On k-Ary Spanning Trees
of Tournaments

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Abstract: It is well known that every tournament contains a Hamiltonian path, which can be restated as that every tournament contains a unary spanning tree. The purpose of this article is to study the general problem of whether a tournament contains a k-ary spanning tree. In particular, we prove that, for any fixed positive integer k, there exists a minimum number $h(k)$ such that every tournament of order at least $h(k)$ contains a k-ary spanning tree. The existence of a Hamiltonian path for any tournament is the same as $h(1) = 1$. We then show that $h(2) = 4$ and $h(3) = 8$. The values of $h(k)$ remain unknown for $k \geq 4$. © 1999 John Wiley & Sons, Inc. J Graph Theory 30: 167–176, 1999

1. INTRODUCTION

A *tournament* $T = (V, E)$ of order n is an orientation of the complete graph of order n, where V is the vertex set and E the (directed) edge set of T. We sometimes use *n-tournament* to denote a tournament of order n. For an edge $xy \in E$, we also write $x \to y$. If $x \to y$, we say that x *beats* y. For $A \subseteq V$ and $B \subseteq V$ with $A \cap B = \emptyset$, we write $[A, B]_T = \{xy \in E : x \in A \text{ and } y \in B\}$. We also use $A \Rightarrow B$ to denote that every vertex in A beats every vertex in B; $x \Rightarrow B$ stands for $\{x\} \Rightarrow B$, and $A \Rightarrow y$ for $A \Rightarrow \{y\}$.

Let $G = (V, E)$ be a directed graph with vertex set V and edge set E. If $x \to y$, we call y an *out-neighbor* of x, and x an *in-neighbor* of y. The *out-neighborhood* of a vertex x is the set $N_G^+(x) = \{y : x \to y \text{ in } G\}$ and the *in-neighborhood* of x is $N_G^-(x) = \{y : y \to x \text{ in } G\}$. The *out-degree* of x is $d_G^+(x) = |N_G^+(x)|$, and the *in-degree* of x is $d_G^-(x) = |N_G^-(x)|$.

A *rooted tree* is a directed tree with a special vertex, called the *root*, such that there exists a unique (directed) path from the root to any other vertex. A *leaf* is a vertex of out-degree zero. If $u \to v$ is an edge, then u is called the *parent* $p(v)$ of v and v a *child* of u. Note that any vertex other than the root has exactly one parent, and a vertex may have many children. *Descendants* of a vertex are those vertices who can be reached from it by a path. The *depth* of a vertex is the length of the unique path from the root to it. The *height* of a rooted tree is the length of a longest path from the root.

A rooted tree is called a *k-ary tree*, if all nonleaf vertices have exactly k children, except possibly one nonleaf vertex has at most $k - 1$ children. If all nonleaf vertices have exactly k children, then we call it a *full k-ary tree*. When $k = 1, 2, 3$, the corresponding k-ary trees are called *unary*, *binary*, and *ternary* trees, respectively. Note that a k-ary tree of order $km + 1$ is a full k-ary tree. In particular, a unary tree is always full. A *star* is a rooted tree of height 1. A *k-star* is a star with k vertices, and, thus, consists of $k - 1$ edges with a common tail.

It is a fundamental and important problem in the study of tournaments to determine if a tournament contains a special structure. There are many results of this

kind in the literature. For example, it is well known that every tournament contains a Hamiltonian path ([11]), and every tournament contains a spanning tree of height at most 2 ([3]). Further results can be found in [1–2, 4–10, 13–14].

The fact that every tournament contains a Hamiltonian path can be restated as that every tournament contains a unary spanning tree. The purpose of this article is to study the general problem of whether a tournament contains a k-ary spanning tree. We first establish that, for any positive integer k, there exists a number $h'(k)$ such that every n-tournament contains a k-ary spanning tree if $n \geq h'(k)$. According to this result, we may define $h(k)$ as the least number such that every n-tournament contains a k-ary spanning tree if $n \geq h(k)$. The well-known result for the Hamiltonian path is then the same as that $h(1) = 1$. We then show that $h(2) = 4$ and $h(3) = 8$. The exact values of $h(k)$ remain unknown for $k \geq 4$.

2. k-ARY SPANNING TREES

The goal of this section is to establish a fundamental result that, for any positive integer k, there exists a number $h'(k)$ such that every n-tournament contains a k-ary spanning tree if $n \geq h'(k)$. For this purpose, we need the following two theorems.

A tournament is *transitive* if $x \to y$ and $y \to z$ imply that $x \to z$, in other words, its vertices can be linearly ordered such that each vertex beats all later vertices.

Given a positive integer m and an n-tournament T, Erdős (see [12]) proved the following result concerning the number of vertex-disjoint transitive sub-tournaments of order mT can have.

Theorem 1. *For any fixed positive integer m, there exists a number $f(m)$ such that every n-tournament contains $\lfloor n/m \rfloor$ vertex-disjoint transitive sub-tournaments of order m if $n \geq f(m)$.*

It is not hard to see that, for n sufficiently large, every n-tournament contains all oriented trees of order m, where m is a fixed positive integer. Sumner (see [2]) conjectured that this is true for $n \geq 2m - 2$. Although this conjecture is still open, Häggkvist and Thomason [2] proved the following result.

Theorem 2. *Every oriented tree of order m is contained in every tournament of order $12m$.*

The following two lemmas are frequently used in this article.

Lemma 1 (Extension Lemma). *For any fixed positive integer k, there exists a number $g(k) \leq 2(k+1)^{k-2}$ such that, for any full k-ary tree R of order at least $g(k)$ and any transitive sub-tournament S of order $s \leq k$, where R and S are vertex disjoint, R can be extended to a k-ary tree R' with $V(R') = V(R) \cup V(S)$. (Note that, if $s = k$, then R' is full.)*

The proof of this lemma heavily depends on the idea used in the proof of Lemma 6 in Section 4 and so is given at the end of this article.

Lemma 2 (Induction Lemma).   *Let R be a k-ary tree of T and S a k-star of T with root u, where R and S are vertex disjoint. If $\|[u, V(R)]\| \geq 1$, then T contains a k-ary tree R' with $V(R') = V(R) \cup V(S)$. Moreover, if u is an out-neighbor of the root of R, then R' can be chosen to have the same root as R.*

Proof.   Let v be the root of R. If $u \to v$, then $E(R) \cup E(S) \cup \{uv\}$ induces a k-ary tree. Thus, we may assume that $v \to u$. Since $\|[u, V(R)]\| \geq 1$, there exists a vertex x such that $u \to x$. Let P be the path of R from v to x, and y the first vertex in P such that $u \to y$. Then $p(y) \to u$, and, thus, $(E(R) - \{p(y)y\}) \cup E(S) \cup \{p(y)u, uy\}$ induces a desired k-ary tree.

The "moreover" part is evident by the argument above.   ∎

Note that the Induction Lemma is also true, if we assume that R is a full k-ary tree and S an s-star of T with $s \leq k$.

Now, we are ready to prove the following fundamental theorem for the existence of a k-ary spanning tree of a tournament, which has a natural similarity to Theorems 1 and 2.

Theorem 3.   *For any fixed positive integer k, there exists a number $h'(k)$ such that every n-tournament contains a k-ary spanning tree if $n \geq h'(k)$.*

Proof.   Let $h'(k) = (g(k) + k) + \max\{f(k) + k, 12(g(k) + k)\} + k$, where $f(k)$ and $g(k)$ are the numbers in Theorem 1 and the Extension Lemma, respectively. For $n \geq h'(k)$, let $a = \lceil (g(k) - 1)/k \rceil, b = \lfloor (n - 2)/k \rfloor - a$, and $c = n - (ak + 1) - bk$. Then $ak + 1 \geq g(k), bk \geq f(k), 1 \leq c \leq k, n = (ak + 1) + bk + c$, and $n - c \geq 12(ak + 1)$.

We first choose a transitive sub-tournament Q of order c of T. Note that the existence of such Q follows from Theorem 1. Since $n - c \geq 12(ak + 1)$, by Theorem 2, $T - Q$ contains all oriented trees of order $ak + 1$, especially a full k-ary tree R of order $ak + 1$. Since $bk \geq f(k)$, by Theorem 1, the sub-tournament $T - R - Q$ contains b vertex-disjoint transitive sub-tournaments of order k, say, $S_1, S_2, \ldots, S_b$. Since R has at least $g(k)$ vertices, by the Extension Lemma, R can be extended to a full k-ary tree R_1 with $V(R_1) = V(R) \cup V(S_1)$. By the same lemma, we can extend R_1 to a full k-ary tree R_2 with $V(R_2) = V(R_1) \cup V(S_2)$. Continuing this process, we can finally obtain a full k-ary tree R_b with $V(R_b) = V(R) \cup V(S_1) \cup \cdots \cup V(S_b)$. Using the Extension Lemma again for R_b and Q, we obtain a k-ary spanning tree of T.   ∎

According to this result, we may define $h(k)$ as the least number such that every n-tournament contains a k-ary spanning tree as long as $n \geq h(k)$. It is well known that $h(1) = 1$. We shall show that $h(2) = 4$ and $h(3) = 8$ in the next two sections, respectively. The values of $h(k)$ remain unknown for $k \geq 4$.

3. BINARY SPANNING TREES

The purpose of this section is to determine the value of $h(2)$. The following theorem is the main result.

Theorem 4.   *Every tournament T of order $n \geq 4$ contains a binary spanning tree.*

Let T_3 be the tournament with $V(T_3) = \{0, 1, 2\}$ and $E(T) = \{01, 12, 20\}$. Then T_3 does not contain a binary spanning tree. Therefore, Theorem 4 implies that $h(2) = 4$.

We first establish two lemmas before proving Theorem 4.

Lemma 3.   *If every $(km + 1)$-tournament has a k-ary spanning tree, then so does every km-tournament.*

Proof.   Suppose that T is a km-tournament. Construct a $(km + 1)$-tournament T' from T by adding a new vertex w such that $V(T) \Rightarrow w$. If R' is a k-ary spanning tree of T', then $R = R' - w$ is a k-ary spanning tree of T.   ∎

For convenience, we use (u, v, w) to denote a transitive 3-tournament with u beats v and w and v beats w. The following fact is evident.

Fact 1.   *Any tournament of order at least 4 contains a transitive 3-tournament.*

Lemma 4.   *Every 5-tournament T contains a binary spanning tree.*

Proof.   Let x be a vertex of T with a maximum out-degree, and T' the sub-tournament induced by $N_T^+(x)$. We consider the following three cases.

Case 1. $d_T^+(x) = 4$. By Fact 1, T' contains a transitive 3-tournament (u, v, w). Let y be the remaining vertex of $N_T^+(x)$. Then $\{xy, xu, uv, uw\}$ induces a binary spanning tree.

Case 2. $d_T^+(x) = 3$.   Say, $N_T^+(x) = \{u, v, w\}$ and $N_T^-(x) = \{t\}$.   If $\|[t, \{u, v, w\}]\| \geq 1$, say, $t \to u$, then $\{tu, tx, xv, xw\}$ induces a binary spanning tree. So we may assume that $\{u, v, w\} \Rightarrow t$. Without loss of generality, we may assume that $u \to v$. Then $\{xw, xu, uv, ut\}$ induces a binary spanning tree.

Case 3. $d_T^+(x) = 2$. In this case, T is regular. Say, $N_T^+(x) = \{u, v\}$ and $N_T^-(x) = \{s, t\}$ with $s \to t$. Then $\{st, sx, xu, xv\}$ induces a binary spanning tree.   ∎

Proof of Theorem 4.   According to Lemma 3, we need to prove only the theorem for $n = 2m + 1 \geq 5$. We use an induction on m. The base case $m = 2$ is proved as Lemma 4. We now assume that $m \geq 3$, and the theorem holds for all $m' < m$.

Let x be a vertex with a maximum out-degree in T, and y one of its out-neighbors. Note that $d_T^+(x) \geq 2$. Now the tournament $T - \{x, y\}$ has order $2(m - 1) + 1$. By the induction hypothesis, it contains a binary spanning tree R. By the choice of x, we must have $\|[x, V(R)]\| \geq 1$. According to the Induction Lemma, the tournament T has a binary spanning tree.   ∎

4. TERNARY SPANNING TREES

The purpose of this section is to determine the value of $h(3)$. The following theorem is the main result.

Theorem 5. *Every tournament T of order $n \geq 8$ contains a ternary spanning tree.*

Let T_7 be the 7-tournament with $V(T_7) = \{0, 1, 2, 3, 4, 5, 6\}$ and $E(T_7) = \{ij : i - j \equiv 1, 2$ or $4 \pmod 7\}$. It is straightforward to check that T_7 does not contain a ternary spanning tree. Therefore, $h(3) \geq 8$, and so Theorem 5 implies that $h(3) = 8$.

For proving Theorem 5, we first establish the following five lemmas.

Although there exists a 7-tournament containing no ternary spanning tree, most 7-tournaments do contain ternary spanning trees.

Lemma 5. *Any nonregular 7-tournament T contains a ternary spanning tree.*

Proof. Let u be a vertex of T with a maximum out-degree, and T' the sub-tournament of T induced by $N_T^+(u)$. Note that $3 \leq d_T^+(u) \leq 6$. We consider the following four cases.

Case 1. $d_T^+(u) = 6$. Let v be a vertex of T' with a maximum out-degree. Then $d_{T'}^+(v) \geq 3$. Say, $v \Rightarrow \{x, y, z\}$. Let the remaining vertices of T' be s and t. Then $\{us, ut, uv, vx, vy, vz\}$ induces a ternary spanning tree.

Case 2. $d_T^+(u) = 5$. Say, $N_T^+(u) = \{s, t, x, y, z\}$ and $N_T^-(u) = \{v\}$. If $|[v, N_T^+(u)]| \geq 2$, say, $v \Rightarrow \{s, t\}$, then $\{vs, vt, vu, ux, uy, uz\}$ induces a ternary spanning tree. We may, therefore, assume that $\{s, t, x, y\} \Rightarrow v$. In the sub-tournament induced by $\{s, t, x, y\}$, there is a transitive 3-tournament, say, (t, x, y). Then $\{uz, us, ut, tx, ty, tv\}$ induces a ternary spanning tree.

Case 3. $d_T^+(u) = 4$. Say, $N_T^+(u) = \{v, x, y, z\}$ and $N_T^-(u) = \{s, t\}$ with $s \to t$. If $|[s, \{v, x, y, z\}]| \geq 1$, say, $s \to v$, then $\{st, sv, su, ux, uy, uz\}$ induces a ternary spanning tree. So we may assume that $\{v, x, y, z\} \Rightarrow s$. If $|[\{v, x, y, z\}, t]| \geq 2$, say, $\{v, x\} \Rightarrow t$ with $v \to x$, then $\{uy, uz, uv, vx, vs, vt\}$ induces a ternary spanning tree. We, thus, may assume that $|[\{v, x, y, z\}, t]| \leq 1$. From $d_T^+(t) \leq d_T^+(u)$, we have $|[\{v, x, y, z\}, t]| \geq 1$, and so $|[\{v, x, y, z\}, t]| = 1$. Say, $v \to t$. If $|[v, \{x, y, z\}]| \geq 1$, say, $v \to x$, then $\{uy, uz, uv, vx, vs, vt\}$ induces a ternary spanning tree. So we may assume that $\{x, y, z\} \Rightarrow v$. Without loss of generality, we may assume that $y \to x$. Then $\{tu, tz, ty, yx, yv, ys\}$ induces a ternary spanning tree.

Case 4. $d_T^+(u) = 3$. In this case, the tournament T is regular.

∎

Let H be a subgraph and x a vertex not in H. The vertex x is said to be *good with respect to H*, if $x \Rightarrow V(H)$. Let R be a tree that is vertex-disjoint from H. A vertex x of R is *minimally good with respect to H*, if x is good but none of its descendants in R is good with respect to H. The following lemma is useful in

proving that every tournament of order 10 contains a ternary spanning tree, and can also be used as an induction step in the ternary case. The proof technique is also useful in the proof of the Extension Lemma in the next section.

Lemma 6.   *Let (x, y, z) be a transitive 3-tournament in a tournament T. If R is a full ternary tree in $T - \{x, y, z\}$ with root g such that g is minimally good with respect to $\{x, y, z\}$, then R can be extended to a ternary tree R' with root g and vertex set $V(R) \cup \{x, y, z\}$.*

Proof.   Suppose that $\|[x, V(R)]\| \geq 1$. Then, by the Induction Lemma, the desired R' exists. Therefore, we may assume $\|[x, V(R)]\| = 0$, i.e., $V(R) \Rightarrow x$.

Suppose that $\|[y, V(R)]\| \geq 2$. Choose a vertex $u \in R$ with a smallest depth such that $y \to u$, and another vertex v of R such that $y \to v$. Since $y \to u$ and $g \to y$, it must be the case that $u \neq g$. Note that $p(u) \to y$ by the choice of u. Then, $(E(R) - \{p(u)u, p(v)v\}) \cup \{yu, yv, yz, p(u)y, p(v)x\}$ induces a desired ternary tree. Therefore, we may assume that $\|[y, V(R)]\| \leq 1$.

Suppose that $\|[z, V(R)]\| \geq 5$. We then choose 5 distinct vertices u, v_1, v_2, v_3, v_4 such that $z \Rightarrow \{u, v_1, v_2, v_3, v_4\}$. Note that u is not the root g of R, since $z \to u$ but $g \to z$. We may assume that u is the vertex with a smallest depth such that $z \to u$ and so $p(u) \to z$. Since a vertex of R has at most three children, $\{p(v_1), p(v_2), p(v_3), p(v_4)\}$ contains at least two distinct vertices, one of them must not be an out-neighbor of y, say, $p(v_2) \to y$. Then $(E(R) - \{p(u)u, p(v_1)v_1, p(v_2)v_2\}) \cup \{zu, zv_1, zv_2, p(u)z, p(v_1)x, p(v_2)y\}$ induces a desired ternary tree. Therefore, we may assume that $\|[z, V(R)]\| \leq 4$.

Note that, to be a full ternary tree, the order of R must be 1 (mod 3). If R has at least 7 vertices, there must be a vertex $u \neq g$ in R that beats all three vertices $x, y,$ and z, a contradiction to the assumption that g is minimally good. In the case that R has 4 vertices, let $u, v,$ and w be the children of g. Then $g \Rightarrow \{u, v, w, x, y, z\}$. Then the sub-tournament induced by $\{u, v, w, x, y, z\}$ has a vertex t of out-degree at least 3. One can find a desired ternary tree with t as the other nonleaf vertex. The case when R has just one vertex is trivial.

Therefore, the lemma holds.   ∎

Lemma 7.   *Every 10-tournament T contains a ternary spanning tree.*

Proof.   A transitive 3-tournament (u, v, w) is called a *nice* transitive 3-tournament if $u, v,$ and w have a common in-neighbor in T and $T - \{u, v, w\}$ is not regular. Since $T - \{u, v, w\}$ is not regular, it has a k-ary spanning tree R by Lemma 5. Since u, v, w has a common in-neighbor, we can choose a vertex x in R that is minimally good with respect to $\{u, v, w\}$. Let R_1 be the subtree of R induced by x and its descendants. Then, by Lemma 6, R_1 can be extended to a ternary tree R_1' with root x and vertex set $V(R_1) \cup \{u, v, w\}$. Let R' be the tree obtained from R by replacing R_1 with R_1'. Then R' is a desired ternary tree of T. Therefore, the existence of a nice transitive 3-tournament implies the existence of a ternary spanning tree.

If T has a vertex x with an out-degree in the set $\{4, 5, 7, 8, 9\}$, then x has an out-neighbor u with two out-neighbors v and w, which are also in $N_T^+(x)$. Without

loss of generality, we may assume that $v \to w$. As x has an out-degree that differs from 3 in $T - \{u, v, w\}$, we have that (u, v, w) is a nice transitive 3-tournament.

So we may assume that for any $u \in V(T), d_T^+(u) \in \{0, 1, 2, 3, 6\}$. As the sum of the out-degrees of all vertices in T is 45, there exist two vertices u and v of out-degree 6 such that $u \to v$. Let x and y be two out-neighbors of u that differ from v. Then $T - \{u, x, y\}$ is a 7-tournament in which v is of out-degree at least 4. So, $T - \{u, x, y\}$ is not regular. By Lemma 5, $T - \{u, x, y\}$ has a ternary spanning tree. Then, by the Induction Lemma, T has a ternary spanning tree. ∎

Lemma 8. *Every 9-tournament contains a ternary spanning tree.*

Proof. The lemma follows immediately from Lemmas 7 and 3. ∎

Lemma 9. *Every 8-tournament T contains a ternary spanning tree.*

Proof. Choose a vertex x in the tournament T with an out-degree not equal to 4. If $d_T^+(x) \neq 0$, then $T - u$ is not regular for any $u \in N_T^+(x)$. If $d_T^+(x) = 0$, then $T - u$ is not regular for any $u \neq x$. In any case, we can find a vertex u such that $T - u$ is not regular. By Lemma 5, $T - u$ has a ternary spanning tree. One can easily extend this tree to a ternary spanning tree of T. ∎

Proof of Theorem 5. Lemmas 9, 8, and 7 prove the cases for $n = 8, 9, 10$. Suppose that $n \geq 11$ and the theorem is true for all $n' < n$. By first choosing a vertex u of out-degree at least 3, it is not difficult to see that there is a transitive 3-tournament (u, v, w) such that u has an out-neighbor in $T' = T - \{u, v, w\}$. By the induction hypothesis, T' has a ternary spanning tree. By the Induction Lemma, T has a ternary spanning tree. ∎

5. PROOF OF THE EXTENSION LEMMA

This section gives a proof of the Extension Lemma by using the idea in the proof of Lemma 6.

Let R be a full k-ary tree of T with root u, and S a transitive sub-tournament of T of order $s \leq k$, where R and S are vertex disjoint. Write $V(S) = \{x_1, x_2, \ldots, x_s\}$ such that x_i beats x_j whenever $i < j$. Set $A_i = N_T^+(x_i) \cap V(R)$ and $B_i = A_1 \cup A_2 \cup \cdots \cup A_i$ for $1 \leq i \leq s$. For convenience, let $B_0 = \emptyset$.

Lemma 10. *If $|B_i - B_{i-1}| \geq |B_{i-1}|k + i$ for some i, then R can be extended to cover S. In other words, there exists a k-ary tree with vertex set $V(R) \cup V(S)$.*

Proof. Note that $|A_i| \geq |B_i - B_{i-1}| \geq |B_{i-1}|k + i \geq 1$. If $u \to x_i$, let y_i be a vertex in R such that $x_i \to y_i$ and $p(y_i) \to x_i$ (such a vertex clearly exists); if $x_i \to u$, let $y_i = u$. We next observe that it follows from the above inequality and the fact that the set B_{i-1} has at most $|B_{i-1}|k$ children, A_i has $i - 1$ vertices $y_1, y_2, \ldots, y_{i-1}$ (distinct from y_i) whose parents are not in B_{i-1}, i.e., $x_i \Rightarrow \{y_1, y_2, \ldots, y_{i-1}\}$ and $\{p(y_1), p(y_2), \ldots, p(y_{i-1})\} \Rightarrow \{x_1, x_2, \ldots, x_{i-1}\}$. If $u \to x_i$, then

$$(E(R) - \{p(y_j)y_j : 1 \leq j \leq i\}) \quad \cup \{p(y_j)x_j, x_iy_j : 1 \leq j \leq i\}$$
$$\cup \{x_ix_{i+1}, x_ix_{i+2}, \ldots, x_ix_s\}$$

determines a k-ary tree with vertex set $V(R) \cup V(S)$. And if $x_i \to u$, then

$$(E(R) - \{p(y_j)y_j : 1 \leq j \leq i - 1\}) \quad \cup \{p(y_j)x_j, x_iy_j : 1 \leq j \leq i - 1\}$$
$$\cup \{x_iu, x_ix_{i+1}, x_ix_{i+2}, \ldots, x_ix_s\}$$

determines a k-ary tree with vertex set $V(R) \cup V(S)$. ∎

Lemma 11. *If there is a leaf $y \in V(R) - B_s$, then R can be extended to cover S.*

Proof. In this case, $y \Rightarrow \{x_1, x_2, \ldots, x_s\}$. Then $E(R) \cup [y, V(S)]$ induces a desired spanning tree. ∎

Proof of the Extension Lemma. Suppose that there exists no k-ary tree with vertex set $V(R) \cup V(S)$. Then, by Lemma 10, $|B_i - B_{i-1}| \leq |B_{i-1}|k + i - 1$ and so $|B_i| \leq |B_{i-1}|(k + 1) + i - 1$ for $1 \leq i \leq s$. Using $|B_0| = 0$ and an induction, we have $|B_s| \leq \sum_{i=1}^{s-1} i(k + 1)^{s-1-i} < 2(k + 1)^{k-2}$. Certainly, if R has at least $2(k + 1)^{k-2}$ vertices, we can find a leaf y in $V(R) - B_s$ and use Lemma 11 to complete the proof. ∎

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