International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

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Published online: 08 Nov 2010.

To cite this article: Jwusheng Hu (1999) Active impedance control of linear one-dimensional wave equations, International Journal of Control, 72:3, 247-257

To link to this article: http://dx.doi.org/10.1080/002071799221226

INT. J. CONTROL, 1999, VOL. 72, NO. 3, 247–257

Active impedance control of linear one-dimensional wave equations

JWUSHENG HU†* and JYH-FENG LIN†

Active impedance control of linear one-dimensional wave equations is investigated. The proposed control algorithms are based on the concepts of wave propagation and impedance matching. Two control objectives are considered. The first objective is to obtain total reflection, and the second objective is to achieve total absorption (e.g. matched impedance). Both control laws utilize some interesting properties of the wave-type partial differential equation and do not require information about the disturbance and boundary conditions. The resulting controllers, although boundary-independent, contain an infinite number of poles on the imaginary axis and the closed-loop systems are not internally stable. A simple modification is added to the control laws and the stability is analysed. An example of active sound cancellation in ducts with a moving medium is given to demonstrate one of the control algorithms.

1. Introduction

A large class of distributed-parameter systems (DPS) such as acoustic, elastic and electromagnetic systems are governed by wave equations. Applications of active control techniques to those systems can be found in many engineering practices (e.g. active vibration control). Since the development of various control algorithms is much more advanced for finite-dimensional systems, it is natural to apply those algorithms directly to DPS by assuming a finite-dimensional approximation (e.g. modal expansion and truncation). Consequently, issues such as the stability, robustness and spill-over become very important.

Other than the problems described above, some physical properties explicitly appearing in partial differential equations (PDE) may be lost after finite-dimensional approximation. However, dealing directly with PDEs (no approximation) is not trivial, especially when non-linearity is involved. An active research area using PDE representation is boundary control (see, for example, Wang and Chen 1989, Bucci 1992, Mbodje and Montseng 1995). Extension of the optimal control approach to DPS has also been studied (Pohjolainen 1987, Morgül 1994). Recently, Helmicki *et al.* (1992) published their work on the ill-posed problem about modelling a DPS by PDE. Rather than deriving a suitable control law, many research work also focuses on stability, controllability/observability and robustness issues (Fabre 1992, Li and Ahmed 1992, Rebarber 1993, Cioranescu *et al.* 1994, Hu 1995).

This paper presents a different controller design approach for one-dimensional wave equations. It is motivated by the concepts of wave propagation and impedance matching. Impedance matching techniques are widely used in transmission lines to ensure maximum power transfer. In active vibration control, impedance matching means energy absorption. For example, to attenuate vibration of a finite-length string, one can actively change the impedance of a boundary to absorb the incident wave (Lu *et al.* 1989). This is equivalent to a matched impedance which results in zero energy reflection. Other related works using the idea of wave propagation include the transfer matrix method (von Flotow and Schäfer 1986) and boundary stabilization (Chen and Zhou 1990), as well as applications such as active noise control and cylindrical shell vibration suppression (Brévart and Fuller 1993).

The plant considered in this paper is governed by a class of linear one-dimensional wave equations with two boundary points. It is assumed that the boundary conditions can be described by simple impedance functions in the frequency domain. All disturbance and control sources are considered as point-type. The objective is to design control laws such that the wave generated by the noise source is blocked (e.g. total reflection) or passed (e.g. no reflection) at the control source location. Further, it is not necessary to know the boundary conditions of the plant.

It should be emphasized that this paper only presents an alternative controller design method using the property of wave propagation and the idea of impedance matching. Other important issues such as controllability/observability and robustness are not discussed here. Moreover, the proposed controller itself is also distributed in nature. Although the ideal control law is boundary-independent, the closed-loop system induces infinitely many marginal stable poles. A simple modification is presented and the corresponding stability requirement is analysed. The control law can be implemented by using delay operators. The main purpose of this paper is to reveal some interesting properties hidden in general linear wave equations, and show how those properties could be used to construct simple feedback control laws.

Received 24 February 1998. Revised 30 June 1998.
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2. The dynamic model

A class of stable, one-dimensional wave equations can be formulated non-dimensionally as

$$\left(A\frac{\partial^2}{\partial x^2}+B\frac{\partial^2}{\partial x\,\partial t}+C\frac{\partial^2}{\partial t^2}\right)P(x,t)=Q(x,t),\quad x\in(0,1),t>0 \tag{1}$$

where A, B, C are constants, $A\cdot C<0$ and $B^2-4AC>0$. $P(x,t)$ is the system response (output) and $Q(x,t)$ represents the external forcing term (input) of the system. For example, in acoustic systems, equation (1) is a one-dimensional approximation of the dynamic of sound in finite-length ducts. $P(x,t)$ is the acoustic pressure and $Q(x,t)$ may be the strength of a monopole or dipole.

The Laplace transform of equation (1) with zero initial conditions is

$$\left[A\frac{\partial^2}{\partial x^2}+Bs\frac{\partial}{\partial x}+Cs^2\right]\bar{P}(x,s)=\bar{Q}(x,s) \tag{2}$$

where $\bar{P}(x,s)$ and $\bar{Q}(x,s)$ are the Laplace transform of $P(x,t)$ and $Q(x,t)$, respectively. We introduce the homogeneous boundary conditions in the frequency domain as

$$\bar{P}(0,s)=Z_0(s)\frac{\partial\bar{P}(0,s)}{\partial x},\qquad \bar{P}(1,s)=-Z_1(s)\frac{\partial\bar{P}(1,s)}{\partial x} \tag{3}$$

where $Z_0(s)$ and $Z_1(s)$ are influence functions of each boundary. In fact, they are closely related to impedance functions. The impedance function is defined as the ratio of velocity (voltage) to force (current). For example, consider the longitudinal vibration of a uniform bar (Rao 1995) with cross-sectional area A and Young's modulus E. Both ends of the bar are connected to springs (with spring constants k_1 and k_2) and dampers (with damping constants c_1 and c_2), as shown in figure 1. The boundary conditions can be derived as

$$AE\frac{\partial P(0,t)}{\partial x}=k_1P(0,t)+c_1\frac{\partial P(0,t)}{\partial t}$$

at the left-hand end and

$$AE\frac{\partial P(1,t)}{\partial x}=-k_2P(1,t)-c_2\frac{\partial P(1,t)}{\partial t}$$

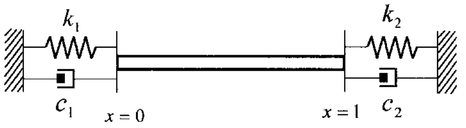

Figure 1. Schematic diagram of the longitudinal vibration of a bar.

at the right-hand end, where $P(\cdot,\cdot)$ is the axial displacement. Taking Laplace transforms of the above equations, we have the influence functions $Z_0(s)=AE/(k_1+c_1s)$ and $Z_1(s)=AE/(k_2+c_2s)$. In this case, the corresponding impedance functions are $sZ_0(s)/AE$ and $sZ_1(s)/AE$, i.e.

$$\frac{\bar{P}_t(0,s)}{AE\bar{P}_x(0,s}=\frac{sZ_0(s)}{AE},\qquad \frac{\bar{P}_1(1,s)}{AE\bar{P}_x(1,s)}=-\frac{sZ_1(s)}{AE}$$

For the sake of simplicity, we use influence functions (equation (3)) to derive the system model. Further, since the boundaries are considered homogeneous (i.e. no extra forcing terms), the influence function is different from the corresponding impedance function by only one factor.

Solving for the Green's function of equations (2) and (3) (Butkovskiy 1982, Yang and Tan 1992), we have

$$\bar{P}(x,s)=\frac{1}{\Delta}\int_0^1 G(x,\xi,s)\bar{Q}(\xi,s)\,\mathrm{d}\xi \tag{4}$$

where $\Delta = A$ (see bottom of page), λ_1 and λ_2 are distinct characteristic roots (i.e. $\lambda_1-\lambda_2\neq 0$) of the system and are derived by solving

$$A\lambda^2+Bs\lambda+Cs^2=0 \tag{5}$$

i.e.

$$\lambda_{1,2}=\left(\frac{-B\pm\sqrt{B^2-4AC}}{2A}\right)s$$

Note that both λ_1 and λ_2 are independent of the boundary conditions. If the distribution of the source $\bar{Q}(x,s)$ in space is known, integration in equation (4) can be carried out. Without loss of generality, we assume that the system is excited by point sources, i.e. for a point source located at $x=a$,

$$\bar{Q}(x,s)=\bar{Q}_a(s)\delta(x-a) \tag{6}$$

$$G(x,\xi,s)=\begin{cases}\text{for } x\geq\xi \\ \dfrac{[(1-Z_0(s)\lambda_1)\,e^{-\lambda_1\xi}-(1-Z_0(s)\lambda_2)\,e^{-\lambda_2\xi}][(1+Z_1(s)\lambda_1)\,e^{\lambda_1+\lambda_2x}-(1+Z_1(s)\lambda_2)\,e^{\lambda_2+\lambda_1x}]}{(\lambda_1-\lambda_2)[(1-Z_0(s)\lambda_2)(1+Z_1(s)\lambda_1)\,e^{\lambda_1}-(1-Z_0(s)\lambda_1)(1+Z_1(s)\lambda_2)\,e^{\lambda_2}]} \\ \text{for } x\leq\xi \\ \dfrac{[(1-Z_0(s)\lambda_1)\,e^{\lambda_2x}-(1-Z_0(s)\lambda_2)\,e^{\lambda_1x}][(1+Z_1(s)\lambda_1)\,e^{\lambda_1(1-\xi)}-(1+Z_1(s)\lambda_2)\,e^{\lambda_2(1-\xi)}]}{(\lambda_1-\lambda_2)[(1-Z_0(s)\lambda_2)(1+Z_1(s)\lambda_1)\,e^{\lambda_1}-(1-Z_0(s)\lambda_1)(1+Z_1(s)\lambda_2)\,e^{\lambda_2}]}\end{cases}$$

where $\bar{Q}_a(s)$ is the source strength. As a result, the transfer function of the plant described by equations (1)–(3) and (6) is (see (7)). Define the boundary reflection coefficients as

$$\theta_0(s) = \frac{1-\lambda_1 Z_0(s)}{1-\lambda_2 Z_0(s)}, \qquad \theta_1(s) = \frac{1+\lambda_2 Z_1(s)}{1+\lambda_1 Z_1(s)} \tag{8}$$

Using the definition of reflection coefficient in equation (8), the transfer functions given by equation (7) can be rearranged as (see (9)). To be able to realize proposed control laws, λ_1 and λ_2 are selected such that all the exponential terms on the right-hand side of equation (9) are causal operators. This point will be illustrated further in an example.

For a clear presentation of the proposed control law, the following notations are defined first.

$$G_D^+(x,a,s) = \frac{1}{(\lambda_1-\lambda_2)\Delta} \times [-e^{\lambda_2(x-a)} + \theta_0(s)\, e^{\lambda_2 x - \lambda_1 a}] \tag{10a}$$

$$G_D^-(x,a,s) = \frac{1}{(\lambda_1-\lambda_2)\Delta} \times [\theta_1(s)\, e^{\lambda_2(1-a)-\lambda_1(1-x)} - \theta_0(s)\theta_1(s)\, e^{\lambda_2-\lambda_1(1+a-x)}] \tag{10b}$$

$$G_U^+(x,a,s) = \frac{1}{(\lambda_1-\lambda_2)\Delta} [\theta_0(s)\, e^{\lambda_2 x - \lambda_1 a} - \theta_0(s)\theta_1(s)\, e^{\lambda_2(1+x-a)-\lambda_1}] \tag{10c}$$

$$G_U^-(x,a,s) = \frac{1}{(\lambda_1-\lambda_2)\Delta} \times [-e^{-\lambda_1(a-x)} + \theta_1(s)\, e^{\lambda_2(1-a)-\lambda_1(1-x)}] \tag{10d}$$

The subscripts/superscripts used in equations (10) are defined based on the nature of wave propagation. Subscripts D and U denote a downstream ($x > a$ in this case) and upstream position ($x < a$), respectively, while superscripts − and + relate energy (or wave) transmission in the upstream and downstream direction, respectively.

Suppose there are two point sources, $\bar{N}_d(s)$ (located at $x = d$) and $\bar{Q}_a(s)$ (at $x = a$), in the system. In what follows, $\bar{N}_d(s)$ is referred to as the primary source, which may represent a disturbance input, and $\bar{Q}_a(s)$ is called the secondary source, denoting a control input. For convenience, it is assumed that $d < a$. Based on the notations defined in equations (10), when $x < a$ the response of the system is written as

$$\bar{P}(x,s) = \frac{[G_U^+(x,a,s) + G_U^-(x,a,s)]}{1-\theta_0(s)\theta_1(s)\, e^{\lambda_2-\lambda_1}} \bar{Q}_a(s) + \frac{[G_U^+(x,d,s) + G_U^-(x,d,s)]}{1-\theta_0(s)\theta_1(s)\, e^{\lambda_2-\lambda_1}} \bar{N}_d(s) \tag{11a}$$

when $d < x < a$,

$$\bar{P}(x,s) = \frac{[G_U^+(x,a,s) + G_U^-(x,a,s)]}{1-\theta_0(s)\theta_1(s)\, e^{\lambda_2-\lambda_1}} \bar{Q}_a(s) + \frac{[G_D^+(x,d,s) + G_D^-(x,d,s)]}{1-\theta_0(s)\theta_1(s)\, e^{\lambda_2-\lambda_1}} \bar{N}_d(s) \tag{11b}$$

and when $x > a$,

$$\bar{P}(x,s) = \frac{[G_D^+(x,a,s) + G_D^-(x,a,s)]}{1-\theta_0(s)\theta_1(s)\, e^{\lambda_2-\lambda_1}} \bar{Q}_a(s) + \frac{[G_D^+(x,d,s) + G_D^-(x,d,s)]}{1-\theta_0(s)\theta_1(s)\, e^{\lambda_2-\lambda_1}} \bar{N}_d(s) \tag{11c}$$

Further, equations (10) possess some interesting properties which are important in deriving the proposed control laws. Letting x and y be two arbitrary points and $x, y \geq a$, we have

$$\bar{P}(x,s) = \begin{cases} \text{for } x \geq a \\ \dfrac{[(1-Z_0(s)\lambda_1)\, e^{-\lambda_1 a} - (1-Z_0(s)\lambda_2)\, e^{-\lambda_2 a}][(1+Z_1(s)\lambda_1)\, e^{\lambda_1+\lambda_2 x} - (1+Z_1(s)\lambda_2)\, e^{\lambda_2+\lambda_1 x}]}{(\lambda_1-\lambda_2)\Delta\,[(1-Z_0(s)\lambda_2)(1+Z_1(s)\lambda_1)\, e^{\lambda_1} - (1-Z_0(s)\lambda_1)(1+Z_1(s)\lambda_2)\, e^{\lambda_2}]} \bar{Q}_a(s) \\ \text{for } x \leq a \\ \dfrac{[(1-Z_0(s)\lambda_1)\, e^{\lambda_2 x} - (1-Z_0(s)\lambda_2)\, e^{\lambda_1 x}][(1+Z_1(s)\lambda_1)\, e^{\lambda_1(1-a)} - (1+Z_1(s)\lambda_2)\, e^{\lambda_2(1-a)}]}{(\lambda_1-\lambda_2)\Delta\,[(1-Z_0(s)\lambda_2)(1+Z_1(s)\lambda_1)\, e^{\lambda_1} - (1-Z_0(s)\lambda_1)(1+Z_1(s)\lambda_2)\, e^{\lambda_2}]} \bar{Q}_a(s) \end{cases} \tag{7}$$

$$\bar{P}(x,s) = \begin{cases} \text{for } x \geq a \\ \dfrac{[-e^{\lambda_2(x-a)} + \theta_0(s)\, e^{\lambda_2 x-\lambda_1 a}] + [\theta_1(s)\, e^{\lambda_2(1-a)-\lambda_1(1-x)} - \theta_0(s)\theta_1(s)\, e^{\lambda_2-\lambda_1(1+a-x)}]}{(\lambda_1-\lambda_2)\Delta\,(1-\theta_2(s)\theta_1(s)\, e^{\lambda_2-\lambda_1})} \bar{Q}_a(s) \\ \text{for } x \leq a \\ \dfrac{[\theta_0(s)\, e^{\lambda_2 x-\lambda_1 a} - \theta_0(s)\theta_1(s)\, e^{\lambda_2(1+x-a)-\lambda_1}] + [-e^{-\lambda_1(a-x)} + \theta_1(s)\, e^{\lambda_2(1-a)-\lambda_1(1-x)}]}{(\lambda_1-\lambda_2)\Delta\,(1-\theta_0(s)\theta_1(s)\, e^{\lambda_2-\lambda_1})} \bar{Q}_a(s) \end{cases} \tag{9}$$

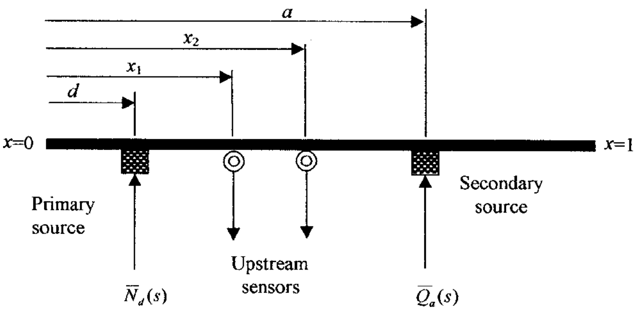

Figure 2. Schematic diagram of active impedance control (total reflection).

$$-G_D^-(x,a,s)G_D^+(x,d,s) + G_D^-(x,d,s)G_D^+(x,a,s) = 0 \tag{12a}$$

$$-G_D^+(x,a,s)G_D^+(y,d,s) + G_D^+(x,d,s)G_D^+(y,a,s) = 0 \tag{12b}$$

$$-G_D^-(x,a,s)G_D^+(y,d,s) + G_D^-(x,d,s)G_D^+(y,a,s) = 0 \tag{12c}$$

These equalities can be verified by simply plugging in the corresponding definitions.

3. Active control of impedance

A schematic diagram of the system described by equations (11) is shown in figure 2. The control objective is simple: determine a control law such that the impedance at $x = a$ is changed into a specified value. In the first case, explained later, the primary source is assumed to be a disturbance input and its excitation is required to be blocked at $x = a$. This means that the impedance (or the influence function) is either infinity or zero. A typical engineering application for this problem is the active noise cancellation in finite-length ducts (Hu 1995). For the second case, the impedance must be matched at $x = a$, i.e. no upstream reflection of the primary source excitation. This impedance matching problem is common in transmission lines exhibiting wave propagation phenomena. For example, mismatched impedance in fluid transport piping systems may lead to pressure build-up or structural vibration. In particular, knowledge of the impedance functions at both ends is not required since they are usually quite difficult to measure precisely.

3.1. *Total reflection*

A zero or infinite impedance at $x = a$ means that the corresponding reflection coefficient is one. To accomplish total reflection, the control law at $x = a$ is designed as

$$\bar{Q}_a(s) = \frac{-G_D^+(y,d,s)}{G_D^+(y,a,s)}\bar{N}_d(s) \tag{13}$$

where y is any location satisfying $y > a$. Further, $G_D^+(y,d,s)$ and $G_D^+(y,a,s)$ have the properties

$$G_D^+(y,d,s) = G_D^+(y_1,d,s)\,e^{\lambda_2(y-y_1)}, \qquad y_1 > d$$

and

$$G_D^+(y,a,s) = G_D^+(y_1,a,s)\,e^{\lambda_2(y-y_1)}, \qquad y_1 > a$$

Putting the above two equations into equation (13) shows that the control law is independent of y. Further, the response $\bar{P}(x,s)$ at any downstream location (i.e. $x > a$) will satisfy

$$\bar{P}(x,s) = 0 \tag{14}$$

Equation (14) can easily be verified by substituting equation (13) into equation (11 *c*) and using the equality of equations (12 *b*) and (12 *c*) (see Appendix). Since the plant is assumed to be stable, equation (14) means that the response at any downstream location will go to zero. In other words, the excitation of the primary source is completely blocked at $x = a$. Moreover, the response at the upstream area becomes (see (15)) when $x < d$, and (see (16)) when $d < x < a$. Denoting $\lambda_1 = \lambda_1'/a$ and $\lambda_2 = \lambda_2'/a$, equations (15) and (16) are changed into the same type as equations (11 *a*) and (11 *b*) by scaling. Therefore, it can be verified that equations (15) and (16) are the exact solution of the PDE,

$$\left(A\frac{\partial^2}{\partial x^2} + B + \frac{\partial^2}{\partial x\,\partial t} + C\frac{\partial^2}{\partial t^2}\right)P(x,t) = N(t)\delta(x-d), \qquad x \in (0,a), t > 0 \tag{17}$$

with boundary conditions

$$\bar{P}(0,s) = Z_0(s)\frac{\partial \bar{P}(0,s)}{\partial x}, \qquad \bar{P}(a,s) = 0 \tag{18}$$

In other words, the boundary impedance at $x = a$ is made zero, which is totally reflective.

Equation (13) can be realized by using signals measured from two upstream locations. Considering two sen-

$$\bar{P}(x,s) = \frac{\theta_0(s)\,e^{\lambda_2 x-\lambda_1 d} - \theta_0(s)\,e^{\lambda_2(x+a-d)-\lambda_1 a} - e^{-\lambda_1(d-x)} + e^{-\lambda_2(a-d)-\lambda_1(a-x)}}{(\lambda_1-\lambda_2)\Delta\,(1-\theta_0(s)\,e^{(\lambda_2-\lambda_1)a})}\bar{N}_d(s) \tag{15}$$

$$\bar{P}(x,s) = \frac{-e^{\lambda_2(x-d)} + \theta_0(s)\,e^{\lambda_2 x-\lambda_1 d} + e^{\lambda_2(a-d)-\lambda_1(a-x)} - \theta_0(s)\,e^{\lambda_2 a-\lambda_1(a+d-x)}}{(\lambda_1-\lambda_2)\Delta\,(1-\theta_0(s)\,e^{(\lambda_2-\lambda_1)a})}\bar{N}_d(s) \tag{16}$$

sors placed at x_1 and x_2, as shown in figure 2 whose measurements are denoted as $\bar{P}(x_1, s)$ and $\bar{P}(x_2, s)$, respectively, we have

$$\hat{P}_1 - \hat{P}_2 \, e^{-\lambda_1(x_2-x_1)} = [1 - e^{(\lambda_2-\lambda_1)(x_2-x_1)}] \times [G_U^+(x_1, a, s)\bar{Q}_a(s) + G_D^+(x_1, d, s)\bar{N}_d(s)] \quad (19)$$

where

$$\hat{P}_1 = \bar{P}(x_1, s) - \theta_0(s)\theta_1(s)\, e^{\lambda_2-\lambda_1}\, \bar{P}(x_1, s) \quad (20\,a)$$

and

$$\hat{P}_2 = \bar{P}(x_2, s) - \theta_0(s)\theta_1(s)\, e^{\lambda_2-\lambda_1}\, \bar{P}(x_2, s) \quad (20\,b)$$

Substituting equation (13) into equations (20) results in

$$\bar{Q}_a(s) = \frac{(\lambda_1 - \lambda_2)\Delta\,[\bar{P}(x_1, s) - \bar{P}(x_2, s)\, e^{-\lambda_1(x_2-x_1)}]\, e^{\lambda_2(a-x_1)}}{[1 - e^{(\lambda_2-\lambda_1)(x_2-x_2)}]} \quad (21)$$

The most important point about equation (21) is that the control law does not require information about the disturbance signal and boundary conditions of the plant. Further, the time-domain counterpart of equation (21) can easily be found and implemented by using delay operators (see section 5).

3.2. *Impedance matching*

In contrast to total reflection, it is also possible to design the control law such that no reflection occurs for wave propagating toward the boundary of $\theta_1(s) \neq 0$. The control law proposed is

$$\bar{Q}_a(s) = \frac{-G_D^-(y, d, s)}{G_U^-(y, a, s)}\bar{N}_d(s) \quad (22)$$

where $d < y < a$. Notice that the control law $\bar{Q}_a(s)$ has the same property as equation (13) in that it is independent of the spatial variable. Substituting equaton (22) into equations (11 *a*) and (11 *b*), the response $\bar{P}(x, s)$ satisfies

$$\left[\bar{P}(x, s) - \frac{1}{(\lambda_1 - \lambda_2)\Delta}(\theta_0(s)\, e^{\lambda_2 x - \lambda_1 d} - e^{-\lambda_1(d-x)})\bar{N}_d(s)\right] = 0 \quad (23)$$

when $x < d$, and

$$[\bar{P}(x, s) - G_D^+(x, d, s)\bar{N}_d(s)] = 0 \quad (24)$$

when $d < x < a$.
In other words, after adding the secondary source $\bar{Q}_a(s)$, the wave travelling downstream will be absorbed exactly, regardless of what value $\theta(s)$ has at $x = 1$, i.e. the system response becomes

$$\bar{P}(x, s) = \frac{1}{(\lambda_1 - \lambda_2)\Delta}[\theta_0(s)\, e^{\lambda_2 x - \lambda_1 d} - e^{-\lambda_1(d-x)}]\bar{N}_d(s) = [G_U^+(x, d, s)|_{\theta_1(s)=0} + G_U^-(x, d, s)|_{\theta_1(s)=0}]\bar{N}_d(s) \quad (25)$$

when $x < d$, and

$$\bar{P}(x, s) = G_D^+(x, d, s)\bar{N}_d(s) \quad (26)$$

when $d < x < a$. It can be verified that equations (25) and (26) are the PDE solution of equation (17), with the following boundary conditions:

$$\bar{P}(0, s) = Z_0(s)\frac{\partial \bar{P}(0, s)}{\partial x}, \qquad \bar{P}(a, s) = \frac{1}{\lambda_2}\frac{\partial \bar{P}(a, s)}{\partial x} \quad (\text{i.e. } \theta = a(s) = 0) \quad (27)$$

where $\theta_a(s)$ is the boundary reflection coefficient at $x = a$. Equation (22) can be realized as in section 3.1 by using signals measured from two downstream locations ($x > a$).

Considering two sensors placed at x_1 and x_2, as shown in figure 3, with corresponding measurements denoted as $\bar{P}(x_1, s)$ and $\bar{P}(x_2, s)$, we have

$$\hat{P}_1\, e^{\lambda_2(x_2-x_1)} - \hat{P}_2 = [e^{(\lambda_2-\lambda_1)(x_2-x_1)} - 1] \times [G_D^-(x_2, a, s)\bar{Q}_a(s) + G_D^-(x_2, d, s)\bar{N}_d(s)] \quad (28)$$

where $\hat{P}_1, \hat{P}_2$ are the same type as equatons (20 *a*) and (20 *b*). Substituting equation (22) into equation (28) results in

$$\bar{Q}_a(s) = \frac{(\lambda_1 - \lambda_2)\Delta\,(\bar{P}(x_2, s) - \bar{P}(x_1, s)\, e^{\lambda_2(x_2-x_1)})\, e^{-\lambda_1(x_2-a)}}{[1 - e^{(\lambda_2-\lambda_1)(x_2-x_1)}]} \quad (29)$$

Equation (29) has the same property as equaiton (21), i.e. the control law does not need information about the primary source and boundary conditions of the plant.

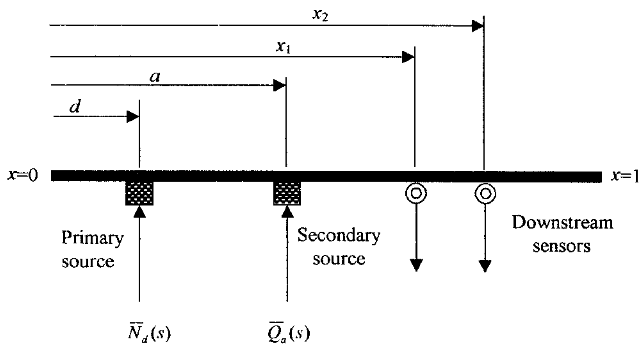

Figure 3. Schematic diagram of active impedance control (impedance matching).

The system response in the downstream area becomes

$$\bar{P}(x,s) = \frac{[G_{\mathrm{D}}^{+}(x,d,s) + G_{\mathrm{D}}^{-}(x,d,s)]}{1-\theta_1(s)\,\mathrm{e}^{(\lambda_2-\lambda_1)(1-a)}}\bar{N}_d(s) \qquad (30)$$

4. Stability analysis and controller modification

Suppose that both boundaries of the system are considered as passive, i.e. no reflecting wave is generated in the absence of an incident wave. From the physical viewpoint, the system is either stable or marginally stable. In this paper, we consider only the stable case by making the following assumption.

Assumption 1: *The boundary reflection coefficients (equation (8)) satisfy* $|\theta_0(j\omega)| < 1$ *and* $|\theta_1(j\omega)| < 1$, $\forall\omega > 0$.

Under the above assumption, the following characteristic equations have stable roots no matter what $T > 0$ is:

(1) $1-\theta_0(s)\theta_1(s)\,\mathrm{e}^{-Ts} = 0$
(2) $1-\theta_0(s)\,\mathrm{e}^{-Ts} = 0$
(3) $1-\theta_1(s)\,\mathrm{e}^{-Ts} = 0$

4.1. *Total reflection*

The closed-loop schematic diagram of the active impedance control problem, i.e. total reflection, is shown in figure 4, and the system can be further represented by the block diagram shown in figure 5. From equations (11 *a*)–(11 *c*) and (21), it can be seen that the composition of each block can be written as

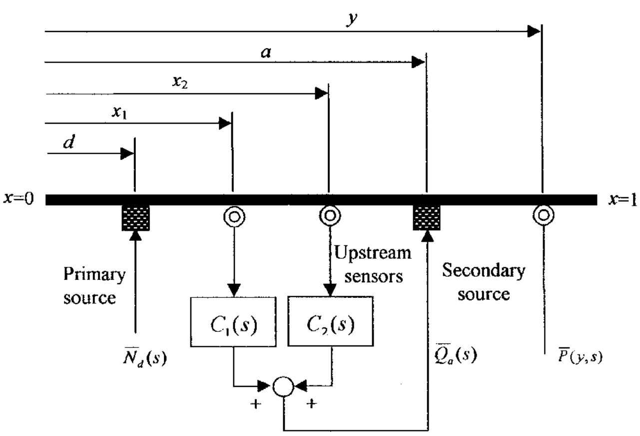

Figure 4. Schematic diagram of active impedance control.

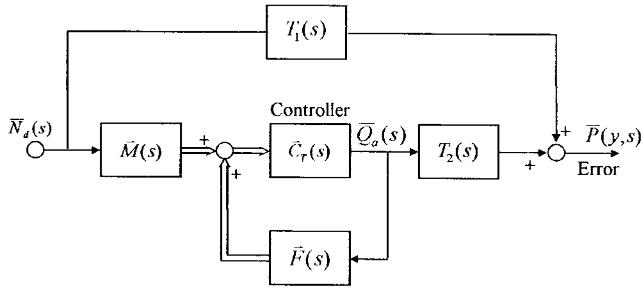

Figure 5. Block diagram of an active control system with two sensors in the downstream area.

$$T_1(s) = \frac{G_{\mathrm{D}}^{+}(y,d,s) + G_{\mathrm{D}}^{-}(y,d,s)}{X(s)} \qquad (31)$$

$$T_2(s) = \frac{G_{\mathrm{D}}^{+}(y,a,s) + G_{\mathrm{D}}^{-}(y,a,s)}{X(s)} \qquad (32)$$

$$C_T(s) \equiv [C_1(s) \;\; C_2(s)] = \frac{(\lambda_1-\lambda_2)\Delta\,\mathrm{e}^{\lambda_2(a-x_1)}}{1-\mathrm{e}^{(\lambda_2-\lambda_1)(x_2-x_2)}}[1-\mathrm{e}^{-\lambda_1(x_2-x_1)}] \qquad (33)$$

$$\bar{M}(s) \equiv \begin{bmatrix} M_1(s) \\ M_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{G_{\mathrm{D}}^{+}(x_1,d,s)+G_{\mathrm{D}}^{-}(x_1,d,s)}{X(s)} \\ \dfrac{G_{\mathrm{D}}^{+}(x_2,d,s)+G_{\mathrm{D}}^{-}(x_2,d,s)}{X(s)} \end{bmatrix} \qquad (34)$$

$$\bar{F}(s) \equiv \begin{bmatrix} F_1(s) \\ F_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{G_{\mathrm{U}}^{+}(x_1,a,s)+G_{\mathrm{U}}^{-}(x_1,a,s)}{X(s)} \\ \dfrac{G_{\mathrm{U}}^{+}(x_2,a,s)+G_{\mathrm{U}}^{-}(x_2,a,s)}{X(s)} \end{bmatrix} \qquad (35)$$

$$X(s) = 1-\theta_0(s)\theta_1(s)\,\mathrm{e}^{\lambda_2-\lambda_1} \qquad (36)$$

The stability problem of the closed-loop system with a two-sensor control law in figure 5 can be analysed by using the following lemma:

Lemma 1: *The closed-loop feedback system in figure 5 is internally stable if and only if* $[1-\bar{C}^T(s)\bar{F}(s)]^{-1}\bar{C}_T(s)$ *is stable.*

Proof: The stability problem considered in figure 5 is equivalent to figure 6 under Assumption 1 that $\bar{F}(s)$,

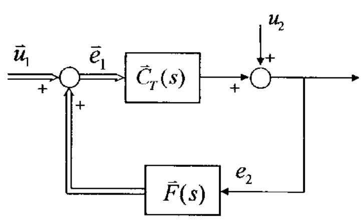

Figure 6. Diagram of an active controller subblock.

$\bar{M}(s)$, $T_1(s)$ and $T_2(s)$ in figure 5 are stable, and the feedback system in figure 6 can be described by

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \equiv \bar{H}(s) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} [I - \bar{F}(s)\bar{C}_T(s)]^{-1} & [I - \bar{F}(s)\bar{C}_T(s)]^{-1}\bar{F}(s) \\ [1 - \bar{C}_T(s)\bar{F}(s)]^{-1}\bar{C}_T(s) & [1 - \bar{C}_T(s)\bar{F}(s)]^{-1} \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (37)$$

From equation (37), the stability of the four transfer matrices in $\bar{H}(s)$ can be ascertained if and only if the transfer matrix $[1 - \bar{C}_T(s)\bar{F}(s)]^{-1}\bar{C}_T(s)$ is stable (Vidyasagar 1985).

From equations (33) and (35) we have

$$[1 - \bar{C}_T(s)\bar{F}(s)]^{-1}\bar{C}_T(s) = \frac{1 - \theta_0(s)\theta_1(s)\, e^{(\lambda_2-\lambda_1)}}{1 - \theta_0(s)e^{(\lambda_2-\lambda_1)a}} \frac{(\lambda_1 - \lambda_2)\Delta\, e^{\lambda_2(a-x_1)}}{1 - e^{(\lambda_2-\lambda_1)(x_2-x_1)}} \times [1 - e^{-\lambda_1(x_2-x_1)}]$$

By Lemma 1, the closed-loop feedback system in figure 5 is not internally stable because $[1 - \bar{C}_T(s)\bar{F}(s)]^{-1}\bar{C}_T(s)$ is unstable (marginally stable). To resolve the stability issue, the optimal controller is modified by adding a 'notch filter' $R_\varepsilon(s)$, i.e.

$$\bar{C}_m(s) = [C_{1m}(s) \;\; C_{2m}(s)] = [C_1(s)R_\varepsilon(s) \;\; C_2(s)R_\varepsilon(s)] \quad (38)$$

where the notch filter is

$$R_\varepsilon(s) = \frac{1 - e^{(\lambda_2-\lambda_1)(x_2-x_1)}}{1 - (1-\varepsilon)\, e^{(\lambda_2-\lambda_1)(x_2-x_1)}}, \quad \varepsilon = \{\varepsilon | \varepsilon > 0, \varepsilon \ll 1\} \quad (39)$$

Substituting λ_1 and λ_2 (equation (5)) into equation (39), it is obvious that $R_\varepsilon(s)$ have an infinite number of 'notches' located at

$$\omega_k = \frac{A}{\sqrt{B^2 - 4AC}} \frac{2k\pi}{(x_2 - x_1)}, \qquad k = 1, 2, 3, \ldots$$

The value ε in equation (39) determines the amount of maximum amplitude attenuation at the point of 'notch'. Substituting the modified controller into the block diagram, the performance degradation (response at y, $y > a$) becomes

$$\bar{P}(y,s) = \frac{\varepsilon\left(\frac{1 - \theta_0(s)\theta_1(s)\, e^{(\lambda_2-\lambda_1)}}{1 - \theta_0(s)\, e^{(\lambda_2-\lambda_1)a}}\right) e^{(\lambda_2-\lambda_1)(x_2-x_1)}}{1 - \left(1 - \varepsilon\frac{1 - \theta_0(s)\theta_1(s)\, e^{(\lambda_2-\lambda_1)}}{1 - \theta_0(s)\, e^{(\lambda_2-\lambda_1)a}}\right) e^{(\lambda_2-\lambda_1)(x_2-x_1)}} T_1(s)\bar{N}_d(s) \quad (40a)$$

where $T_1(s)$ is given in equation (31). At notch frequencies, equation (40 a) becomes (substituting equations (31) and (36))

$$\bar{P}(y, j\omega_k) = T_1(j\omega_k)N_d(j\omega_k) = \frac{[G_D^+(y, d, j\omega_k) + G_D^-(y, d, j\omega_k)]}{1 - \theta_0(j\omega_k)\theta_1(j\omega_k)\, e^{\lambda_2-\lambda_1}} \bar{N}_d(j\omega_k) \quad (40b)$$

Referring to equation (11 c), equation (40 b) shows that the response at those notch frequencies is as if there is no control applied.

The stability of the feedback system using $\bar{C}_m(s)$ can be tested by Lemma 1. Since $\bar{C}_m(s)$ is stable, the internal stability is guaranteed if $[1 - \bar{C}_m(s)\bar{F}(s)]^{-1}$ is stable, where

$$[1 - \bar{C}_m(s)\bar{F}(s)]^{-1} = \frac{1}{1 - C_1(s)F_1(s)R_\varepsilon(s) - C_2(s)F_2(s)R_\varepsilon(s)} = \frac{1 - (1-\varepsilon)\, e^{(\lambda_2-\lambda_1)(x_2-x_1)}}{S(s)} \times \frac{1}{1 - \left(1 - \frac{\varepsilon}{S(s)}\right) e^{(\lambda_2-\lambda_1)(x_2-x_1)}} \quad (41)$$

where

$$S(s) = \frac{1 - \theta_0(s)\, e^{(\lambda_2-\lambda_1)a}}{1 - \theta_0(s)\theta_1(s)\, e^{(\lambda_2-\lambda_1)}} \quad (42)$$

Since $S(s)$ and $1/S(s)$ are stable by Assumption 1, using the Nyquist criterion, equation (41) is stable if $|1 - [\varepsilon/S(j\omega)]| < 1$, $\forall\omega$. The following corollary gives a sufficient condition. □

Corollary 1: *Under Assumption* 1, $[1 - \bar{C}_m(s)\bar{F}(s)]^{-1}$ *is stable if the following conditions hold:*

(1) $$\sup_{\omega>0}\{\sin^{-1}|\theta_0(j\omega)|\} + \sup_{\omega>0}\{\sin^{-1}|\theta_0(j\omega)\theta_1(j\omega)|\} = \alpha_1 + \alpha_2 < \frac{\pi}{2} \quad (43)$$

(2) $$0 < \varepsilon < \bar{\varepsilon}, \quad \varepsilon < \bar{\varepsilon} = \inf_{\omega>0}\{2\mathrm{Re}(S(j\omega))\} = \inf_{\omega>0}\{S(j\omega) + S(-j\omega)\} \quad (44)$$

where

$$S(j\omega) = \frac{1 - \theta_0(j\omega)\, e^{(\lambda_2-\lambda_1)a}}{1 - \theta_0(j\omega)\theta_1(j\omega)\, e^{(\lambda_2-\lambda_1)}} \quad (45)$$

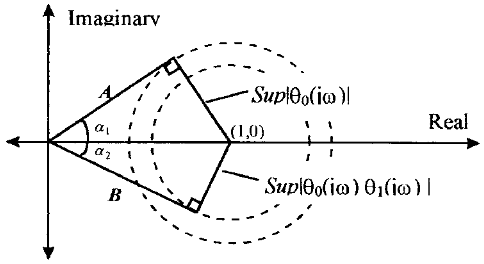

Figure 7. The possible maximum difference angle between $\angle(1-\theta_0(j\omega)\,e^{(\lambda_2-\lambda_1)a})$ and $\angle(1-\theta_0(j\omega)\theta_1(j\omega)\,e^{(\lambda_2-\lambda_1)})$. Note that A represents the candidate vector of $(1-\theta_0(j\omega)\,e^{(\lambda_2-\lambda_1)a})$ and B represents the candidate vector of $(1-\theta_0(j\omega)\theta_1(j\omega)\,e^{(\lambda_2-\lambda_1)})$.

Proof: Given $|\theta_0(j\omega)|<1$ and $|\theta_1(j\omega)|<1$, the maximum possible angle of $S(j\omega)$ can be shown graphically as in figure 7. Therefore, equation (43) guarantees that $\mathrm{Re}(S(j\omega))>0$. Further, from equation (44) we have

$$\begin{aligned}\left|1-\frac{\varepsilon}{S(j\omega)}\right|^2 &= 1-\varepsilon\left(\frac{1}{S(j\omega)}+\frac{1}{S(-j\omega)}\right)+\frac{\varepsilon^2}{S(j\omega)S(-j\omega)}\\ &= 1-\frac{\varepsilon}{S(j\omega)S(-j\omega)}[(S(j\omega)+S(-j\omega))-\varepsilon]\\ &\le 1-\frac{\varepsilon}{|S(j\omega)|^2}(\bar{\varepsilon}-\varepsilon)\\ &< 1 \end{aligned} \tag{46}$$

By the Nyquist criterion, we conclude that $[1-C_m(s)F(s)]^{-1}$ is stable. □

Remark 1: The range for which the magnitude of boundary reflection coefficients $|\theta_0(j\omega)|$ and $|\theta_1(j\omega)|$ satisfy equation (43) is relatively large, as indicated in figure 8.

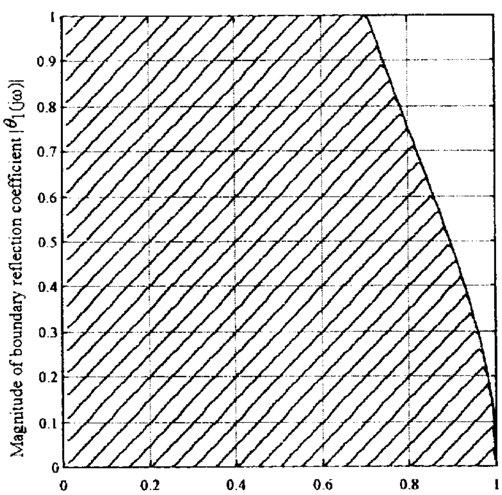

Figure 8. The range of boundary reflection coefficients that satisfy equation (42) (shaded area).

4.2. *Impedance matching*

The stability problem of impedance matching in section 3.2 is similar to the case of total reflection (section 4.1). Therefore, we conclude this section by providing the following corollary:

Corollary 2: *Under Assumption* 1, *the optimal controller in equation* (29) *is modified by adding the same notch filter* $R_\varepsilon(s)$ *as in equation* (39). *Then the closed-loop system is stable if the following conditions hold:*

(1) $$\sup_{\omega>0}\{\sin^{-1}|\theta_1(j\omega)|\}+\sup_{\omega>0}\{\sin^{-1}|\theta_0(j\omega)\theta_1(j\omega)|\} = \alpha_1+\alpha_2<\frac{\pi}{2} \tag{47}$$

(2) $$0<\varepsilon<\bar{\varepsilon},\quad \bar{\varepsilon}=\inf_{\omega>0}\{D(j\omega)+D(-j\omega)\} \tag{48}$$

where

$$D(j\omega)=\frac{1-\theta_1(s)\,e^{(\lambda_2-\lambda_1)(1-a)}}{1-\theta_0(s)\theta_1(s)\,e^{(\lambda_2-\lambda_1)}} \tag{49}$$

5. Example

Many physical systems can be described by the PDE in equation (1), e.g. sound propagation of a finite-length duct, and vibration control of the axially moving string and impedance matching in power transmission lines. We now give an example of noise cancellation in a finite-length duct with a moving medium to demonstrate the proposed control law derived in section 3.1.

5.1. *Finite-length duct with a moving medium*

The one-dimensional sound propagation through a moving medium in a hard-walled duct can be expressed as

$$\left[\frac{\partial^2}{\partial T^2}+2V_0\frac{\partial^2}{\partial X\partial T}-(\upsilon^2-V_0^2)\frac{\partial^2}{\partial X^2}\right]p(X,T)=\rho\upsilon^2 q(X,T),\quad X\in(0,L),\ T>0 \tag{50}$$

where L is length of a duct, ρ and υ are the ambient mass density and speed of sound, respectively, $V_0<\upsilon$ is the speed of the medium travelling in the downstream direction, and $p(X,T)$ and $q(X,T)$ are sound pressure and source strength, respectively, using the following non-dimensional variables:

$$t=\frac{\upsilon}{L}T,\quad x=\frac{X}{L},\quad P=\frac{p}{\rho\upsilon^2},\quad c=\frac{V_0}{\upsilon},\quad Q=\frac{L^2}{c^2}q \tag{51}$$

The non-dimensional hyperbolic PDE is equal to

$$\frac{\partial^2 P(x,t)}{\partial t^2}+2c\frac{\partial^2 P(x,t)}{\partial x\partial t}-(1-c^2)\frac{\partial^2 P(x,t)}{\partial x^2}=Q(x,t) \tag{52}$$

We adopt the concept of acoustic impedance (Morse and Ingard 1968) and introduce homogeneous boundary conditions at $x = 0$ and 1 in the frequency domain as

$$\bar{P}(0,s) = Z_0(s)\frac{\partial \bar{P}(0,s)}{\partial x}, \qquad \bar{P}(1,s) = -Z_1(s)\frac{\partial \bar{P}(1,s)}{\partial x} \tag{53}$$

$Z_0(s)$ and $Z_1(s)$ are the specific boundary impedances at $x = 0$ and $x = 1$, respectively. Supposing that both boundaries are considered as passive, i.e. no reflecting wave is generated in the absence of an incident wave, then both $Z_0(s)$ and $Z_1(s)$ are positive-real transfer functions.

Comparing equations (52) and (1), we have

$$A = -(1-c^2), \qquad B = 2c, \qquad C = 1 \tag{54}$$

Substituting equation (54) into equation (5), we can find the characteristic roots of the system as $s/(1-c)$ and $-s/(1+c)$. Selecting $\lambda_1 = s/1-c$ and $\lambda_2 = -s/1+c$ to satisfy causality requirements, equations (10 *a*)–(10 *d*) can be rewritten as

$$G_D^+(x,a,s) = \frac{-1}{2s}\left[-e^{-\frac{s}{1+c}(x-a)} + \theta_0(s)\,e^{-\frac{s}{1+c}x-\frac{s}{1-c}a}\right] \tag{55 a}$$

$$G_D^-(x,a,s) = \frac{-1}{2s}\left[\theta_1(s)\,e^{-\frac{s}{1+c}(1-a)-\frac{s}{1-c}(1-x)} - \theta_0(s)\theta_1(s)\,e^{-\frac{s}{1+c}-\frac{s}{1-c}(1+a-x)}\right] \tag{55 b}$$

$$G_U^+(x,a,s) = \frac{-1}{2s}\left[\theta_0(s)\,e^{-\frac{s}{1+c}x-\frac{s}{1-c}a} - \theta_0(s)\theta_1(s)\,e^{-\frac{s}{1+c}(1+x-a)-\frac{s}{1-c}}\right] \tag{55 c}$$

$$G_U^-(x,a,s) = \frac{-1}{2s}\left[-e^{-\frac{s}{1-c}(a-x)} + \theta_1(s)\,e^{-\frac{s}{1+c}(1-a)-\frac{s}{1-c}(1-x)}\right] \tag{55 d}$$

where $\theta_0(s)$ and $\theta_1(s)$ are defined in equation (8). Since both boundaries are passive, the duct itself is stable. It is reasonable to assume that the boundary reflection coefficients satisfy Assumption 1.

As a result, the controller is derived as (equation (38))

$$\bar{Q}_a(s) = [C_{1m}(s) \;\; C_{2m}(s)]\begin{bmatrix}\bar{P}(x_1,s)\\ \bar{P}(x_2,s)\end{bmatrix} = \frac{-2s\left[\bar{P}(x_1,s) - \bar{P}(x_2,s)\,e^{-\frac{s}{1-c}(x_2-x_1)}\right]e^{-\frac{s}{1+c}(a-x_1)}}{1-(1-\varepsilon)e^{-\frac{2s}{1-c^2}(x_1-x_1}} \tag{56}$$

Taking the inverse Laplace transform, equation (56) is realized in the time domain as

$$Q_a(t) = (1-\varepsilon)Q_a(t-t_1) - 2\left\{\frac{d}{dt}[P(x_1,t-t_2) - P(x_2,t-t_2-t_3)]\right\} \tag{57}$$

where

$$t_1 = \frac{2}{1-c^2}(x_2-x_1), \qquad t_2 = \frac{1}{1+c}(a-x_1), \qquad t_3 = \frac{1}{1-c}(x_2-x_1)$$

We conclude this section by providing the following example.

5.2. *Numerical simulation*

A point external disturbance $N_d(t) = \sin(20\pi t)$ is applied at $d = 0.2$ during the time-interval $0.0 \le t \le 0.1$, an active controller is supplied at $a = 0.7$ and the sensors are located at $x_1 = 0.4$ and $x_2 = 0.415$. For convenience, let the transport velocity $c = 0.5$. The homogeneous boundary condition $\theta_0(s) = \theta_1(s) = 0.7$ is assumed, from equation (43) we have $\bar{\varepsilon} \cong 0.37$ and the control law developed in equation (57) can be written as (let $\varepsilon = 0.01$)

$$Q_a(t) = 0.99 \cdot Q_a(t-0.04) - 2\cdot\{P_t(x_1, t-0.2) - P_t(x_2, t-0.23)\}$$

The frequency response of the noise in the duct with and without active control is shown in figure 9. From figure 9 we can see that the first notch frequency is located at

$$\omega_1 = \frac{(1-c^2)\pi}{x_2-x_1} \cong 157.0$$

6. Conclusion

Active impedance control of one-dimensional wave equations is discussed. Both total reflection and impedance matching control are considered. Using the control laws proposed, we can achieve total reflection or impedance matching conditions in the upstream section. It is shown that different types of source excitation result in different blocking or matching conditions. An example of active noise control in ducts with a moving medium is given to demonstrate the proposed control law, and a simulation result is shown.

Acknowledgment

This research work was supported in part by the National Science Council of Taiwan under grant number NSC86-2218-E009-027.

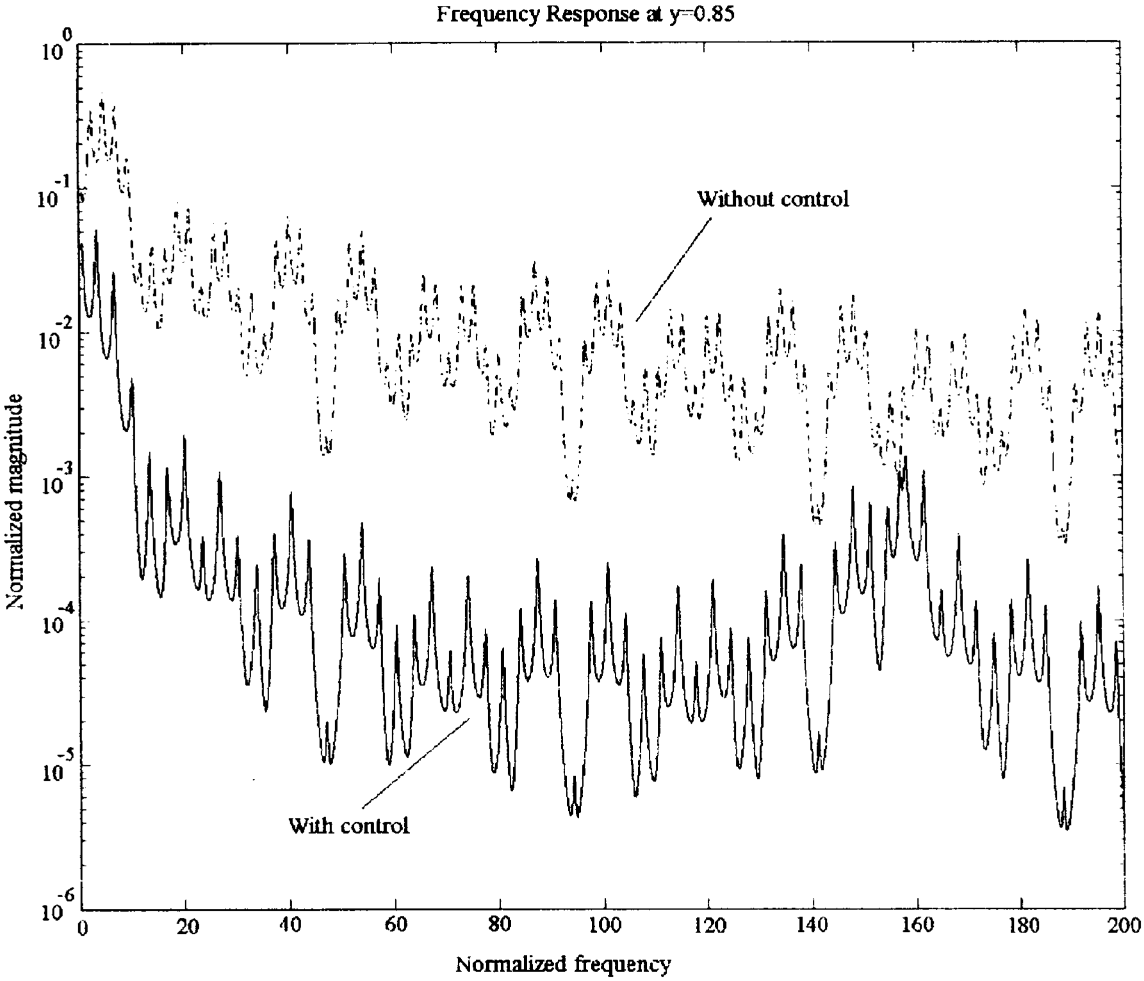

Figure 9. The frequency response of active noise cancellation measured at $\bar{P}(0.85, s)$. Broken line, without active control; solid line, after active control.

Appendix

Proof of equation (14): Substituting equation (13) into equation (11 c) we have

$$\bar{P}(x,s) = \frac{-G_D^+(y,d,s)G_D^+(x,a,s) - G_D^+(y,d,s)G_D^-(x,a,s)}{(1-\theta_0(s)\theta_1(s)\,e^{\lambda_2-\lambda_1})G_D^+(y,a,s)}\bar{N}_d(s)$$
$$+\frac{G_D^+(x,d,s)G_D^+(y,a,s) + G_D^-(x,d,s)G_D^+(y,a,s)}{(1-\theta_0(s)\theta_1(s)\,e^{\lambda_2-\lambda_1})G_D^+(y,a,s)}\bar{N}_d(s)$$
$$= \frac{-G_D^+(y,d,s)G_D^+(x,a,s) + G_D^+(x,d,s)G_D^+(y,a,s)}{(1-\theta_0(s)\theta_1(s)\,e^{\lambda_2-\lambda_1})G_D^+(y,a,s)}\bar{N}_d(s)$$
$$+\frac{-G_D^+(y,d,s)G_D^-(x,a,s) + G_D^-(x,d,s)G_D^+(y,a,s)}{(1-\theta_0(s)\theta_1(s)\,e^{\lambda_2-\lambda_1})G_D^+(y,a,s)}\bar{N}_d(s)$$

By equations (12 b) and (12 c) we have

$$\bar{P}(x,s) = 0 \qquad \square$$

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