\left(\frac{m}{k-1}+1\right)\right\rfloor t\right\} \leqslant \chi(Z, D_{m,k}) \leqslant \min\left\{m, \left\lfloor \frac{1}{2}\left(\frac{m}{k}+3\right)\right\rfloor k\right\},$$

where $t=2$ if $k=3$, and $t=k-2$ if $k \geqslant 4$. The same result for the case $k=1$ was also proven by Kemnitz and Kolberg in [21] by a different approach. The lower bound of $\chi(Z, D_{m,k})$ in the above has been improved to $\lceil (m+k+1)/2 \rceil$ by Liu ([26]), who also showed that the new bound is sharp for all pairs of integers (m, k) where k is odd. Furthermore, complete solutions for $k=2$ and 4, and partial solutions for other even integers k are given in [26].

The main results of this paper are complete solutions for the chromatic numbers and the fractional chromatic numbers of the distance graphs $G(Z, D_{m,k})$ for *all* values m and k. These results are also applied to study the circular chromatic number of distance graphs.

Suppose k and d are positive integers such that $k \geqslant 2d$. A *(k, d)-coloring* of a graph $G = (V, E)$ is a mapping c from V to $\{0, 1, ..., k-1\}$ such that $\|c(x) - c(y)\|_k \geqslant d$ for any edge xy in G, where $\|a\|_k = \min\{a, k-a\}$. The *circular chromatic number* $\chi_c(G)$ of G is the infimum of k/d for all (k, d)-colorings of G. The circular chromatic number is also known as the *star-chromatic number* in the literature (see [36, 42, 43]).

For any graph G, it is well known that

$$\max\left\{\omega(G), \frac{|V(G)|}{\alpha(G)}\right\} \leqslant \chi_f(G) \leqslant \chi_c(G) \leqslant \lceil \chi_c(G) \rceil = \chi(G). \qquad (*)$$

The parameters involved in $(*)$ for distance graphs are explored in this paper. For simplicity, let $\omega(Z, D)$, $\alpha(Z, D)$, $\chi_f(Z, D)$, and $\chi_c(Z, D)$ denote the clique number, the independence number, the fractional chromatic number, and the circular chromatic number of $G(Z, D)$, respectively.

2. RELATIONSHIPS BETWEEN $R(T)$ AND $\chi_f(Z, D)$

This section shows that for any T-set, the asymptotic T-coloring ratio $R(T)$ is equal to the fractional chromatic number of the distance graph $G(Z, D)$, if $D = T - \{0\}$. Based upon this result, we give simpler and different proofs of some known facts regarding $R(T)$.

THEOREM 1. *For any finite T-set, if* $D = T - \{0\}$, *then* $R(T) = \chi_f(Z, D)$.

Proof. Suppose c is an optimal T-coloring of K_n, where $0 = c(1) < c(2) < \cdots < c(n) = \sigma_n$. Let $m = 1 + \max_{d \in D} d$. For $1 \leqslant i \leqslant c(n) + m$, let

$$I_i = \{ j \in Z : j \equiv i + c(k) \ (\mathrm{mod}\ c(n) + m) \text{ for some } k \text{ with } 1 \leqslant k \leqslant n \}.$$

It is straightforward to verify that each I_i is an independent set in $G(Z, D)$, and that every integer belongs to exactly n of the independent sets I_i $(1 \leqslant i \leqslant c(n) + m)$. Define a mapping c': $\mathscr{I}(G(Z, D)) \to [0, 1]$ as

$$c'(I) = \begin{cases} \dfrac{1}{n}, & \text{if} \quad I = I_i \quad \text{for } 1 \leqslant i \leqslant c(n) + m; \\ 0, & \text{otherwise.} \end{cases}$$

Then c' is a fractional coloring of $G(Z, D)$. This implies that for any positive integer n, $\chi_f(Z, D) \leqslant (c(n) + m)/n$. Hence, we have

$$\chi_f(Z, D) \leqslant \lim_{n \to \infty} \frac{c(n) + m}{n} = \lim_{n \to \infty} \frac{\sigma_n}{n} = R(T).$$

To show $\chi_f(Z, D) \geqslant R(T)$, let G_n be the subgraph of $G(Z, D)$ induced by the vertex set $\{0, 1, 2, ..., c(n)\}$; i.e., $G_n = G(\{0, 1, 2, ..., c(n)\}, D)$. Then the set of vertices $\{c(1), c(2), ..., c(n)\}$ is a maximum independent set in G_n. So, for any positive integer n, we have

$$\chi_f(Z, D) \geqslant \chi_f(G_n) \geqslant \frac{|V(G_n)|}{\alpha(G_n)} = \frac{c(n) + 1}{n}.$$

This implies

$$\chi_f(Z, D) \geqslant \lim_{n \to \infty} \frac{c(n)+1}{n} = \lim_{n \to \infty} \frac{\sigma_n}{n} = R(T).$$

Therefore, $R(T) = \chi_f(Z, D)$. Q.E.D

The theorem above provides new insights into the asymptotic T-coloring ratio $R(T)$. Some previous results concerning $R(T)$ can be obtained from this approach. For example, it is well known that $R(T) \geqslant 2$, provided $T \neq \{0\}$ (see [1, 16, 30]). This is straightforward when we consider fractional chromatic numbers, since the fractional chromatic number of any non-trivial graph is at least 2. Moreover, for a non-trivial graph G, $\chi_f(G) = 2$ if and only if G is bipartite. Since $G(Z, D)$ is bipartite if and only if D contains no even integers (assuming that $\gcd(T) = 1$), we have the following result.

COROLLARY 2. *For any T-set with* $\gcd(T) = 1$, $R(T) = 2$ *if and only if T contains only odd integers except* 0.

Theorem 1 can also be applied to some other known results about $R(T)$ which are closely related to an earlier number theory problem, namely, sequences with missing differences. Given a T-set, a *T-sequence* is an increasing sequence S of nonnegative integers such that $x - y \notin T$ for any $x, y \in S$. Motzkin [29] proposed studying the supremum $\mu(T)$ of the asymptotic upper densities of these sequences S. Cantor and Gordon [1] determined the exact values of $\mu(T)$ when $|T| = 2$ and 3. Haralambis [19] gave partial solutions when $|T| = 4$ or 5. It is known that $\mu(T)$ is equal to the reciprocal of $R(T)$ (see [16]). Therefore results on sequences with missing differences can be applied to T-colorings. Cantor and Gordon [1] and Rabinowitz and Proulx [30] proved that if $T = \{0, a, b\}$, $\gcd(a, b) = 1$, and a and b are of different parity, then $R(T) = 2(a+b)/(a+b-1)$. The original argument in proving the inequality $R(T) \geqslant 2(a+b)/(a+b-1)$ in [1] was quite complicated. However, applying some facts about fractional chromatic number, one can obtain the following simpler proof. Since a and b are of opposite parity, $G(Z, D)$ with $D = \{a, b\}$ contains an odd cycle C_{a+b}. As $\chi_f(C_{2m+1}) = 2 + 1/m$, it follows that $\chi_f(Z, D) \geqslant \chi_f(C_{a+b}) = 2(a+b)/(a+b-1)$. Thus, by Theorem 1, $R(T) \geqslant 2(a+b)/(a+b-1)$. Indeed, combining Theorem 1 with the fact that $\chi_f(H) \leqslant \chi_f(G)$ if H is a subgraph of G, the following is obvious.

COROLLARY 3. *If H is a subgraph of the distance graph* $G(Z, D)$, *then* $\chi_f(H) \leqslant R(T)$, *where* $T = D \cup \{0\}$.

3. VALUES OF $\chi_f(Z, D_{m,k})$, $\chi(Z, D_{m,k})$, AND $\chi_c(Z, D_{m,k})$

In this section, we first calculate $\chi_f(Z, D_{m,k})$, which, according to (∗), is a lower bound of $\chi(Z, D_{m,k})$. We then determine $\chi(Z, D_{m,k})$ for all values of m and k. Using this approach, circular chromatic numbers of $G(Z, D_{m,k})$ for special values of m and k are obtained as well.

The values of $R(T)$ as $T = D_{m,k} \cup \{0\}$ are given in [26]. Thus, the following two results can also be obtained by using Theorem 1. Here we include methods of calculating $\chi_f(Z, D_{m,k})$ directly.

THEOREM 4. *If* $2k > m$, *then*

$$\omega(Z, D_{m,k}) = \chi_f(Z, D_{m,k}) = \chi_c(Z, D_{m,k}) = \chi(Z, D_{m,k}) = k.$$

Proof. Since the set of vertices $\{1, 2, ..., k\}$ forms a clique in $G(Z, D_{m,k})$, $k \leqslant \omega(Z, D_{m,k})$.

By (∗), it is sufficient to show $\chi(Z, D_{m,k}) \leqslant k$. Define a vertex-coloring f on Z as $f(i) = (i \bmod k)$. Then f is a proper coloring, since $D_{m,k}$ contains no multiple of k. Q.E.D

THEOREM 5. *If* $2k \leqslant m$, *then* $\chi_f(Z, D_{m,k}) = (m+k+1)/2$.

Proof. For any i with $0 \leqslant i \leqslant m+k$, $I_i = \{j \in Z : j - i \equiv 0 \text{ or } k \pmod{m+k+1}\}$ is an independent set in $G(Z, D_{m,k})$. Furthermore, each integer is contained in exactly two such independent sets. Define a mapping $c \colon \mathscr{I}(G(Z, D_{m,k})) \to [0, 1]$ as

$$c(I) = \begin{cases} \frac{1}{2}, & \text{if } \ I = I_i \text{ for } 0 \leqslant i \leqslant m+k; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that c is a fractional coloring of $G(Z, D_{m,k})$. Thus, $\chi_f(Z, D_{m,k}) \leqslant (m+k+1)/2$.

Since $\alpha(\{0, 1, ..., m+k\}, D_{m,k}) = 2$ for $2k \leqslant m$, by (∗), $\chi_f(\{0, 1, ..., m+k\}, D_{m,k}) \geqslant (m+k+1)/2$. Therefore, the proof is complete. Q.E.D

Because $\chi(G)$ is an integer, Theorem 5 and (∗) imply the following:

COROLLARY 6 [26]. *If* $2k \leqslant m$ *then* $\chi(Z, D_{m,k}) \geqslant \lceil (m+k+1)/2 \rceil$.

We are now in a position to given the complete solutions to $\chi(Z, D_{m,k})$ for all values of m and k. This is accomplished in the next two results. As will be shown, $\chi(Z, D_{m,k})$ is either $\lceil (m+k+1)/2 \rceil$ or $\lceil (m+k+1)/2 \rceil + 1$.

LEMMA 7. *Suppose* $2k \leqslant m$. *Write* $m+k+1=2^r m'$ *and* $k=2^s k'$, *where* r *and* s *are non-negative integers and* m' *and* k' *are odd integers. If* $1 \leqslant r \leqslant s$, *then* $\chi(Z, D_{m,k}) > (m+k+1)/2$.

Proof. Since $1 \leqslant r$, $m+k+1$ is even. Assume to the contrary that $\chi(Z, D_{m,k}) \leqslant (m+k+1)/2$. By Corollary 6, $\chi(Z, D_{m,k}) = (m+k+1)/2$. Color $G(Z, D_{m,k})$ by using $(m+k+1)/2$ colors.

For each integer i, consider the subgraph of $G(Z, D_{m,k})$ induced by the $m+k+1$ vertices $\{i, i+1, ..., i+m+k\}$. This subgraph has independence number 2. Hence, each of $(m+k+1)/2$ colors is used at most, and hence exactly, twice in this subgraph. Thus, each color is used exactly twice in any consecutive $m+k+1$ vertices. Consequently, vertices i and $i+m+k+1$ have the same colors for all $i \in Z$. Therefore, for each $i \in S := \{0, 1, ..., m+k\}$, the only possible vertices in S having the same color as i are $i+k$ and $i-k$ $(\text{mod}\ m+k+1)$.

Consider the *circulant graph* $C(m+k+1, k)$, with vertex set S and in which vertex i is adjacent to vertex j if and only if $j \equiv i+k$ or $i-k$ $(\text{mod}\ m+k+1)$. It follows from the discussion in the preceding paragraph that two vertices x and y of S have the same color only if xy is an edge of the circulant graph $C(m+k+1, k)$. Since the intersection of each color class with S contains exactly two vertices, the coloring induces a perfect matching of $C(m+k+1, k)$. However, $C(m+k+1, k)$ is the disjoint union of d cycles of length $(m+k+1)/d$, where $d = \gcd(m+k+1, k)$. Since $C(m+k+1, k)$ has a perfect matching, each cycle has an even length. This implies that $r > s$, contrary to the assumption $r \leqslant s$. Q.E.D

The next theorem determines the chromatic number $\chi(Z, D_{m,k})$ for all values of m and k. Incidentally, it also shows that the converse of Lemma 7 is true.

THEOREM 8. *Suppose* $2k \leqslant m$. *Write* $m+k+1=2^r m'$ *and* $k=2^s k'$, *where* r *and* s *are non-negative integers and* m' *and* k' *are odd integers. Then*

$$\chi(Z, D_{m,k}) = \begin{cases} \dfrac{m+k+1}{2}, & \textit{if} \quad r > s; \\ \left\lfloor \dfrac{m+k+2}{2} \right\rfloor, & \textit{otherwise}. \end{cases}$$

Proof. It follows from Corollary 6 and Lemma 7 that if $r > s$, then $\chi(Z, D_{m,k}) \geqslant (m+k+1)/2$; if $r \leqslant s$, then $\chi(Z, D_{m,k}) \geqslant \lceil (m+k+2)/2 \rceil$. Therefor it suffices to show that $G(Z, D_{m,k})$ is $(m+k+1)/2$-colorable, if $r > s$; and $G(Z, D_{m,k})$ is $\lceil (m+k+2)/2 \rceil$-colorable, if $r \leqslant s$. It is known that $\chi(Z, D) = \chi(Z^+ \cup \{0\}, D)$ [14]. Therefore, it is sufficient to find a proper coloring for the subgraph of $G(Z, D_{m,k})$ induced by all non-negative integers.

We first decompose k into the sum of an odd number of integers, $k = a_1 + a_2 + \cdots + a_p$, as follows:

Case 1. For $r > s$ or $s = 0$, let $p = k'$ and $a_j = 2^s$ for $1 \leqslant j \leqslant p$.

Case 2. For $r \leqslant s \neq 0$, let $p = k-1$ and $a_j = 1$ for $1 \leqslant j < p$ and $a_p = 2$.

Next, partition the set Z into consecutive blocks of sizes $a_1, a_2, ..., a_p$ periodically. Then "pre-color" the blocks, alternating RED and BLUE. We call a vertex RED (or BLUE) if it falls within a RED (or BLUE) block.

Define a coloring f on the vertices of all non-negative integers of $G(Z, D_{m,k})$ according to the following three rules:

(R1) $f(i) = i$, if $0 \leqslant i \leqslant k-1$;

(R2) $f(i) = f(i-k)$, if i is BLUE and $i \geqslant k$;

(R3) $f(i) =$ the smallest non-negative integer that has not been used as a color in the m vertices preceding i, if i is RED and $i \geqslant k$.

To show that f is a proper coloring, we claim that for any vertex i, $f(i) \neq f(j)$ for all $j \neq i-k$ with $i-m \leqslant j < i$. It is easy to see that the claim is true when (R1) or (R3) is performed. Suppose (R2) is executed, i.e., i is BLUE, $i \geqslant k$, and $f(i) = f(i-k)$. Since k is divided into an odd number of blocks, $i-k$ is a RED vertex. By (R1) or (R3), $f(i-k)$ is different from any of the colors of the m vertices preceding $i-k$. Thus, it is sufficient to show that $f(i) \neq f(j)$ for all j with $i-k < j < i$.

If j is RED, by (R1) or (R3), $f(j) \neq f(i-k)$ and so $f(i) \neq f(j)$. If j is BLUE, by (R1) of (R2), $f(j) = f(j-k)$. One has $i-k-m < j-k < i-k$ (because $2k \leqslant m$), so $f(i-k) \neq f(j-k)$. This implies $f(i) \neq f(j)$.

To complete the proof of the theorem, it is sufficient to show that f is an $((m+k+1)/2)$-coloring if $r > s$; and f is an $\lceil (m+k+2)/2 \rceil$-coloring if $r \leqslant s$. One can accomplish this by counting the number of colors that have been used for the m vertices preceding a RED vertex i for which $i \geqslant k$. The first k vertices need at most k colors. For the remaining $m-k$ vertices, only those RED vertices need new colors.

If $r > s$, then $m-k+1$ is a multiple of 2^{s+1}. Any consecutive 2^{s+1} vertices have 2^s BLUE vertices and 2^s RED ones, so there are exactly $(m-k-1)/2$ RED vertices in the remaining $m-k$ vertices. Therefore, the total number of colors used in f is at most $k + (m-k-1)/2 + 1 = (m+k+1)/2$.

If $r \leqslant s$ (with $s = 0$ in Case 1 and $s \neq 0$ in Case 2), then there are at most $\lceil (m-k)/2 \rceil$ RED vertices in the remaining $m-k$ vertices. Thus, the total number of colors used in f is at most $k + \lceil (m-k)/2 \rceil + 1 = \lceil (m+k+2)/2 \rceil$. This completes the proof of Theorem 8.

We now present the following two results concerning the circular chromatic number of $G(Z, D_{m,k})$. The first one follows from $(*)$ and Theorems 5 and 8.

COROLLARY 9. *Suppose* $2k \leqslant m$. *Write* $m+k+1=2^r m'$ *and* $k=2^s k'$, *where* r *and* s *are non-negative integers and* m' *and* k' *are odd integers. If* $r>s$, *then* $\chi_f(Z, D_{m,k})=\chi_c(Z, D_{m,k})=\chi(Z, D_{m,k})=(m+k+1)/2$.

THEOREM 10. *If* $2k \leqslant m$ *and* k *is relatively prime to* $m+k+1$, *then* $\chi_f(Z, D_{m,k})=\chi_c(Z, D_{m,k})=(m+k+1)/2$.

Proof. Since k is relatively prime to $m+k+1$, there exists an integer n such that $nk \equiv 1 \pmod{m+k+1}$. Consider the mapping c defined by $c(i)=(in \bmod m+k+1)$ for all $i \in Z$. For any edge ij in $G(Z, D_{m,k})$, we shall prove that $\|c(i)-c(j)\|_{m+k+1} \geqslant 2$. Suppose to the contrary, that $\|c(i)-c(j)\|_{m+k+1} \leqslant 1$; i.e., $c(i)-c(j) \equiv 0$ or 1 or $-1 \pmod{m+k+1}$. Then $i-j \equiv 0$ or k or $-k \pmod{m+k+1}$, which contradicts the fact that i is adjacent to j. Thus c is an $(m+k+1, 2)$-coloring of $G(Z, D_{m,k})$. This along with Theorem 5 and $(*)$ implies the theorem. Q.E.D

Remarks. Many new results related to this topic have been obtained since the submission of this paper. In [3], the circular chromatic numbers of all the graphs $G(Z, D_{m,k})$ are determined. The chromatic number, circular chromatic number and fractional chromatic number of distance graphs with distance sets of the form $D_{m,k,s}=\{1, 2, ..., m\}-\{k, 2k, ..., sk\}$ have been studied in [8, 20, 27, 44]. (Accordingly, the distance graphs discussed in this paper are $G(Z, D_{m,k,1})$.) In [27], the chromatic numbers of all the graphs $G(Z, D_{m,k,2})$ are determined. The same paper also determined the fractional chromatic numbers of all the graphs $G(Z, D_{m,k,s})$. In [8], the following was proved:

$$\lceil (m+sk+1)/(s+1) \rceil \leqslant \chi(G(Z, D_{m,ks})) \leqslant \lceil (m+sk+1)/(s+1) \rceil + 1.$$

Moreover, both the upper bound and the lower bound are attainable. Then in [20], the chromatic numbers of all the graphs $G(Z, D_{m,k,s})$ are completely determined. Finally and most recently, [44] determines the circular chromatic numbers of all the graphs $G(Z, D_{m,k,s})$.

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REFERENCES

1. D. G. Cantor and B. Gordon, Sequences of integers with missing differences, *J. Combin. Theory Ser. A* **14** (1973), 281–287.
2. J. J. Chen, G. J. Chang, and K. C. Huang, Integral distance graphs, *J. Graph Theory* **25** (1997), 287–294.
3. G. J. Chang, L. Huang, and X. Zhu, Circular chromatic numbers and fractional chromatic numbers of distance graphs, *European J. Combin.* **19** (1998), 423–431.
4. M. B. Cozzens and F. S. Roberts, *T*-colorings of graphs and the channel assignment problem, *Congr. Numer.* **35** (1982), 191–208.
5. M. B. Cozzens and F. S. Roberts, Greedy algorithms for *T*-colorings of complete graphs and the meaningfulness of conclusions about them, *J. Combin. Inform. System Sci.* **16** (1991), 286–299.
6. H. T. Croft, K. J. Falconer, and R. K. Guy, "Unsolved Problems in Geometry," Springer-Verlag, New York, 1991.
7. W. Deuber and X. Zhu, The chromatic number of distance graphs, *Discrete Math.* **165/166** (1997), 195–204.
8. W. Deuber and X. Zhu, Chromatic numbers of distance graphs with distance sets missing multiples, manuscript, 1997.
9. R. B. Eggleton, New results on 3-chromatic prime distance graphs, *Ars Combin.* **26B** (1988), 153–180.
10. R. B. Eggleton, Distance graphs and their chromatic numbers, Workshop Notes of the Conference of Graph Theory and Combinatorics, Manila, 1991.
11. R. B. Eggleton, P. Erdős, and D. K. Skilton, Coloring the real line, *J. Combin. Theory Ser. B* **39** (1985), 86–100.
12. R. B. Eggleton, P. Erdős, and D. K. Skilton, Research problem 77, *Discrete Math.* **58** (1986), 323.
13. R. B. Eggleton, P. Erdős, and D. K. Skilton, Update information on research problem 77, *Discrete Math.* **69** (1988), 105.
14. R. B. Eggleton, P. Erdős, and D. K. Skilton, Coloring prime distance graphs, *Graphs Combin.* **32** (1990), 17–32.
15. D. Fisher, Fractional coloring with large denominators, *J. Graph Theory* **20** (1995), 403–409.
16. J. R. Griggs and D. D.-F. Liu, The channel assignment problem for mutually adjacent sites, *J. Combin. Theory Ser. A* **68** (1994), 169–183.
17. H. Hadwiger, H. Debrunner, and V. Klee, "Combinatorial Geometry in the Plane," Holt, Rinehard, Winston, New York, 1964.
18. W. K. Hale, Frequency assignment: theory and applications, *Proc. IEEE* **68** (1980), 1497–1514.
19. N. M. Haralambis, Sets of integers with missing differences, *J. Combin. Theory Ser. A* **23** (1977), 22–33.
20. L. Huang and G. J. Chang, Circular chromatic number of distance graphs with distance set missing multiples, *European J. Combin.*, to appear.
21. A. Kemnitz and H. Kolberg, Coloring of integer distance graphs, *Discrete Math.* **191** (1998), 113–123.
22. I. Leader, The fractional chromatic number of infinite graphs, *J. Graph Theory* **20** (1995), 411–417.
23. D. D.-F. Liu, *T*-colorings of graphs, *Discrete Math.* **101** (1992), 203–212.
24. D. D-F. Liu, On a conjecture of *T*-colorings, *Congr. Numer.* **103** (1994), 27–31.
25. D. D.-F. Liu, *T*-graphs and the channel assignment problem, *Discrete Math.* **161** (1996), 197–205.

26. D. D.-F. Liu, *T*-coloring and chromatic number of distance graphs, *Ars Comb.*, to appear.
27. D. D.-F. Liu and X. Zhu, Distance graphs with missing multiples in the distance sets, *J. Graph Theory*, to appear.
28. L. Moser and W. Moser, Solution to problem 10, *Canad. Math. Bull.* **4** (1961), 187–189.
29. T. S. Motzkin, unpublished problem collection.
30. J. H. Rabinowitz and V. K. Proulx, An asymptotic approach to the channel assignment problem, *SIAM J. Algebraic Discrete Math.* **6** (1985), 507–518.
31. A. Raychaudhuri, Further results on *T*-coloring and frequency assignment problem, *SIAM J. Discrete Math.* **7** (1994), 605–613.
32. F. Roberts, *T*-colorings of graphs: recent results and open problems, *Discrete Math.* **93** (1991), 229–245.
33. E. R. Scheinerman and D. H. Ullman, "Fractional Graph Theory," Interscience Series in Discrete Mathematics and Optimization, Wiley–Interscience, New York, 1997.
34. S. H. Scott, "Multiple Node Colourings of Finite Graphs," Doctoral dissertation, University of Reading, England, March, 1975.
35. B. A. Tesman, "*T*-Colorings, List *T*-Colorings and set *T*-Colorings of Graphs," Ph.D. dissertation, Dept. Math., Rutgers University, 1989.
36. A. Vince, Star chromatic number, *J. Graph Theory* **12** (1988), 551–559.
37. M. Voigt and H. Walther, Chromatic number of prime distance graphs, *Discrete Appl. Math.* **51** (1994), 197–209.
38. M. Voigt, Coloring of distance graphs, *Ars Comb.*, to appear.
39. X. Zhu, Distance graphs on the real line, manuscript, 1996.
40. X. Zhu, Non-integral distance graphs on the real line, manuscript, 1996.
41. X. Zhu, Pattern-periodic coloring of distance graphs, *J. Combin. Theory Ser. B* **73** (1998), 195–206.
42. X. Zhu, Circular chromatic number: a survey, manuscript, 1997.
43. X. Zhu, The star chromatic number and graph product, *J. Graph Theory* **16** (1992), 557–569.
44. X. Zhu, Circular chromatic numbers of a class of distance graphs, manuscript, 1988.