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Dolbeault cohomology of $G/(P,P)$

Meng-Kiat Chuah, Lee-Peng Teo

Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, Taiwan (e-mail: chuah@math.nctu.edu.tw)

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Abstract. Let G be a complex connected semi-simple Lie group, with parabolic subgroup P. Let (P,P) be its commutator subgroup. The generalized Borel-Weil theorem on flag manifolds has an analogous result on the Dolbeault cohomology $H^{0,q}(G/(P,P))$. Consequently, the dimension of $H^{0,q}(G/(P,P))$ is either 0 or ∞. In this paper, we show that the Dolbeault operator $\bar{\partial}$ has closed image, and apply the Peter-Weyl theorem to show how q determines the value 0 or ∞. For the case when P is maximal, we apply our result to compute the Dolbeault cohomology of certain examples, such as the punctured determinant bundle over the Grassmannian.

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1 Introduction

Let G be a complex connected semi-simple Lie group with compact real form K, and $G = KAN$ an Iwasawa decomposition. Let T be the centralizer of A in K, so that $H = TA$ is a Cartan subgroup of G. The Lie algebras of H, T, A are denoted by $\mathfrak{h}, \mathfrak{t}, \mathfrak{a}$. Let P be a parabolic subgroup of G containing the Borel subgroup $B = HN$. For the compact flag manifold G/P, the generalized Borel-Weil theorem [1] computes the cohomology $H^q(G/P, L_\lambda)$, where L_λ is the homogeneous line bundle corresponding to the weight $\lambda \in \mathfrak{h}^*$. Let (P,P) be the commutator subgroup of P. Since H normalizes (P,P), it acts on $G/(P,P)$ on the right, and we let $H^{0,q}_\lambda(G/(P,P))$ denote the Dolbeault cohomology of $(0,q)$-forms that transform by λ under the right H-action. Since the right H-action commutes with the left K-action, it is a K-representation space. In [3], the K-spaces

$H^q(G/P, L_\lambda)$ and $H^{0,q}_\lambda(G/(P,P))$ are shown to be isomorphic. This gives yet another version of the generalized Borel-Weil theorem, among others [6]. Consequently, the Dolbeault cohomology $H^{0,q}(G/(P,P))$ is either 0 or infinite dimensional.

Let $\Omega^{0,q}(G/(P,P))$ denote the complex $(0,q)$-forms on $G/(P,P)$. We equip it with the following topology. Given a sequence of $(0,q)$-forms $\alpha_i \in \Omega^{0,q}(G/(P,P))$, we express them as $\alpha_i = \sum_I f^i_I dx_I$ on a compact coordinate patch, where $|I| = q$ is the multiple index notation. Then we say that $\alpha_i \to 0$ in $\Omega^{0,q}(G/(P,P))$ if and only if on every such coordinate neighborhood and every index I, all derivatives of $\{f^i_I\}_i$ converge uniformly to 0 as $i \to \infty$ ([5], p.820). The following theorem says that the Dolbeault operator $\bar\partial$ behaves nicely under this topology.

Theorem 1 *The image of* $\bar\partial : \Omega^{0,q}(G/(P,P)) \longrightarrow \Omega^{0,q+1}(G/(P,P))$ *is closed.*

We shall prove Theorem 1 in §2. The Iwasawa decomposition $G = KAN$ determines a positive system Δ^+ for the roots in $\mathfrak{h}^*$, where the Lie algebra of N consists of positive root spaces. This way, the Killing form $(-,-)$ defines a closed Weyl chamber D, consisting of the vectors $\lambda \in \mathfrak{h}^*$ satisfying $(\lambda, \Delta^+) \geq 0$. Let Δ^+_0 be the simple roots for Δ^+. We shall call $\sigma \subset D$ a cell if there exists a subset $S \subset \Delta^+_0$ such that

$$(\sigma, S) > 0\ ,\ (\sigma, \Delta^+_0 \backslash S) = 0. \tag{1.1}$$

Then D is a disjoint union of cells. Each cell spans $\mathfrak{h}^*_\sigma \subset \mathfrak{h}^*$, which can be identified with $\mathfrak{h}_\sigma$ under the Killing form. Taking its intersection with $\mathfrak{t}, \mathfrak{a}$ give $\mathfrak{t}_\sigma, \mathfrak{a}_\sigma$ respectively. These Lie subalgebras correspond to the subgroups T_σ, A_σ. There is a bijective correspondence between the cells $\{\sigma\ ;\ \sigma \subset D\}$ and the parabolic subgroups $\{P\ ;\ B \subset P\}$. This is given by Langlands decomposition ([5], p.659)

$$P = M_\sigma A_\sigma N_\sigma, \tag{1.2}$$

where A_σ is the subgroup determined by the cell σ described above. Let ρ denote half the sum of all positive roots, and W the Weyl group.

Theorem 2 $H^{0,q}(G/(P,P))$ *is infinite dimensional if there exists* $w \in W$ *of length q, and a weight* $\lambda \in \mathfrak{h}^*_\sigma$*, such that* $w(\lambda+\rho)-\rho$ *is dominant. The cohomology vanishes otherwise.*

Actually, much of Theorem 2 has been proved in [3], except for its last statement on the vanishing of cohomology. In §3, we apply the Peter-Weyl theorem to Theorem 1, and show that the right T_σ-representation on $H^{0,q}(G/(P,P))$ has no infinite dimensional irreducible subrepresentation, which proves the last statement of Theorem 2.

Suppose that P is a maximal parabolic subgroup of G. We shall show that Theorem 2 becomes simpler, and can be stated in terms of the dimension of the flag manifold G/P:

Theorem 3 *Let P be a maximal parabolic subgroup of G. Then*

$$\dim H^{0,q}(G/(P,P)) = \begin{cases} \infty & q = 0, \dim G/P \\ 0 & \text{otherwise.} \end{cases}$$

For $G = SL(n, \mathbf{C})$ the homogeneous spaces $G/(P,P)$ with P as in Theorem 3 admit a direct geometric description. They include the space of non-zero vectors in $\mathbf{C}^n$ where Theorem 3 yields an alternative approach to known results.

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2 Image of $\bar{\partial}$

Let X be a complex manifold. Let $\Omega^q(X)$ and $\Omega^{0,q}(X)$ respectively denote the spaces of complex q-forms and $(0,q)$-forms on X. Equip $\Omega^q(X)$ with the topology as described in §1 before ([5], p.820). Then $\Omega^{0,q}(X)$ is a closed subspace of $\Omega^q(X)$.

Recall that G is a complex connected semi-simple Lie group, P a parabolic subgroup of G, and (P,P) its commutator subgroup. The purpose of this section is to show that the image of the Dolbeault operator

$$\bar{\partial} : \Omega^{0,q}(G/(P,P)) \longrightarrow \Omega^{0,q+1}(G/(P,P)) \tag{2.1}$$

is closed, and prove Theorem 1. However, Proposition 2.1, Corollary 2.2 and Proposition 2.3 below hold for general connected complex manifold X.

Proposition 2.1 *If X is a connected manifold, then the image of $d : C^\infty(X) \longrightarrow \Omega^1(X)$ is closed.*

Proof. Let ω_j be a sequence of exact 1-forms on X. Suppose that they converge to a 1-form ω. We need to show that ω is exact too.

The 1-form ω is certainly closed. To prove that it is exact, it suffices that for each smooth loop $\gamma \subset X$, the integral $\int_\gamma \omega$ vanishes [7]. But since ω_j converges uniformly to ω,

$$\int_\gamma \omega = \int_\gamma \lim_{j\to\infty} \omega_j = \lim_{j\to\infty} \int_\gamma \omega_j.$$

Since each ω_j is exact, Stoke's theorem says that the last integral vanishes for each smooth loop γ. We conclude that ω is exact. □

Since the natural projection $\pi : \Omega^1(X) \longrightarrow \Omega^{0,1}(X)$ is a closed map, and since $\bar\partial = \pi \cdot d$, it follows that

Corollary 2.2 *Let X be a connected complex manifold. Then the image of* $\bar\partial : C^\infty(X) \longrightarrow \Omega^{0,1}(X)$ *is closed.*

The following general result provides a sufficient condition for the map

$$\bar\partial : \Omega^{0,q}(X) \longrightarrow \Omega^{0,q+1}(X) \tag{2.2}$$

to have closed image.

Proposition 2.3 *Let X be a connected complex manifold. Suppose that* $\Omega^{0,q}(X)$ *is a free module over* $C^\infty(X)$ *with basis* $\xi_i, 1 \le i \le a$*, and that* $\Omega^{0,q+1}(X)$ *is a free module over* $C^\infty(X)$ *with basis* $\xi'_j, 1 \le j \le b$*. If* $\bar\partial(\xi_i) \in \sum_j \mathbf{C}\xi'_j$ *for all i, then the image of (2.2) is closed.*

Proof. Suppose that the above conditions are satisfied. We want to show that (2.2) has closed image. Write the Dolbeault operator as $\bar\partial = (\sum_{i=1}^a \bar\partial^i) + \bar\partial'$, where

$$\bar\partial^i(\sum_k f_k\xi_k) = (\bar\partial f_i) \wedge \xi_i \ , \ \bar\partial'(\sum_k f_k\xi_k) = \sum_k f_k \bar\partial(\xi_k);$$

for $f_k \in C^\infty(X)$. It suffices to show that the images of $\bar\partial'$ and all $\bar\partial^i$ are closed.

For $\bar\partial^i$, consider

$$V_i = \{\beta \wedge \xi_i \ ; \ \beta \in \Omega^{0,1}(X)\} \ \subset \ \Omega^{0,q+1}(X).$$

By Corollary 2.2, the image of $\bar\partial^i$ is closed in V_i. Since $V_i \subset \Omega^{0,q+1}(X)$ is closed, it follows that the image of $\bar\partial^i$ is closed in $\Omega^{0,q+1}(X)$.

Next let I be the image of $\bar\partial'$, and we want to show that $I \subset \Omega^{0,q+1}(X)$ is closed. Let Θ be a $\mathbf{C}$-basis of $\sum_i \mathbf{C}\bar\partial(\xi_i)$. The assumption of this proposition says that Θ can be imbedded into a $\mathbf{C}$-basis of $\sum_j \mathbf{C}\xi'_j$, and hence into a basis Θ' of $\Omega^{0,q+1}(X)$ over $C^\infty(X)$. From $\Theta \subset \Theta'$, order the elements of Θ' so that Θ consists of the beginning ones. Then identify $\Omega^{0,q+1}(X)$ with $C^\infty(X)^b$ via the basis Θ'. Since Θ is also a basis for I over $C^\infty(X)$, I has the form

$$I = \{(f_1, ..., f_t, 0, ..., 0) \ ; \ f_i \in C^\infty(X)\} \subset C^\infty(X)^b, \tag{2.3}$$

where $t = |\Theta|$. It follows from (2.3) that the image of $\bar\partial'$ is closed. This proves the proposition. □

Now let X be the specific space $G/(P,P)$. Recall from (1.1) and (1.2) that P determines a subset S of the simple roots Δ_0^+, as well as a cell σ. They satisfy $(\sigma, S) > 0$ under the Killing form. Let S' be the positive roots generated by S, so that $S \subset S' \subset \Delta^+$. Let $r = |S|$ and $s = |S'|$. Clearly $r \leq s$. Note that when P is the minimal parabolic subgroup B, then $r =$ rank G and $s = |\Delta^+|$. From Langlands decomposition ([3] and [5], p.659), it follows that the dimension of $G/(P,P)$ is $r+s$. The cell σ determined by P is of dimension r. We order the positive roots so that

$$S = \{\alpha_1, ..., \alpha_r\}\ ,\ S' = \{\alpha_1, ..., \alpha_s\}\ , (\sigma, S') > 0.$$

The space $\Omega^{0,q}(G/(P,P))$ is a module over the ring $C^\infty(G/(P,P))$, with dimension C_q^{r+s}. The next proposition constructs a basis for each module $\Omega^{0,q}(G/(P,P))$, and shows that $\bar\partial$ sends one basis into another. In other words, such basis satisfy the assumptions of Proposition 2.3. We use multiple index notation in the usual way. For instance if $I = (1, ..., q)$, then ξ_I denotes $\xi_1 \wedge ... \wedge \xi_q$.

Proposition 2.4 *There exist* $\xi_1, ..., \xi_s, \eta_1, ..., \eta_r \in \Omega^{0,1}(G/(P,P))$, *indexed according to the positive roots* $\alpha_1, ..., \alpha_s$, *such that:*

(a) *For each q,* $\{\xi_I \wedge \eta_J\}$ *is a basis of the module* $\Omega^{0,q}(G/(P,P))$, *where* $|I| + |J| = q, I \subset \{1, ..., s\}$ *and* $J \subset \{1, ..., r\}$.

(b) $\bar\partial \xi_I = \sum \xi_Q \wedge \xi_u \wedge \xi_v$, *summed over* $\alpha_u + \alpha_v = \alpha_t$, *and* t *satisfies* $(Q, t) = I$.

(c) $\bar\partial \eta_J = 0$ *for all* J.

Proof. Although Lemmas 3.2 and 3.3 of [3] work on the special case $P = B$ and $G/(P,P) = G/N$, the idea is essentially the same. Simply let ξ_i be as given there, and note equations (3.11), (3.12) there. Let η_i be as given by u_i of Lemma 3.2 there. The $(0,1)$-forms $\{\xi_i, \eta_j\}$ then satisfy

$$\bar\partial \xi_i = \sum_{\alpha_k + \alpha_l = \alpha_i} \xi_k \wedge \xi_l \quad \text{and} \quad \bar\partial \eta_j = 0 \tag{2.4}$$

for all $i = 1, ..., s$ and $j = 1, ..., r$. Parts (b), (c) of proposition follow from (2.4).

The values of $\{\xi_i, \eta_j\}$ at $e \in G/(P,P)$ form a basis of the cotangent space $\wedge^{0,1} T_e^* G/(P,P)$. Since they are all K-invariant and transform nicely under the right A_σ-action, their values at every $p \in G/(P,P)$ also form a basis. Hence $\{\xi_i, \eta_j\}$ is a basis of the module $\Omega^{0,1}(G/(P,P))$. Consequently, their exterior products become the basis of the various modules $\Omega^{0,q}(G/(P,P))$. Hence part (a) of the proposition. □

We now use Propositions 2.3 and 2.4 to show that the image of the Dolbeault operator (2.1) is closed, and prove Theorem 1.

Proof of Theorem 1. By Proposition 2.4(a), $\Omega^{0,q}(G/(P,P))$ and $\Omega^{0,q+1}(G/(P,P))$ are free modules over $C^\infty(G/(P,P))$, with basis $\{\xi_I \wedge \eta_J\}$ given in that proposition. Further, Proposition 2.4(b-c) says that such basis satisfy the assumptions of Proposition 2.3. Therefore, Proposition 2.3 implies that the image of the Dolbeault operator (2.1) is closed. Theorem 1 follows. □

3 Dolbeault cohomology

Let us consider the Dolbeault cohomology $H^{0,q}(G/(P,P))$. Recall from (1.2) that P determines a subalgebra $\mathfrak{h}_\sigma$, via Langlands decomposition. It has been proved in [3] that the dimension of $H^{0,q}(G/(P,P))$ is either 0 or infinite. In particular, it is infinite dimensional if there exists a Weyl group element $w \in W$ of length q and an integral weight $\lambda \in \mathfrak{h}_\sigma^*$ such that $w(\lambda+\rho)-\rho$ is dominant. Here ρ denotes half the sum of all positive roots. We now prove Theorem 2 by showing that conversely, if such w and λ are absent, then $H^{0,q}(G/(P,P)) = 0$.

Proof of Theorem 2. Let T_σ be the toral subgroup determined by P via (1.1) and (1.2). Since T_σ normalizes (P,P), it acts on $G/(P,P)$ on the right. The natural $K \times T_\sigma$-action on $G/(P,P)$ gives rise to a $K \times T_\sigma$-representation on $H^{0,q}(G/(P,P))$. Theorem 1 says that the representation space $H^{0,q}(G/(P,P))$ is a complete locally convex Hausdorff space. Since T_σ is compact, the Peter-Weyl theorem says that the right representation of T_σ contains no infinite dimensional irreducible subrepresentations ([2], p.141). Therefore, since T_σ is abelian, its irreducible subrepresentations are 1-dimensional. Each of them is contained in $H_\lambda^{0,q}(G/(P,P))$, consisting of cohomology classes that transform by $\lambda \in \mathfrak{h}_\sigma^*$ under the right T_σ-action. Suppose now that for given q, there exists no $w \in W$ of length q and integral weight $\lambda \in \mathfrak{h}_\sigma^*$ such that $w(\lambda+\rho)-\rho$ is dominant. By Theorem 2(ii) of [3], $H_\lambda^{0,q}(G/(P,P)) = 0$ for all $\lambda \in \mathfrak{h}_\sigma^*$. Consequently, since $H^{0,q}(G/(P,P))$ is a T_σ-representation with no infinite or 1-dimensional irreducible subrepresentation, it has to vanish. This proves Theorem 2. □

Suppose now that P is a maximal parabolic subgroup of G. We shall show that Theorem 2 simplifies considerably. From the simple roots $\{\alpha_i\}$, we get the corresponding dominant fundamental weights $\{\lambda_i\}$ satisfying $(\alpha_i, \lambda_j) = \delta_{ij}$. Let Δ^+ and Δ^- denote the positive and negative roots respectively, and let W be the Weyl group. Let D be the dominant Weyl chamber, and D° its interior.

Proof of Theorem 3. It is clear from the Borel-Weil theorem that $H^{0,q}(G/(P,P))$ is infinite dimensional for $q = 0$. We now consider the case where $q = \dim G/P$. By (1.1) and (1.2), the maximal parabolic subgroup P corresponds to a unique fundamental weight λ_k. Let $S_1 \subset \Delta^+$ be the positive roots spanned by the simple roots different from α_k, and let $S_2 = \Delta^+ \backslash S_1$. Since $(\lambda_k, S_1) = 0$ and $(\rho, S_1) > 0$, for all $n \in \mathbf{Z}$,

(3.1) $$(n\lambda_k + \rho, S_1) > 0.$$

We shall write $n \ll 0$ to denote negative numbers n of sufficiently large magnitude. Since $(\lambda_k, S_2) > 0$, for all integers $n \ll 0$,

(3.2) $$(n\lambda_k + \rho, S_2) < 0.$$

Let $w \in W$ be the element in which $w(n\lambda_k + \rho) \in D$ for all integers $n \ll 0$. In fact, since $n\lambda_k + \rho$ is regular, we get

(3.3) $$w(n\lambda_k + \rho) \in D^\circ \quad \text{and} \quad w(n\lambda_k + \rho) - \rho \in D.$$

By (3.1) and (3.3),

(3.4) $$(w(n\lambda_k + \rho), w(S_1)) > 0 \implies w(S_1) \subset \Delta^+.$$

Similarly, for integers $n \ll 0$, (3.2) and (3.3) imply that

(3.5) $$(w(n\lambda_k + \rho), w(S_2)) < 0 \implies w(S_2) \subset \Delta^-.$$

By (3.4) and (3.5),

(3.6) $$\text{length } w = |S_2| = \dim G/P.$$

Since (3.3) says that $w(n\lambda_k + \rho) - \rho$ is dominant, we can apply Theorem 2 to (3.6) and conclude that $H^{0,q}(G/(P,P))$ is infinite dimensional for $q = \dim G/P$.

Consider the lattice points in

$$\{n\lambda_k \; ; \; n \in \mathbf{Z}\} = \mathbf{Z}(\lambda_k) \subset \mathfrak{h}_\sigma^*.$$

For $n \geq 0$ and $n \ll 0$ respectively, there exist $w = 1$ and w in (3.6) such that $w(n\lambda_k + \rho) - \rho$ is dominant. For the remaining finitely many lattice points $\lambda \in \mathbf{Z}(\lambda_k)$, we cannot find w which make $w(\lambda + \rho) - \rho$ dominant: This is because there has to be either none or infinitely many $\lambda \in \mathbf{Z}(\lambda_k)$ such that $w(\lambda + \rho) - \rho$ is dominant [3]. Therefore, $H^{0,q}(G/(P,P)) = 0$ if q differs from 0 or $\dim G/P$. Theorem 3 follows. $\square$

If $G = SL(n, \mathbf{C})$ and if P is a maximal parabolic subgroup in G such that $G/P = \mathbf{CP}^{n-1}$, then $G/(P,P)$ identifies with $\mathbf{C}_0^n$, the non-zero vectors in $\mathbf{C}^n$. In this case Theorem 3 yields the known results on Dolbeault cohomology of $\mathbf{C}_0^n$ ([4], p.49). If P is an arbitrary maximal parabolic subgroup in $G = SL(n, \mathbf{C})$, then G/P is the Grassmannian X of all k-dimensional subspaces in $\mathbf{C}^n$ for some k with $1 \leq k < n$. Let $E \longrightarrow X$ be the universal bundle. It is a rank-k bundle, where the fiber over $p \in X$ is just the subspace $p \subset \mathbf{C}^n$ itself. Define the corresponding determinant line bundle by $L = \wedge^k E$, and let $L^\times$ denote L with the 0-section omitted. Now $G/(P,P)$ identifies with $L^\times$ and therefore Theorem 3 shows that

$$\dim H^{0,q}(L^\times) = \begin{cases} \infty & q = 0, k(n-k) \\ 0 & \text{otherwise.} \end{cases}$$

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