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Asymptotic stabilization of driftless systems

DER-CHERNG LIAW†* and YEW-WEN LIANG†

Issues of asymptotic stabilization of a class of non-linear driftless systems are presented. In addition to the necessary and sufficient condition for the existence of a smooth time-invariant asymptotic stabilizer, sufficient condition for the existence of a quadratic-type Lyapunov function candidate is also proposed herein to alleviate the construction of stabilizing control laws. Following the deduction of the equivalence of the sufficient condition and the determination of the local definiteness of a defined scalar function, the stabilizability checking conditions are then derived in terms of system dynamics and its derivatives at the origin only. These are achieved by taking Taylor's series expansion on system dynamics. The derived conditions are shown to be consistent with those obtained by Brockett. Comparative results of Liaw and Liang are also included. Finally, examples are given to demonstrate the use of the main results.

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1. Introduction

Feedback stabilization of non-linear systems, specifically non-linear critical systems, have recently attracted much attention (e.g. Aeyel 1985, Behtash and Sastry 1988, Liaw and Abed 1991, Liaw 1993, 1998, Fu and Abed 1993). Critical systems occur at which the linearized model of non-linear systems possess eigenvalues lying on the imaginary axis with the remaining eigenvalues in the open left half of the complex plane. For the most degenerated case, the linearized model of the uncontrolled version of non-linear systems may possess only zero eigenvalues. One class of such systems is the so-called 'non-linear driftless system'. A practical example is the control model of a synchronous satellite's orbital motion (Ahmed and Sen 1980, 1981). In addition, non-holonomic systems in chain form or in power form can also be treated as driftless systems (e.g. Walsh *et al.* 1994, Samson 1995, Sordalen and Egeland 1995, Godhavn and Egeland 1997, M'Closkey and Murray 1997).

The study of the asymptotic stabilization of non-linear driftless systems includes the existence conditions of time-invariant smooth stabilizers (Brockett 1983, Liaw and Liang 1993, 1997), design of time-varying stabilizers (Coron 1992, Pomet 1992, Samson 1995, Sordalen and Egeland 1995, Godhavn and Egeland 1997, M'Closkey and Murray 1997), design of time-invariant piecewise smooth stabilizers (Canudas de Wit and Sordalen 1992), and applications to the study of orbital motion of satellites (Ahmed and Sen 1980, 1981) and car-like robot systems (Walsh *et al.* 1994). In Liaw and Liang (1997), conditions for the existence of the quadratic-type Lyapunov function were proposed to relax the assumption of stabilizability of the system as proposed by Brockett (1983) for non-linear driftless systems. However, in general, these conditions are not easy to verify, especially when the system dynamics is highly non-linear. The main goals of this paper are to establish the asymptotic stabilizability checking conditions for the non-linear driftless system and to propose the corresponding polynomial asymptotic stabilizers for easier implementation. In this paper the checking conditions for system stabilization are proposed, which require the information of system dynamics and its derivative at the origin only.

The organization of this paper is as follows. In § 2, we study the asymptotic stabilizability of driftless systems. The corresponding asymptotic stabilizers are obtained either in terms of the whole system dynamics or in terms of its Taylor's series approximations In § 3, Taylor's series expansion on system dynamics are employed to derive the conditions of the local definiteness of a defined scalar-valued function, which are equivalent to system stabilizability conditions. Results are compared with those of Brockett (1983) and Liaw and Liang (1997). In § 4, examples are given to demonstrate the use of the main results. Finally, § 5 gives the conclusion.

2. Set up

Consider a class of non-linear control driftless systems as given by

$$\dot{x} = g(x)u = \sum_{k=1}^{m} u_k g_k(x) \tag{1}$$

where $x \in \mathbb{R}^n$, $u = (u_1, \ldots, u_m)^{\mathrm{T}} \in \mathbb{R}^m$ and $g(x) = [g_1(x), \ldots, g_m(x)] \in \mathbb{R}^{n \times m}$. In addition, $g_i(x)$ are assumed to be smooth vector fields of $\mathbb{R}^n$ for $1 \le i \le m$. It is clear that system (1) is a special class of affine systems (see, e.g. Bacciotti 1992, p. 16). Various results have been presented regarding the asymptotic stabilization of the operating point of system (1) (Brockett 1983, Lafferriere 1991, Canudas de Wit and

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Sordalen 1992, Coron 1992, Pomet 1992, Walsh *et al.* 1994, Samson 1995, Sordalen and Egeland 1995, Godhavn and Egeland 1997, M'Closkey and Murray 1997). Two practical examples of driftless systems can be found in Ahmed and Sen (1980, 1981) and Walsh *et al.* (1994). The former cites the control model of the orbital motion of synchronous satellites; while the latter cites the motion equation of car-like robots. For simplicity and without loss of generality, the origin is assumed to be the operating point of interest.

An existence condition of smooth time-invariant stabilizing control laws for system (1) obtained by Brockett (1983) is shown in Lemma 1.

Lemma 1 (Brockett 1983): *Suppose all the vectors* $g_k(0)$ *and* (1) *are linearly independent. Then there exists a smooth time-invariant asymptotic stabilizer for the origin of system* (1) *if and only if* $m = n$.

Lemma 1 provides a necessary and sufficient condition for the existence of a smooth stabilizing controller for system (1) while all the vectors $g_k(0)$ are assumed to be linearly independent. However, as discussed in Liaw and Liang (1997), the linear independency of $g_k(0)$ is not a necessary condition to identify the stabilizability of system (1).

By applying Lyapunov stability theory and converse theorem on the uniformly asymptotic stability (e.g. Vidyasagar 1993), the origin of system (1) is locally asymptotically stabilizable by time-invariant control law $u = u(x)$ if and only if there exists a locally positive definite function (lpdf) $V(x)$ such that $\nabla_x^{\mathrm{T}} V(x) g(x) u(x) < 0$ for all x around a deleted neighbourhood Ω of the origin (i.e. $\Omega \backslash \{0\}$). We therefore have the next necessary and sufficient condition for the asymptotic stabilization of the origin of system (1).

Theorem 1: *The origin of system* (1) *is locally asymptotically stabilizable by* C^1 *time-invariant control law* $u = u(x)$ *if and only if there exists a smooth* lpdf $V(x)$ *such that*

$$\nabla_x^{\mathrm{T}} V(x) g(x) \neq 0 \tag{2}$$

for all x *around a deleted neighbourhood* Ω *of the origin. Moreover, the result can be extended for guaranteeing the global stabilizability if equation* (2) *holds for all* $x \neq 0$ *and* V *is a positive definite function satisfying a radially unbounded assumption, that is,* $V(x) \to \infty$ *and* $|x| \to \infty$.

Though Theorem 1 provides a necessary and sufficient condition for determining the asymptotic stabilizability of system (1), the proposed condition, however, strongly depends on the selected Lyapunov function $V(x)$. In general, such a Lyapunov function is not easy to construct. A potential candidate is given in the next lemma to illustrate its usage.

Lemma 2 (Liaw and Liang, 1997): *Suppose there exists a symmetric positive definite matrix* P *such that*

$$x^{\mathrm{T}} P g(x) \neq 0 \quad \text{for all } x \text{ in a deleted neighbourhood } \Omega \text{ of the origin} \tag{3}$$

Then the origin of system (1) *is asymptotically stabilizable. Moreover, the stabilizing control law can be in the state feedback form as in* (4) *or in a bang–bang form as in* (5)

$$u_i = -\gamma_i \cdot x^{\mathrm{T}} P g_i(x), \quad \text{for } i = 1, \ldots, m \tag{4}$$

or

$$u_i = -\gamma_i \cdot \operatorname{sgn}[x^{\mathrm{T}} P g_i(x)], \quad \text{for } i = 1, \ldots, m \tag{5}$$

Here, $\gamma_i > 0$ *for all* $i \geq 1$.

Note that it is obvious that the condition in equation (3) of Lemma 2 holds for some $P > 0$ if $g(0)$ is of full rank with $m = n$. The sufficient condition of system stabilization for system (1) as in Lemma 1 can therefore be abstracted from Lemma 2.

In general, conditions (3) are not easy to verify. In the following we will transform these conditions into equivalent checking conditions. For this purpose we introduce the scalar functions $h(x)$ as

$$h(x) = \begin{cases} x^{\mathrm{T}} P g(x) & \text{for the single-input case} \\ x^{\mathrm{T}} P g(x) g^{\mathrm{T}}(x) P x & \text{for the multi-input case} \end{cases} \tag{6}$$

Note that, though the definition of $h(x)$ for the multi-input case can also be applied to the study of the single input case, as presented in § 3.1, the definition of $h(x)$ in (6) for the single input case can induce more fruitful results.

The next result investigates the relationship between the local definiteness of $h(x)$ as in (6) and the checking condition (3) in Lemma 2.

Lemma 3:

(*a*) *For single-input case* (*i.e.* $m = 1$), *however, condition* (3) *holds if* $h(x)$ *is a locally definite function* (ldf) (*i.e.* $h(x)$ *or* $-h(x)$ *is an* lpdf).

(*b*) *For multi-input case* (*i.e.* $m > 1$), *condition* (3) *holds if and only if the scalar function* $h(x)$ *as given by* (6) *is an* lpdf.

Remark 1: Converse of the statement in Lemma 3(*a*) might not be true. A trivial counterexample is given by $g(x) = x^2$. However, it is not difficult to show by Intermediate-Value Theorem that the condition of $x^{\mathrm{T}} P g(x)$ being an ldf as required in Lemma 3(*a*) is equivalent to the condition as in equation (3) for $n > 1$. Moreover, for the single-input case, $g(0) = 0$ is an inevitable result if $h(x)$ is an ldf. The reason is that the lowest or-

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der term of an ldf cannot be an odd number. This agrees with the result of Lemma 1.

It is observed that the control law as given by (4) or (5) might be highly non-linear. To demonstrate that the control law can be easily implemented, the next attempt will be to investigate the possibility of the existence of a polynomial asymptotic stabilizer for system (1). Taking Taylor's series expansion of $g(x)$ at the origin, we have

$$g(x) = g_a(x) + o(\|x\|^k) \tag{7}$$

where $g_a(x)$ denotes Taylor's polynomial of $g(x)$ up to order k and $o(\|x\|^k)$ denotes terms of order higher than k. It is clear that function $h(x)$ as given in (6) is an ldf if $h_a(x) = x^{\mathrm{T}} P g_a(x)$ for a single-input case is an ldf or $h_a(x) = x^{\mathrm{T}} P g_a(x) g_a^{\mathrm{T}}(x) P x$ for multi-input case is an lpdf. Moreover, in such a case, system (1) possesses an asymptotic stabilizer in the form of (4) or (5) with $g(x)$ being replaced by $g_a(x)$. In the following, we give a result on the determination of the local definiteness of a scalar function defined in (8) below to facilitate the checking of local definiteness of $h_a(x)$. Note that, in the following, for simplicity, $\|\cdot\|$ denotes the L_2-norm of vector or matrix, $o(\|\cdot\|^k)$ denotes terms of order higher than k and $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue.

Consider a scalar function as given by

$$\delta(y,z) = y^{\mathrm{T}} R y + \rho_{yzz} + \rho_{zzz} + \rho_{zzzz} + R_3(y,z) + R_4(y,z) + o(\|(y,z)\|^4) \tag{8}$$

where $y \in \mathbb{R}^r$, $z \in \mathbb{R}^{n-r}$, $R \in \mathbb{R}^{r\times r}$, ρ_{yzz} is a scalar polynomial function of order in (y,z) exactly 1 and 2, ρ_{zzz} is a 3-linear function in z (the definition of k-linear function can be referred to, e.g. Fu and Abed 1993), ρ_{zzzz} is a 4-linear function in z, $R_3(y,z)$ is a scalar polynomial function in (y,z) of order 3 except for the terms ρ_{yzz} and ρ_{zzz}, and $R_4(y,z)$ is a scalar polynomial function in (y,z) of order 4 except for the term ρ_{zzzz}.

The next result provides a sufficient condition on determining whether the real-values function $\delta(y,z)$ is an lpdf.

Lemma 4: *Suppose there exist* $\alpha_1 > 0$, $\alpha_2 > 0$ *and* $\beta_1 > 0$ *such that*

$$y^{\mathrm{T}} R y \geq \alpha_1 \|y\|^2 \tag{9}$$

$$|\rho_{yzz}| \leq \beta_1 \|y\| \cdot \|z\|^2 \tag{10}$$

$$\rho_{zzz} = 0 \tag{11}$$

and

$$\rho_{zzzz} \geq \alpha_2 \|z\|^4 \tag{12}$$

If $4\alpha_1\alpha_2 > \beta_1^2$, *then* $\delta(y,z)$ *given in* (8) *is an* lpdf.

Proof: It is known that there exist $\epsilon_1 > 0$, $\beta_2 > 0$ such that

$$|R_3(y,z)| \leq \beta_2(\|z\| + \|y\|) \cdot \|y\|^2 \tag{13}$$

and

$$|R_4(y,z)| \leq \beta_2(\|z\|^2 + \|y\| \cdot \|y\| + \|y\|^2) \cdot \|y\|^2 + \beta_2 \cdot \|y\| \cdot \|z\|^3 \tag{14}$$

for all $\|y\|$, $\|z\| < \epsilon_1$. From (9)–(12), we then have

$$\begin{aligned}\delta(y,z) - o(\|(y,z)\|^4) &\geq \Delta_1\|y\|^2 - \Delta_2\|y\| \cdot \|z\|^2 + \alpha_2\|z\|^4 \\ &= \Delta_1\left(\|y\| - \frac{\Delta_2}{2\Delta_1}\|z\|^2\right)^2 \\ &\quad + \frac{1}{4\Delta_1}(4\alpha_1\alpha_2 - \beta_1^2 \\ &\quad + o(\|(y,z)\|)) \cdot \|z\|^4\end{aligned} \tag{15}$$

Here

$$\Delta_1 := \alpha_1 - \beta_2(\|y\| + \|z\| + \|y\|^2 + \|y\| \cdot \|z\| + \|z\|^2)$$

and

$$\Delta_2 := \beta_1 + \beta_2\|z\|$$

It is clear that there exists ϵ_2 with $0 < \epsilon_2 \leq \epsilon_1$ such that $\delta(y,z) > 0$ for all $\|y\|$, $\|z\| < \epsilon_2$ if $4\alpha_1\alpha_2 > \beta_1^2$. The conclusion of the lemma is hence implied. □

Remark 2: The conditions for local definiteness of two-variable functions obtained in Fu and Abed (1993) can be abstracted from Lemma 4.

3. Main results

In this section we will take Taylor's series expansion of $g(x)$ and apply Lemma 4 to determine the local definiteness of the function $h(x)$ as defined in (6). In § 3.1, we consider the single-input case, while the multi-input case is studied in § 3.2. Details are given as follows.

3.1. *The single-input case* (*i.e.* $m = 1$)

First, we consider the single-input case. It is clear that $g(0) \neq 0$ implies that $g(0)$ is of full rank, which has been discussed in Lemma 1. In the following, we will discuss the case of $g(0) = 0$ only.

Taking Taylor's series expansion on $g(x)$ at the origin up to third order and choosing P to be the identity matrix, from equation (6) we then have

$$\begin{aligned}h(x) &= x^{\mathrm{T}} g(x) \\ &= x^{\mathrm{T}}\{Lx + Q(x,x) + C(x,x,x)\} + o(\|x\|^4)\end{aligned} \tag{16}$$

where L, $Q(\cdot,\cdot)$ and $C(\cdot,\cdot,\cdot)$ denote the Jacobian matrix, quadratic and cubic terms, respectively. The next result follows readily from Lemma 3.

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Theorem 2: *If $x^{\mathrm{T}}Lx$ is an* ldf*, then the origin of the system* (1) *is asymptotically stabilizable by a constant or quadratic feedback.*

In the following, we suppose the Jacobian matrix $L = \nabla g(0)$ has rank $r < n$. For simplicity, we assume that

$$L = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \tag{17}$$

where $R \in \mathbb{R}^{r\times r}$ is a non-singular matrix. Otherwise, a change of variable will lead L to this form. It is known that $h(x)$ as in (16) cannot be an ldf if R is an indefinite matrix. For simplicity and without loss of generality, we may assume that R is a positive definite matrix. Results for the case of which R is a negative definite can be obtained by a similar approach. Details are omitted.

Let $y \in \mathbb{R}^r$, $z \in \mathbb{R}^{n-r}$ such that $x = (y^{\mathrm{T}}, z^{\mathrm{T}})^{\mathrm{T}}$ and partition the quadratic and the cubic terms in the form

$$\left.\begin{aligned} Q(x,x) &= \begin{pmatrix} Q_1(x,x) \\ Q_2(x,x) \end{pmatrix} \\ &\text{and} \\ C(x,x,x) &= \begin{pmatrix} C_1(x,x,x) \\ C_2(x,x,x) \end{pmatrix} \end{aligned}\right\} \tag{18}$$

where $Q_1(x,x)$, $C_1(x,x,x) \in \mathbb{R}^r$ and $Q_2(x,x)$, $C_2(x,x,x) \in \mathbb{R}^{n-r}$. Equation (16) can then be rewritten as

$$\begin{aligned} h(x) &= y^{\mathrm{T}}Ry + y^{\mathrm{T}}Q_1(x,x) + z^{\mathrm{T}}Q_2(x,x) \\ &\quad + y^{\mathrm{T}}C_1(x,x,x) + z^{\mathrm{T}}C_2(x,x,x) + o(\|x\|^4) \end{aligned} \tag{19}$$

Comparing equation (19) with the notation as in (8), we have

$$\rho_{zzz} = z^{\mathrm{T}}Q_2(z,z) \tag{20}$$

$$\rho_{yzz} = y^{\mathrm{T}}Q_1(z,z) + z^{\mathrm{T}}Q_2(y,z) \tag{21}$$

and

$$\rho_{zzzz} = z^{\mathrm{T}}C_2(z,z,z) \tag{22}$$

Denote α_1 the smallest eigenvalue of the matrix R, i.e. $\alpha_1 = \lambda_{\min}(R)$. It follows that $y^{\mathrm{T}}Ry \geq \alpha_1\|y\|^2$. In addition, the condition $\rho_{zzz} = 0$ implies $Q_2(z,z) = 0$ for all z.

From Lemmas 3 and 4 and Theorem 1, we then have the next obvious result.

Proposition 1: *Suppose $m = 1$, $g(0) = 0$ and $\nabla g(0)$ is in the form of* (17) *with R being a positive definite matrix. The origin of system* (1) *is asymptotically stabilizable if $Q_2(z,z) = 0$ and*

$$\beta_1^2 < 4\alpha_1\alpha_2 \tag{23}$$

where $\alpha_1 = \lambda_{\min}(R)$, $z^{\mathrm{T}}C_2(z,z,z) \geq \alpha_2\|z\|^4$ and $|y^{\mathrm{T}}Q_1(z,z) + z^{\mathrm{T}}Q_2(y,z)| \leq \beta_1 \cdot \|y\| \cdot \|z\|^2$. Moreover, an asymptotic stabilizer can be chosen in either the form (4) *or* (5)*, or in a polynomial form of* (4) *or* (5) *with $g(x)$ replaced by $Lx + Q(x,x) + C(x,x,x)$.*

One of the choices for β_1 and α_2, as stated in Proposition 1, can be obtained as follows.

Let

$$Q_1(z,z) = \begin{pmatrix} z^{\mathrm{T}}D_1z \\ \vdots \\ z^{\mathrm{T}}D_rz \end{pmatrix} \quad \text{and} \quad Q_2(y,z) = \begin{pmatrix} y^{\mathrm{T}}E_1z \\ \vdots \\ y^{\mathrm{T}}E_{n-r}z \end{pmatrix} \tag{24}$$

where $D_i \in \mathbb{R}^{(n-r)\times(n-r)}$ for $l \leq i \leq r$ and $E_j \in \mathbb{R}^{r\times(n-r)}$ for $1 \leq j \leq n-r$. From (21), we then have

$$\begin{aligned} \rho_{yzz} &\leq \|y\| \cdot \sqrt{\sum_{i=1}^{r}(z^{\mathrm{T}}D_iz)^2} + \|z\| \cdot \sqrt{\sum_{j=r}^{n-r}(y^{\mathrm{T}}E_jz)^2} \\ &\leq \beta_1 \cdot \|y\| \cdot \|z\|^2 \end{aligned} \tag{25}$$

where

$$\beta_i = \left(\sqrt{\sum_{i=1}^{r}\|D_i\|^2} + \sqrt{\sum_{j=r}^{n-r}\|E_j\|^2}\right) \tag{26}$$

Let $z = (z_1, \ldots, z_{n-r})^{\mathrm{T}}$. ρ_{zzzz} given in (22) can then be rewritten as

$$\begin{aligned} \rho_{zzzz} &= z^{\mathrm{T}}C_2(z,z,z) \\ &= \sum_{i=1}^{n-r} z_i^2(z^{\mathrm{T}}\Phi_{ii}z) + \sum_{i<j<k<l} d_{ijkl} \cdot z_iz_jz_kz_l \end{aligned} \tag{27}$$

where $\Phi_{ii} \in \mathbb{R}^{(n-r)\times(n-r)}$. In order to estimate a larger lower bound for ρ_{zzzz}, the (j,j)-entry of Φ_{ii} and the (i,i)-entry of Φ_{jj} are set to be the same value of $\frac{1}{2}\cdot$ (coefficient of $z_i^2z_j^2$ in ρ_{zzzz}) for all $i,j = 1,\ldots,n-r$. It is observed that

$$\sum_{i=1}^{n-r} z_i^2(z^{\mathrm{T}}\Phi_{ii}z) \geq \left(\min_{1\leq i\leq n-r} \lambda_{\min}(\Phi_{ii})\right)\|z\|^4 \tag{28}$$

It is known that the function $f(x) = x_1x_2x_3x_4$, subject to the constraint: $\|x\|^2 = \sum_{i=1}^{n} x_i^2 = \epsilon^2$, has a global minimum value $-(\epsilon^4/16)$ which occurs at the points $x_1^2 = x_2^2 = x_3^2 = x_4^2$ and $x_5 = \cdots = x_n = 0$. Let

$$\alpha_2\left(\min_{1\leq i\leq n-r} \lambda_{\min}(\Phi_{ii}) - \frac{1}{16}\sum_{i<j<k<1} |d_{ijkl}|\right) \tag{29}$$

From (27) and (28) we then have

$$\begin{aligned} \rho_{zzzz} &\geq \left(\min_{1\leq i\leq n-r} \lambda_{\min}(\Phi_{ii}) - \frac{1}{16}\sum_{i<j<k<l} |d_{ijkl}|\right) \cdot \|z\|^4 \\ &\geq \alpha_2 \cdot \|z\|^4 \end{aligned} \tag{30}$$

Thus, we have the next corollary.

Corollary 1: *Suppose $m = 1$, $g(0) = 0$ and $\nabla g(0)$ is in the form of (17) with R being a positive definite matrix. The origin of system (1) is asymptotically stabilizable if $Q_2(z, z) = 0$ and $\beta_1^2 < 4\alpha_1\alpha_2$, where $\alpha_1 = \lambda_{\min}(R)$, β_1 and α_2 are defined in (26) and (29), respectively.*

Remark 3: If there exists an i such that the coefficient of z_i^4 in ρ_{zzzz} is negative or zero, then α_2 cannot be a positive number. This implies that (23) does not hold. Thus, Proposition 1 or Corollary 1 cannot be applied to the determination of the local stabilizability of system (1).

3.2. *The multi-input case (i.e. $m > 1$)*

Next, we consider the case of which $m > 1$. Suppose that the constant matrix $g(0)$ has rank r. For simplicity, we may assume that $g(0)$ is in the form of the right-hand side of (31) below, where $A \in \mathbb{R}^{r\times r}$ is a non-singular matrix. Otherwise, by the use of elementary row and column operations (see, e.g. Noble and Daniel 1988), there exist two non-singular matrices $W_1 \in \mathbb{R}^{n\times n}$ and $W_2 \in \mathbb{R}^{m\times m}$ such that

$$W_1 g(0) W_2 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \tag{31}$$

Let $\eta = W_1 x$ and $u = W_2 v$. System (1) is then transformed into

$$\dot{\eta} = W_1 g(W_1^{-1}\eta) W_2 v \tag{32}$$

which has the desired constant term. Thus, without loss of any generality, we may impose the following assumption.

Assumption 1: *$\operatorname{rank}[g(0)] = r$ and $g(0) \in \mathbb{R}^{n\times m}$ is in the form of the right-hand side of (31) above with non-singular matrix $A \in \mathbb{R}^{r\times r}$.*

Taking the Taylor's series expansion on $g(x)$ at the origin, we have

$$g(x) = g(0) + L(x) + Q(x, x) + C(x, x, x) + o(\|x\|^3) \tag{33}$$

Here, $L(x)$, $Q(x, x)$ and $C(x, x, x)$ denote the linear, bilinear and trilinear terms of $g(x)$, respectively. For simplicity, choose the matrix P as the identity matrix. From (6)

$$\begin{aligned} h(x) = {} & x^{\mathrm{T}} g(0) g^{\mathrm{T}}(0) x + x^{\mathrm{T}}\{g(0) L^{\mathrm{T}}(x) + L(x) g^{\mathrm{T}}(0)\}x \\ & + x^{\mathrm{T}}\{g(0) Q^{\mathrm{T}}(x, x) + L(x) L^{\mathrm{T}}(x) \\ & + Q(x, x) g^{\mathrm{T}}(0)\}x + o(\|x\|^4) \end{aligned} \tag{34}$$

Similarly, let $x = (y^{\mathrm{T}}, z^{\mathrm{T}})^{\mathrm{T}}$, where $y \in \mathbb{R}^r$ and $z \in \mathbb{R}^{n-r}$. Rewrite $L(x)$ and $Q(x, x)$ as

$$L(x) = \begin{pmatrix} L_1(x) & L_2(x) \\ L_3(x) & L_4(x) \end{pmatrix} \tag{35}$$

and

$$Q(x, x) = \begin{pmatrix} Q_1(x, x) & Q_2(x, x) \\ Q_3(x, x) & Q_4(x, x) \end{pmatrix} \tag{36}$$

where

$$L_1(x), Q_1(x, x) \in \mathbb{R}^{r\times r}$$

and

$$L_4(x), Q_4(x, x) \in \mathbb{R}^{(n-r)\times(n-r)}.$$

The dimension of the remaining matrices are obvious. Equation (34) can then be rewritten as

$$\begin{aligned} h(x) = {} & y^{\mathrm{T}}\{AA^{\mathrm{T}} + AL_1^{\mathrm{T}}(x) + L_1(x)A^{\mathrm{T}} + AQ_1^{\mathrm{T}}(x, x) \\ & + L_1(x)L_1^{\mathrm{T}}(x) + L_2(x)L_2^{\mathrm{T}}(x) + Q_1(x, x)A^{\mathrm{T}}\}y \\ & + 2y^{\mathrm{T}}\{AL_3^{\mathrm{T}}(x) + AQ_3^{\mathrm{T}}(x, x) \\ & + L_1(x)L_3^{\mathrm{T}}(x) + L_2(x)L_4^{\mathrm{T}}(x)\}z \\ & + z^{\mathrm{T}}\{L_3(x)L_3^{\mathrm{T}}(x) + L_4(x)L_4^{\mathrm{T}}(x)\}z \\ & + o(\|y, z\|^4) \end{aligned} \tag{37}$$

Now, we employ Lemma 4 to check the local definiteness of $h(x)$. Comparing equation (37) with the notations as in (8), we have

$$R = AA^{\mathrm{T}}, \ \rho_{zzz} = 0, \ \rho_{yzz} = 2y^{\mathrm{T}}\{AL_3^{\mathrm{T}}(z)\}z$$

and

$$\rho_{zzzz} = z^{\mathrm{T}}\{L_3(z)L_3^{\mathrm{T}}(z) + L_4(z)L_4^{\mathrm{T}}(z)\}z.$$

Let $\alpha_1 = \lambda_{\min}(AA^{\mathrm{T}})$. It is clear that we have

$$y^{\mathrm{T}}Ry \geq \alpha_1\|y\|^2 \tag{38}$$

From Lemmas 3 and 4 and Theorem 1, we then have the next result.

Proposition 2: *Suppose Assumption 1 holds. The origin of system (1) is asymptotically stabilizable if*

$$\beta_1^2 < 4\alpha_1\alpha_2 \tag{39}$$

where $\alpha_1 = \lambda_{\min}(AA^{\mathrm{T}})$, $|2y^{\mathrm{T}}\{AL_3^{\mathrm{T}}(z)\}z| \geq \beta_1 \cdot \|y\| \cdot \|z\|^2$ and $|z^{\mathrm{T}}\{L_3(z)L_3^{\mathrm{T}}(z) + L_4(z)L_4^{\mathrm{T}}(z)\}z| \geq \alpha_2 \cdot \|z\|^4$. Moreover, the stabilizing control laws can be obtained in the form of (4) or (5).

To demonstrate that condition (39) in Proposition 2 is not vacuous, we will derive in the following the expressions for the candidate of β_1 and α_2. Let

$$[L_3(z), L_4(z)] = (M_1 z, \ldots, M_r z, M_{r+1} z, \ldots, M_m z) \tag{40}$$

where $M_i \in \mathbb{R}^{(n-r)\times(n-r)}$ for $i = 1, \ldots, m$. ρ_{zzzz} can then be rewritten as

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$$\rho_{zzzz} = z^{\mathrm{T}} \left\{ (M_1 z, \ldots, M_m z) \begin{pmatrix} z^{\mathrm{T}} M_1^{\mathrm{T}} \\ \vdots \\ z^{\mathrm{T}} M_m^{\mathrm{T}} \end{pmatrix} \right\} z$$
$$= \sum_{i=1}^{m} (z^{\mathrm{T}} M_i z) \cdot (z^{\mathrm{T}} M_i^{\mathrm{T}} z)$$
$$\geq \alpha_2 \cdot \|z\|^4 \tag{41}$$

where $\alpha_2 = \sum_{i=1}^{m} \lambda_i^2$ with

$$\lambda_i = \begin{cases} \lambda_{\min}\left(\dfrac{M_i + M_i^{\mathrm{T}}}{2}\right) & \text{if } \dfrac{M_i + M_i^{\mathrm{T}}}{2} \text{ is positive semidefinite} \\ \lambda_{\max}\left(\dfrac{M_i + M_i^{\mathrm{T}}}{2}\right) & \text{if } \dfrac{M_i + M_i^{\mathrm{T}}}{2} \text{ is negative semidefinite} \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

Next, let

$$\beta_1 = 2 \cdot \|A\| \cdot \sqrt{\sum_{i=1}^{r} \|M_i\|^2} \tag{43}$$

From the definition of ρ_{yzz} above and (40), we then have

$$\rho_{yzz} \leq 2 \cdot \|A\| \cdot \|L_3^{\mathrm{T}}(z)\| \cdot \|y\| \cdot \|z\|$$
$$\leq \beta_1 \cdot \|y\| \cdot \|z\|^2 \tag{44}$$

These lead to the next result.

Corollary 2: *Suppose Assumption* 1 *holds. The origin of system* (1) *is asymptotically stabilizable if*

$$\beta_1^2 < 4\alpha_1\alpha_2 \tag{45}$$

where $\alpha_1 = \lambda_{\min}(AA^{\mathrm{T}})$, $\alpha_2 = \sum_{i=1}^{m} \lambda_i^2$ *with* λ_i *defined in* (42) *and* β_1 *defined in* (43).

In the following we study the two special cases which might not be covered by the discussions above.

Case 1: $g(0)$ is of full rank with $m \leq n$.

First, consider the case of which $g(0)$ is of full rank with $m \leq n$. This implies that matrices $L_2(x)$, $L_4(x)$, $Q_2(x,x)$ and $Q_4(x,x)$ in (35) and (36) are all null. For the case of $m < n$, we claim that $|\lambda_i| \leq \|M_i\|$ for each $i = 1, \ldots, r = m$, where λ_i is defined in (42). To see this, let ξ be the unit eigenvector of $(M_i + M_i^{\mathrm{T}})/2$ corresponding to the eigenvalue $\lambda_i = \lambda_{\min}[(M_i + M_i^{\mathrm{T}})/2]$ for $(M_i + M_i^{\mathrm{T}})/2$ being a positive semidefinite matrix. We then have

$$|\lambda_i| = \left| \xi^{\mathrm{T}} \frac{M_i + M_i^{\mathrm{T}}}{2} \xi \right|$$
$$\leq \|M_i\| \tag{46}$$

Similarly, we can prove that $\lambda_i \leq \|M_i\|$ for the case of which $(M_i + M_i^{\mathrm{T}})/2$ is a negative semidefinite matrix. According to the definition of L_2 norm, we have $\|A\|^2 = \lambda_{\max}(AA^{\mathrm{T}})$. From the definitions of α_1, α_2 and β_1 in §3.2, we have

$$4\alpha_1\alpha_2 = 4 \cdot \lambda_{\min}(AA^{\mathrm{T}}) \cdot \sum_{i=1}^{m} \lambda_i^2$$
$$\leq 4 \cdot \lambda_{\max}(AA^{\mathrm{T}}) \cdot \sum_{i=1}^{m} \|M_i\|^2$$
$$= \beta_1^2 \tag{47}$$

Thus, Corollary 2 fails to verify the local stabilizability of system (1).

Note that, in general, the matrix $L_4(x)$ in §3.2 is not null and it will enlarge the magnitude of α_2. That is why Proposition 2 or Corollary 2 can be applied to some of the cases of which $g(0)$ does not have full rank. For the case of $m = n$, the matrix $g(0)g^{\mathrm{T}}(0)$ is a non-singular matrix. This implies that $h(x)$ defined in (6) is an lpdf. The origin of system (1) is therefore concluded by Lemmas 2 and 3 to be asymptotically stabilizable, which agrees with the result of Lemma 1.

Case 2: $g(0) = 0$ with $1 \leq m \leq n$.

For the case of which $1 < m \leq n$ and $g(0) = 0$, we then have $\alpha_1 = 0$. The results of §3.2 cannot be applied since the relation $\beta_1^2 < 4\alpha_1\alpha_2$, as required in Proposition 2 or Corollary 2, cannot hold. Alternatively, in the following we consider the effect of linear terms of $g(x)$ only on the local definiteness of $h(x)$. Details are given below.

For the case for which $m = 1$, Theorem 2 in §3.1 provides a stabilizability condition for system (1). We now investigate the more general case of $m > 1$. Since $g(0) = 0$, matrix A defined in Assumption 1 is null. This implies that state variable $z = x$. Thus, all the matrices $L_1(x)$, $L_2(x)$, $L_4(x)$, $Q_1(x,x)$, $Q_2(x,x)$ and $Q_4(x,x)$ in (35) and (36) are null. Equation (37) can then be rewritten as

$$h(x) = x^{\mathrm{T}} L_3(x) L_3^{\mathrm{T}}(x) x \tag{48}$$

Similarly, let

$$L_3(x) = (M_1 x, \ldots, M_m x) \tag{49}$$

where $M_i \in \mathbb{R}^{n \times n}$ for $i = 1, \ldots, m$. It follows that

$$h(x) = \sum_{i=1}^{m} (x^{\mathrm{T}} M_i x) \cdot (x^{\mathrm{T}} M_i^{\mathrm{T}} x) = \sum_{i=1}^{m} (x^{\mathrm{T}} M_i x)^2 \tag{50}$$

since $x^{\mathrm{T}} M_i x = x^{\mathrm{T}} M_i^{\mathrm{T}} x$ is a scalar. We then have the next theorem.

Theorem 3: *Suppose $g(0) = 0$. Then the origin of system (1) is asymptotically stabilizable if*

$$\sum_{i=1}^{m}(x^{\mathrm{T}} M_i x)^2 > 0 \quad \text{for all } x \neq 0 \tag{51}$$

The next two results follow readily from Theorem 3.

Corollary 3: *Suppose $g(0) = 0$. The origin of system (1) is asymptotically stabilizable if there exists some i such that the symmetric part of M_i is a definite matrix.*

Corollary 4: *Suppose $g(0) = 0$ and none of M_i is a definite matrix. Then the origin of system (1) is asymptotically stabilizable if there exists a semidefinite matrix M_j with simple zero eigenvalue and some $i \in \{1, \ldots, m\}$ such that $\xi_j^{\mathrm{T}} M_i \xi_j \neq 0$, where ξ_j is an eigenvector corresponding to the zero eigenvalue of the symmetric part of M_j.*

Remark 4: The results of Theorem 3 and Corollaries 3 and 4 can also be applied to the case where $g_i(0) = 0$ and $g_j(0) \neq 0$ for some i and j. For instance, if there is only one i such that $g_i(0) = 0$, we can transform system (1) into a single input system by letting $u_j = 0$ for all $j \neq i$. Corollary 3 can then be applied to the determination of local stabilizability of system (1). If there are more than one such i we have $g_i(0) = 0$. Similarly, we can apply Theorem 3 by letting $u_j = 0$ for all j in which $g_j(0) \neq 0$.

Remark 5: The result of Theorem 3 relates to the asymptotic stabilization problem of the bilinear driftless systems, which is different from those of Liaw and Liang (1997, Theorem 8 and Corollary 2). In Liaw and Liang (1997), the checking conditions were obtained for investigating the possibility of the existence of a constant asymptotic stabilizer for the bilinear driftless systems. The results of this paper consider not only the constant stabilizer but also the quadratic asymptotic stabilizer. However, either of these two results can imply the other. In the next section, Example 1 demonstrates that the system might not possess a constant asymptotic stabilizer but satisfies the condition of Theorem 3. Moreover, Example 2 presents a driftless system which possesses a constant asymptotic stabilizer but yet checking condition (51) does not hold.

4. Illustrative examples and simulation results

In this section, we present four examples. Example 1 gives a driftless system whose constant and linear terms satisfy the condition of Theorem 3 but not possessing any constant asymptotic stabilizer. Example 2 presents a driftless system which possesses a constant asymptotic stabilizer yet checking condition (51) does not hold. Examples 3 and 4 are given to demonstrate the use of the checking operations derived in § 3. Simulation results are also given for Example 3.

Example 1: Consider the following two-input driftless system

$$\dot{x} = u_1 g_1(x) + u_2 g_2(x) \tag{52}$$

where $g_i(x) + M_i x + o(\|x\|)$ for $i = 1, 2$ with

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \tag{53}$$

It is observed that both matrices M_1 and M_2 are indefinite and not commutative. For any constants c_1 and c_2 which are not both zero, the trace and the determinant of the matrix $c_1 M_1 + c_2 M_2$ are calculated, respectively, to be zero and $-2c_1^2 - 2c_1c_2 - 5c_2^5 < 0$. This implies the matrix $c_1 M_1 + c_2 M_2$ always possess a positive real eigenvalue unless $c_1 = c_2 = 0$. Thus, system (52) does not possess any constant stabilizer. On the other hand, by direct calculation, we have $x^{\mathrm{T}} M_1 x = 0$ if and only if $x_1 = (-1 \pm \sqrt{2})x_2$ and $x^{\mathrm{T}} M_2 x = 0$ if and only if $x_1 = (2 \pm \sqrt{5})x_2$. It follows that the condition (51) of Theorem 3 holds. The origin of system (52) is hence concluded to be stabilizable by a quadratic-type asymptotic stabilizer.

Example 2: Consider system (1) with $x = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2$ and

$$g(x) = \begin{pmatrix} x_1 - 2x_2 & x_1 - 2x_2 + x_1 x_2 \\ x_2 + x_2 \sin x_1 & x_2 + x_2^2 \end{pmatrix} \tag{54}$$

We have $g(0) = 0$ and

$$M_1 = M_2 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \tag{55}$$

It is clear that the origin of system (1) defined by (54) is asymptotically stabilizable by constant control law: $u_1 = c_1$ and $u_2 = c_2$ with $c_2 + c_2 < 0$. However, the asymptotic stabilizability of the origin cannot be concluded by Theorem 3 since for all x on the line spanned by the vector $(1, 1)^{\mathrm{T}}$ we have

$$\sum_{i=1}^{2}(x^{\mathrm{T}} M_i x)^2 = 0 \tag{56}$$

Example 3: Consider the following system

$$\dot{x}_1 = 2u_1 + x_2 x_3 u_2 \tag{57}$$

$$\dot{x}_2 = (\sin x_2 + x_3^2)u_1 + \sqrt{3}x_2 u_2 \tag{58}$$

$$\dot{x}_3 = (\sin x_3 + x_2^2)u_1 + \sqrt{3}x_3 u_2 \tag{59}$$

where $x = (x_1, x_2, x_3)^{\mathrm{T}} \in \mathbb{R}^3$, $u = (u_1, u_2)^{\mathrm{T}} \in \mathbb{R}^2$ and

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$$g(x) = \begin{pmatrix} 2 & x_2x_3 \\ \sin x_2 + x_3^2 & \sqrt{3}x_2 \\ \sin x_3 + x_2^2 & \sqrt{3}x_3 \end{pmatrix} \tag{60}$$

It is observed that $g(0)$ satisfies Assumption 1 with rank $[g(0)] = 1$ and $A = 2$. Thus, Lemma 1 (Brockett 1983) cannot draw any conclusion about asymptotic stabilizability of the origin. However, the origin of this system can be shown to be asymptotically stabilizable by checking the local definiteness of the defined scalar-valued function $h(x)$ proposed in this paper. Details are given below.

First, we examine the local definiteness of $x^{\mathrm{T}}g_i(x)$ for $i = 1, 2$ by choosing the matrix P to be the identity matrix. Clearly, $x^{\mathrm{T}}g_1(x)$ cannot be an ldf because the order of its lowest order term is an odd number. Also, $x^{\mathrm{T}}g_2(x)$ is not an ldf since it vanishes at the line $x_2 = x_3 = 0$. Thus, condition (3) cannot be applied to each single-input case. On the other hand, it is clear that we have

$$\left.\begin{aligned} & g(0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Lx = \begin{pmatrix} 0 & 0 \\ x_2 & \sqrt{3}x_2 \\ x_3 & \sqrt{3}x_3 \end{pmatrix} \\ & \text{and} \\ & Q(x, x) = \begin{pmatrix} 0 & x_2x_3 \\ x_3^2 & 0 \\ x_2^2 & 0 \end{pmatrix} \end{aligned}\right\} \tag{61}$$

According to the discussions in §3.2, it is easy to have $\alpha_1 = 4$, $\alpha_2 = 4$, $\beta_1 = 4$ and $\beta_1^2 < 4\alpha_1\alpha_2$. Thus, according to Proposition 2 or Corollary 2, the origin of system (57)–(59) is asymptotically stabilizable by the control law (4) or (5) or the polynomial stabilizer in the form of (4) or (5) with $g(x)$ being replaced by $g(0) + Lx + Q(x, x)$.

Simulation results are given in figures 1 and 2 by taking the control input in the form of (4) with $\gamma_1 = \gamma_2 = 1$ and the initial condition $[x_1(0), x_2(0), x_3(0)] = (-0.1, 0.1, 0.2)$. Figure 1 shows the timing response of the state variables, while figure 2 indicates the norm of the state vector. It is observed from these two figures that all the state variables and the norm of state vector are converged to zero, which agrees with the theoretical results. However, since the closed loop system has order greater than one, the convergent rate is getting smaller as states come closer to the origin.

Example 4: Consider a two-input non-linear driftless system as given by

$$\dot{x} = g(x)u = u_1g_1(x) + u_2g_2(x) \tag{62}$$

where $x \in \mathbb{R}^3$, $g_i(x) = M_ix + o(\|x\|)$ for $i = 1, 2$ with

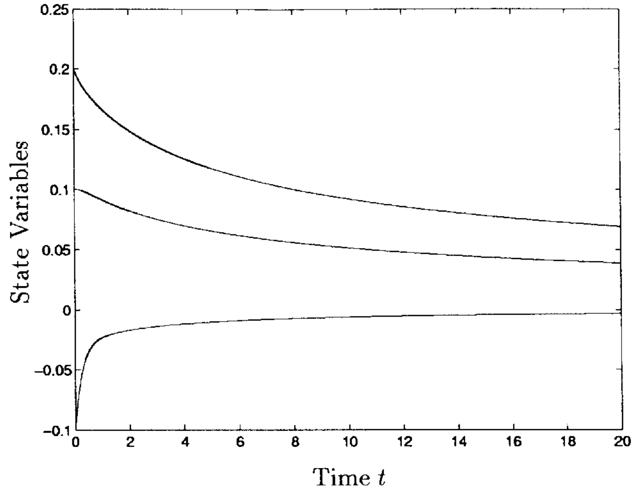

Figure 1. Timing response of state variables for Example 3 with the initial condition $[x_1(0).x_2(0), x_3(0)] = (-0.1, 0.1, 0.2)$.

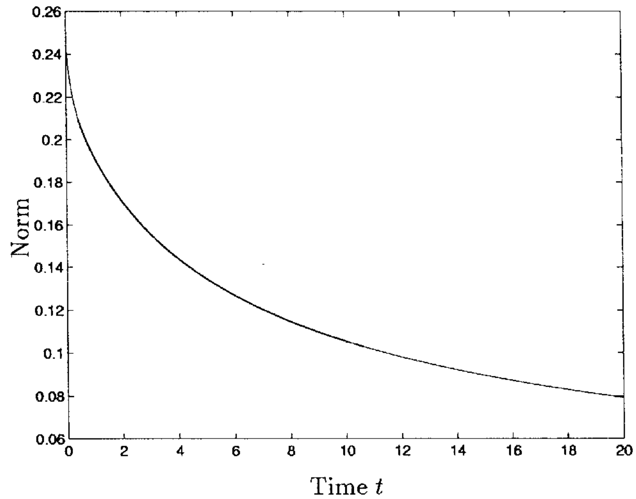

Figure 2. Norm of state variables for Example 3 with the initial condition $[x_1(0), x_2(0), x_3(0)] = (-0.1, 0.1, 0.2)$.

$$M_1 = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 1 & -5 \\ -2 & 5 & 4 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} -1 & 0 & 0 \\ -6 & 4 & -1 \\ 3 & 0 & 2 \end{pmatrix} \tag{63}$$

It is observed that rank $[g(0)] = 0$. Thus, Lemma 1 (Brockett 1983) cannot draw any conclusion about asymptotic stabilizability of the origin. Since $g(0)$ is a zero matrix, the asymptotic stabilizability of system (62) can then be determined by that of bilinear driftless system

$$\dot{x} = u_1M_1x + u_2M_2x \tag{64}$$

It is noted that either of the symmetric part of the two matrices M_1 and M_2 is definite. In addition, matrices M_1 and M_2 are not commutative. Thus, the results of Liaw and Liang (1997) cannot be applied. However, the symmetric part of matrix M_1 is found to be a positive semi-definite matrix with simple zero eigenvalue. By direct calculation, $\xi = (\sqrt{2}, -\sqrt{2}, 0)^{\mathrm{T}}$ is an eigenvector of zero eigenvalue for the symmetric part of M_1 and $\xi^{\mathrm{T}} M_2 \xi = 18 \neq 0$. According to Corollary 4, the origin of system (62) is concluded to be asymptotically stabilizable. Moreover, stabilizing control laws can be chosen in the form of $u = -g^{\mathrm{T}}(x)x$ or $u_i = -x^{\mathrm{T}} M_i^{\mathrm{T}} x$ for $i = 1, 2$.

5. Conclusions

In this paper, we have derived the asymptotic stabilizability conditions for non-linear driftless systems. The asymptotic stabilizers were obtained by checking the local definiteness of a defined real-valued function which is a function of system dynamics. By invoking Taylor's series expansion on system dynamics, the stabilizability conditions and their corresponding asymptotic stabilizers were explicitly attained in terms of system dynamics and its derivatives at the origin only. Moreover, both constant control laws and quadratic-type control laws were proposed in this paper for the stabilization of bilinear systems. These were not covered by our earlier work (1997).

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