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Rabin numbers of Butterfly networks[1]

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Abstract

Reliability and efficiency are important criteria in the design of interconnection networks. Recently, the w-wide diameter $d_w(G)$, the $(w-1)$-fault diameter $D_w(G)$, and the w-Rabin number $r_w(G)$ have been used to measure network reliability and efficiency. In this paper, we study these parameters for an important class of parallel networks — Butterfly networks. The main result of this paper is to determine the Rabin number of Butterfly networks.

Keywords: Diameter; Connectivity; Rabin number; Butterfly network; Banyan network; Level

1. Introduction

Reliability and efficiency are important criteria in the design of interconnection networks. Connectivity is widely used to measure network fault-tolerance capacity, while diameter determines routing efficiency along individual paths. In practice, we are interested in high-connectivity, small-diameter networks.

By a network, we mean a graph. For general notions of graphs, see [3]. The *distance* $d_G(x,y)$ from a vertex x to another vertex y in a network G is the minimum number of edges of a path from x to y. The *diameter* $d(G)$ of a network G is the maximum distance from one vertex to another. The *connectivity* $k(G)$ of a network G is the minimum number of vertices whose removal results in a disconnected or one-vertex network. According to Menger's theorem (see [3], Theorem 2.2.5), there are k internally vertex-disjoint paths (i.e. with disjoint vertices except for the extremities) from a vertex x to another vertex y in a network of connectivity k. Throughout this paper, 'vertex-disjoint' always means 'internally vertex-disjoint'.

For a network G with connectivity $k(G)$ and $w \leqslant k(G)$, the three parameters $d_w(G)$, $D_w(G)$, and $r_w(G)$ (defined below) arise from the study of, respectively, parallel

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routing, fault-tolerant systems, and randomized routing (see [6,9,12–14]). Due to widespread use of (and demand for) reliable, efficient, and fault-tolerant networks, these three parameters have been the subjects of extensive study over the past decade (see [6]).

Give an integer w, the *w-wide diameter* $d_w(G)$ of a network G is the minimum l such that for any two distinct vertices x and y there exist w vertex-disjoint paths of length at most l from x to y. The notion of w-wide diameter was introduced by Hsu [6] to unify the concepts of diameter and connectivity.

The $(w-1)$*-fault diameter* of G is $D_w(G) = \max\{d(G-S)\colon |S| \leqslant w-1\}$ for $w \leqslant k(G)$. This notion was defined by Hsu [6], and the special case in which $w = k(G)$ was first defined by Krishnamoorthy and Krishnamurthy [9], who studied the fault-tolerant properties of graphs and networks.

The *w-Rabin number* $r_w(G)$ of a network G is the minimum l such that for any $w+1$ distinct vertices x, $y_1, \ldots, y_w$ there exist w vertex-disjoint paths of length at most l from x to $y_1, y_2, \ldots, y_w$. This concept was first defined by Hsu [6], and the special case in which $w = k(G)$ was studied by Rabin [14] in conjunction with a randomized routing algorithm.

It is clear that when $w = 1$, $d_1(G) = D_1(G) = r_1(G) = d(G)$ for any network G. On the other hand, these parameters can be very large, as in the case in which $w = k(G)$. For example, Hsu and Luczak [7] showed that $d_k(G) = n/2$ for some regular graphs G having connectivity and degree k and n vertices. The following are basic properties and relationships among $d_w(G)$, $D_w(G)$, and $r_w(G)$.

Lemma 1 (Liaw et al. [11]). *The following statements hold for any network G of connectivity k.*

(1) $D_1(G) \leqslant D_2(G) \leqslant \cdots \leqslant D_k(G)$.

(2) $d_1(G) \leqslant d_2(G) \leqslant \cdots \leqslant d_k(G)$.

(3) $r_1(G) \leqslant r_2(G) \leqslant \cdots \leqslant r_k(G)$.

(4) $D_w(G) \leqslant d_w(G)$ *and* $D_w(G) \leqslant r_w(G)$ *for* $1 \leqslant w \leqslant k$.

This paper examines the above parameters for Butterfly networks, which are also known as banyan networks in the literature, see [2,4,5,15] for discussions of these networks as multistage interconnection networks. The *Butterfly network* B_n is the graph whose vertices are $x = (x_0, x_1, \ldots, x_n)$ with $0 \leqslant x_0 \leqslant n$ and $x_i \in \{0, 1\}$ for $1 \leqslant i \leqslant n$, and two vertices x and y are adjacent if and only if $y_0 = x_0 + 1$ and $x_i = y_i$ for $1 \leqslant i \leqslant n$ with $i \neq y_0$. Note that B_1 is a 4-cycle. For a vertex $x = (x_0, x_1, \ldots, x_n)$ in B_n, we say that x is in *level* x_0 of B_n and call x_i the *ith coordinate* of x. Fig. 1 shows an example of B_3, in which the top row indicates the level numbers and the left column indicates the names $(x_1, x_2, \ldots, x_n)$.

Cao et al. [1] gave the connectivity, the diameter, the fault diameter, and bounds of the wide diameter and the Rabin number of the Butterfly network B_n as follows:

Theorem 2 (Cao et al. [1]). *If* $n \geqslant 2$, *then* $k(B_n) = 2$, $d(B_n) = 2n$, $D_2(B_n) = 2n+2$, $2n+2 \leqslant d_2(B_n) \leqslant 2n+4$, *and* $2n+2 \leqslant r_2(B_n) \leqslant 2n+4$.

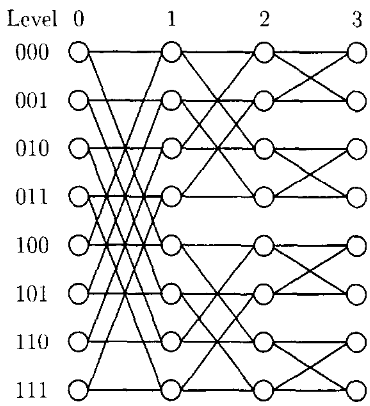

Fig. 1. The Butterfly network B_3.

In a previous paper [13], we determined the exact value of the wide diameter of B_n:

Theorem 3 (Liaw and Chang [13]). *If* $n \geqslant 2$, *then* $d_2(B_n) = 2n + 2$.

In the same paper, we proposed the following conjecture.

Conjecture: If $n \geqslant 2$, then $r_2(B_n) = 2n + 2$.

In this paper, we confirm the conjecture.

2. The Rabin number $r_2(B_n)$

The *inverse* B_n^{-1} of a Butterfly network B_n is the network obtained from B_n by interchange levels i and $n-i$ for $0 \leqslant i \leqslant n$. It is trivial that B_n is isomorphic to B_n^{-1} by the following mapping:

$$(x_0, x_1, x_2, \ldots, x_{n-1}, x_n) \to (n - x_0, x_n, x_{n-1}, \ldots, x_2, x_1).$$

This is useful in the proof of our main result.

For any $a \in \{0, 1\}$, $\bar{a}$ is defined to be $1 - a$. Suppose y and x are two vertices with $y_0 = i \leqslant j = x_0$ and $y_k = x_k$ for $k \in \{1, 2, \ldots, i\} \cup \{j+1, j+2, \ldots, n\}$. Denoted as $P_{i,j}(y, x)$, or $P_{i,j}$ with y and x specified, the following path of length $j - i$ from y to x:

$$\begin{aligned}
&(i, y_1, \ldots, y_i, y_{i+1}, y_{i+2}, y_{i+3}, \ldots, y_j, y_{j+1}, \ldots, y_n) \\
&\quad \to (i+1, y_1, \ldots, y_i, x_{i+1}, y_{i+2}, y_{i+3}, \ldots, y_j, y_{j+1}, \ldots, y_n) \\
&\quad \to (i+2, y_1, \ldots, y_i, x_{i+1}, x_{i+2}, y_{i+3}, \ldots, y_j, y_{j+1}, \ldots, y_n) \\
&\quad \to \cdots \\
&\quad \to (j, y_1, \ldots, y_i, x_{i+1}, x_{i+2}, x_{i+3}, \ldots, x_j, y_{j+1}, \ldots, y_n).
\end{aligned}$$

Similarly, if y and x are two vertices with $y_0 = i \geqslant j = x_0$ and $y_k = x_k$ for $k \in \{1,2,\ldots,j\} \cup \{i+1, i+2, \ldots, n\}$. Denoted as $Q_{i,j}(y,x)$, or $Q_{i,j}$ with y and x specified, the following path of length $i-j$ from y to x:

$$\begin{aligned}&(i, y_1, \ldots, y_j, y_{j+1}, \ldots, y_{i-2}, y_{i-1}, y_i, y_{i+1}, \ldots, y_n)\\ &\quad \to (i-1, y_1, \ldots, y_j, y_{j+1}, \ldots, y_{i-2}, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)\\ &\quad \to (i-2, y_1, \ldots, y_j, y_{j+1}, \ldots, y_{i-2}, x_{i-1}, x_i, y_{i+1}, \ldots, y_n)\\ &\quad \to \cdots\\ &\quad \to (j, y_1, \ldots, y_j, x_{j+1}, \ldots, x_{i-2}, x_{i-1}, x_i, y_{i+1}, \ldots, y_n).\end{aligned}$$

We are now ready to prove the main result.

Theorem 4. *If $n \geqslant 2$, then $r_2(B_n) = 2n+2$.*

Proof. According to Theorem 2, it suffices to show that for any three distinct vertices $y = (y_0, y_1, \ldots, y_n)$, $x^1 = (x^1_0, x^1_1, \ldots, x^1_n)$, $x^2 = (x^2_0, x^2_1, \ldots, x^2_n)$, there exist two vertex-disjoint paths of lengths at most $2n+2$ from y to x^1 and y to x^2, respectively. We, in fact, will construct two vertex-disjoint y–x^1 and y–x^2 walks, based on the following three cases. Without loss of generality, we may assume that $x^1_0 \geqslant x^2_0$.

Case 1: $x^1_0 \geqslant y_0 \geqslant x^2_0$. As $B_n = B_n^{-1}$, we only need to consider the case in which $y_0 > 0$. The y–x^1 walk is

$$W = Q_{y_0,0}(y, u^1) P_{0,n}(u^1, u^2) Q_{n,x^1_0}(u^2, x^1),$$

where

$$\begin{aligned}y &= (y_0, y_1, \ldots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \ldots, y_n),\\ u^1 &= (0, y_1, \ldots, y_{y_0-1}, \overline{x^2_{y_0}}, y_{y_0+1}, \ldots, y_n),\\ u^2 &= (n, x^1_1, \ldots, x^1_{y_0-1}, x^1_{y_0}, x^1_{y_0+1}, \ldots, x^1_n),\\ x^1 &= (x^1_0, x^1_1, \ldots, x^1_{y_0-1}, x^1_{y_0}, x^1_{y_0+1}, \ldots, x^1_n).\end{aligned}$$

Note that the total length of W is $y_0 + n + (n - x^1_0) = 2n + y_0 - x^1_0 \leqslant 2n$. The y–x^2 walk is

$$W' = Q_{y_0, y_0-1}(y, v^1) P_{y_0-1,n}(v^1, v^2) Q_{n,0}(v^2, v^3) P_{0,x^2_0}(v^3, x^2),$$

where

$$\begin{aligned}y &= (y_0, y_1, \ldots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \ldots, y_n),\\ v^1 &= (y_0 - 1, y_1, \ldots, y_{y_0-1}, x^2_{y_0}, y_{y_0+1}, \ldots, y_n),\\ v^2 &= (n, y_1, \ldots, y_{y_0-1}, \overline{x^1_{y_0}}, y_{y_0+1}, \ldots, y_n),\\ v^3 &= (0, x^2_1, \ldots, x^2_{y_0-1}, x^2_{y_0}, x^2_{y_0+1}, \ldots, x^2_n),\\ x^2 &= (x^2_0, x^2_1, \ldots, x^2_{y_0-1}, x^2_{y_0}, x^2_{y_0+1}, \ldots, x^2_n).\end{aligned}$$

Note that the total length of W' is $1+(n-y_0+1)+n+x_0^2=2n+2-y_0+x_0^2\leqslant 2n+2$. Moreover, vertices in W and W' differ at the y_0th coordinate and hence are disjoint, except the special case in which $x^2=(y_0,x_1^1,x_2^1,\dots,x_{y_0}^1,y_{y_0+1},y_{y_0+2},\dots,y_n)$ is a vertex in $P_{0,n}(u^1,u^2)$ in W. For this special case, we may assume $x_0^1=y_0$, otherwise we consider y, x^1, x^2 in B_n^{-1} to avoid the special case. In this case, we only need to exchange the roles of x^1 and x^2 in the above process. From W and W' we can find two vertex-disjoint y–x^1 and y–x^2 paths as desired.

Case 2: $y_0-1=x_0^1\geqslant x_0^2$. The arguments in Case 1 also work except when $x^1=(y_0-1, y_1,y_2,\dots,y_{y_0-1},x_{y_0}^2,x_{y_0+1}^2,\dots,x_n^2)$ or $(y_0-1,y_1,y_2,\dots,y_{y_0-1},x_{y_0}^2,y_{y_0+1},\dots,y_n)$ is a vertex in $Q_{n,0}(v^2,v^3)$ in W' or is equal to v^1. We consider the following two sub-cases.

Case 2.1: $y_0-1=x_0^1>x_0^2$. Let $a=\lceil (y_0+x_0^2-2)/2\rceil$. The y–x^1 walk is

$$W=Q_{y_0,y_0-1}(y,u^1)P_{y_0-1,n}(u^1,u^2)Q_{n,x_0^1}(u^2,x^1),$$

where

$$\begin{aligned}
y&=(y_0,y_1,\dots,y_a,y_{a+1},y_{a+2},\dots,y_{y_0-1},y_{y_0},y_{y_0+1},\dots,y_n),\\
u^1&=(y_0-1,y_1,\dots,y_a,y_{a+1},y_{a+2},\dots,y_{y_0-1},x_{y_0}^2,y_{y_0+1},\dots,y_n),\\
u^2&=(n,y_1,\dots,y_a,y_{a+1},y_{a+2},\dots,y_{y_0-1},x_{y_0}^2,y_{y_0+1},\dots,y_n^1),\\
x^1&=(y_0-1,y_1,\dots,y_a,y_{a+1},y_{a+2},\dots,y_{y_0-1},x_{y_0}^2,x_{y_0+1}^1,\dots,x_n^1).
\end{aligned}$$

Note that the total length of W is $1+(n-x_0^1)+(n-x_0^1)=2n+1-2x_0^1\leqslant 2n+1$. The y–x^2 walk is

$$W'=Q_{y_0,a}(y,v^1)P_{a,n}(v^1,v^2)Q_{n,0}(v^2,v^3)P_{0,x_0^2}(v^3,x^2),$$

where

$$\begin{aligned}
y&=(y_0,y_1,\dots,y_a,y_{a+1},y_{a+2},\dots,y_{y_0-1},y_{y_0},y_{y_0+1},\dots,y_n),\\
v^1&=(a,y_1,\dots,y_a,\overline{y_{a+1}},y_{a+2},\dots,y_{y_0-1},\overline{x_{y_0}^2},y_{y_0+1},\dots,y_n),\\
v^2&=(n,y_1,\dots,y_a,\overline{y_{a+1}},y_{a+2},\dots,y_{y_0-1},\overline{x_{y_0}^2},y_{y_0+1},\dots,y_n),\\
v^3&=(0,x_1^2,\dots,x_a^2,x_{a+1}^2,x_{a+2}^2,\dots,x_{y_0-1}^2,x_{y_0}^2,x_{y_0+1}^2,\dots,x_n^2),\\
x^2&=(x_0^2,x_1^2,\dots,x_a^2,x_{a+1}^2,x_{a+2}^2,\dots,x_{y_0-1}^2,x_{y_0}^2,x_{y_0+1}^2,\dots,x_n^2).
\end{aligned}$$

Note that the total length of W' is $(y_0-a)+(n-a)+n+x_0^2=2n-2a+y_0+x_0^2\leqslant 2n+2$. Moreover, between levels n and y_0, vertices in W and W' differ at $(a+1)$th coordinate; between levels y_0 and 0, vertices in W and W' differ at y_0th coordinate. So, W and W' are vertex-disjoint.

Case 2.2: $y_0-1=x_0^1=x_0^2$. The y–x^1 walk W is the same as in Case 2.1. The y–x^2 walk is

$$W'=Q_{y_0,0}(y,v^1)P_{0,n}(v^1,v^2)Q_{n,x_0^2}(v^2,x^2),$$

where

$$y=(y_0,y_1,\ldots,y_{y_0-1},y_{y_0},y_{y_0+1},\ldots,y_n),$$
$$v^1=(0,y_1,\ldots,y_{y_0-1},\overline{x^2_{y_0}},y_{y_0+1},\ldots,y_n),$$
$$v^2=(n,x^2_1,\ldots,x^2_{y_0-1},\overline{x^2_{y_0}},x^2_{y_0+1},\ldots,x^2_n),$$
$$x^2=(x^2_0,x^2_1,\ldots,x^2_{y_0-1},x^2_{y_0},x^2_{y_0+1},\ldots,x^2_n).$$

Note that the total length of W' is $y_0+n+(n-x^2_0)=2n+y_0-x^2_0=2n+1$. Moreover, vertices in W and W' differ at the y_0th coordinate and hence are disjoint, except the special case in which $x^2=(y_0-1,y_1,y_2,\ldots,y_{y_0-1},x^2_{y_0},y_{y_0+1},\ldots,y_n)$. For this special case, we only need to exchange the roles of x^1 and x^2 in the above process.

Case 3: $y_0-1>x^1_0\geqslant x^2_0$.

Case 3.1: $x^1\neq(x^1_0,y_1,y_2,\ldots,y_{x^1_0},x^2_{x^1_0+1},x^2_{x^1_0+2},\ldots,x^2_n)$. Let $a=\lceil(y_0+x^1_0-2)/2\rceil$. The y–x^1 walk is

$$W=P_{y_0,n}(y,u^1)Q_{n,a}(u^1,u^2)Q_{a,0}(u^2,u^3)P_{0,a+1}(u^3,u^4)Q_{a+1,x^1_0}(u^4,x^1),$$

where

$$y=(y_0,y_1,\ldots,y_{x^1_0},y_{x^1_0+1},y_{x^1_0+2},\ldots,y_a,y_{a+1},y_{a+2},\ldots,y_{y_0-1},y_{y_0},y_{y_0+1},\ldots,y_n),$$
$$u^1=(n,y_1,\ldots,y_{x^1_0},y_{x^1_0+1},y_{x^1_0+2},\ldots,y_a,y_{a+1},y_{a+2},\ldots,y_{y_0-1},y_{y_0},y_{y_0+1},\ldots,y_n),$$
$$u^2=(a,y_1,\ldots,y_{x^1_0},y_{x^1_0+1},y_{x^1_0+2},\ldots,y_a,\overline{x^2_{a+1}},x^1_{a+2},\ldots,x^1_{y_0-1},x^1_{y_0},x^1_{y_0+1},\ldots,x^1_n),$$
$$u^3=(0,x^1_1,\ldots,x^1_{x^1_0},x^1_{x^1_0+1},x^1_{x^1_0+2},\ldots,x^1_a,\overline{x^2_{a+1}},x^1_{a+2},\ldots,x^1_{y_0-1},x^1_{y_0},x^1_{y_0+1},\ldots,x^1_n),$$
$$u^4=(a+1,x^1_1,\ldots,x^1_{x^1_0},\overline{y_{x^1_0+1}},x^1_{x^1_0+2},\ldots,x^1_a,\overline{x^2_{a+1}},x^1_{a+2},\ldots,x^1_{y_0-1},x^1_{y_0},x^1_{y_0+1},\ldots,x^1_n),$$
$$x^1=(x^1_0,x^1_1,\ldots,x^1_{x^1_0},x^1_{x^1_0+1},x^1_{x^1_0+2},\ldots,x^1_a,x^1_{a+1},x^1_{a+2},\ldots,x^1_{y_0-1},x^1_{y_0},x^1_{y_0+1},\ldots,x^1_n).$$

Note that the total length of W is $(n-y_0)+(n-a)+a+(a+1)+(a+1-x^1_0)=2n+2+2a-y_0-x^1_0\leqslant 2n+1$. The y–x^2 walk is

$$W'=Q_{y_0,a}(y,v^1)P_{a,n}(v^1,v^2)Q_{n,0}(v^2,v^3)P_{0,x^2_0}(v^3,x^2),$$

where

$$y=(y_0,y_1,\ldots,y_{x^1_0},y_{x^1_0+1},y_{x^1_0+2},\ldots,y_a,y_{a+1},y_{a+2},\ldots,y_{y_0-1},y_{y_0},y_{y_0+1},\ldots,y_n),$$
$$v^1=(a,y_1,\ldots,y_{x^1_0},y_{x^1_0+1},y_{x^1_0+2},\ldots,y_a,\overline{y_{a+1}},y_{a+2},\ldots,y_{y_0-1},\overline{x^1_{y_0}},y_{y_0+1},\ldots,y_n),$$
$$v^2=(n,y_1,\ldots,y_{x^1_0},y_{x^1_0+1},y_{x^1_0+2},\ldots,y_a,\overline{y_{a+1}},y_{a+2},\ldots,y_{y_0-1},\overline{x^1_{y_0}},y_{y_0+1},\ldots,y_n),$$
$$v^3=(0,x^2_1,\ldots,x^2_{x^1_0},x^2_{x^1_0+1},x^2_{x^1_0+2},\ldots,x^2_a,x^2_{a+1},x^2_{a+2},\ldots,x^2_{y_0-1},x^2_{y_0},x^2_{y_0+1},\ldots,x^2_n),$$
$$x^2=(x^2_0,x^2_1,\ldots,x^2_{x^1_0},x^2_{x^1_0+1},x^2_{x^1_0+2},\ldots,x^2_a,x^2_{a+1},x^2_{a+2},\ldots,x^2_{y_0-1},x^2_{y_0},x^2_{y_0+1},\ldots,x^2_n).$$

Note that the total length of W' is $(y_0-a)+(n-a)+n+x^2_0=2n-2a+y_0+x^2_0\leqslant 2n-2a+y_0+x^1_0\leqslant 2n+2$. Moreover, between levels n and y_0, vertices in W and W'

differ at $(a+1)$th coordinate; between levels y_0 and 0, vertices in W and W' differ at y_0th, $(a+1)$th, or (x_0^1+1)th coordinate. So, W and W' are vertex-disjoint.

Case 3.2: $x^1=(x_0^1, y_1, y_2, \ldots, y_{x_0^1}, x^2_{x_0^1+1}, x^2_{x_0^1+2}, \ldots, x_n^2)$. Let $a=\lceil (y_0+x_0^2-2)/2 \rceil$.

Case 3.2.1: $a \geqslant x_0^1$. The y–x^1 walk is

$$W=Q_{y_0,a}(y,u^1)P_{a,n}(u^1,u^2)Q_{n,x_0^1}(u^2,x^1),$$

where

$$\begin{aligned}
y&=(y_0, y_1, \ldots, y_{x_0^2}, y_{x_0^2+1}, y_{x_0^2+2}, \ldots, y_a, y_{a+1}, y_{a+2}, \ldots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \ldots, y_n),\\
u^1&=(a, y_1, \ldots, y_{x_0^2}, y_{x_0^2+1}, y_{x_0^2+2}, \ldots, y_a, \overline{y_{a+1}}, y_{a+2}, \ldots, y_{y_0-1}, \overline{x^2_{y_0}}, y_{y_0+1}, \ldots, y_n),\\
u^2&=(n, y_1, \ldots, y_{x_0^2}, y_{x_0^2+1}, y_{x_0^2+2}, \ldots, y_a, \overline{y_{a+1}}, y_{a+2}, \ldots, y_{y_0-1}, \overline{x^2_{y_0}}, y_{y_0+1}, \ldots, y_n),\\
x^1&=(x_0^1, y_1, \ldots, y_{x_0^1}, x^2_{x_0^1+1}, x^2_{x_0^1+2}, \ldots, x^2_a, x^2_{a+1}, x^2_{a+2}, \ldots, x^2_{y_0-1}, x^2_{y_0}, x^2_{y_0+1}, \ldots, x^2_n).
\end{aligned}$$

Note that the total length of W is $(y_0-a)+(n-a)+(n-x_0^1)=2n-2a+y_0-x_0^1 \leqslant 2n-2a+y_0+x_0^2 \leqslant 2n+2$. The y–x^2 walk is

$$W'=P_{y_0,n}(y,v^1)Q_{n,a}(v^1,v^2)Q_{a,0}(v^2,v^3)P_{0,a+1}(v^3,v^4)Q_{a+1,x_0^1}(v^4,x^2),$$

where

$$\begin{aligned}
y&=(y_0, y_1, \ldots, y_{x_0^2}, y_{x_0^2+1}, y_{x_0^2+2}, \ldots, y_a, y_{a+1}, y_{a+2}, \ldots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \ldots, y_n),\\
v^1&=(n, y_1, \ldots, y_{x_0^2}, y_{x_0^2+1}, y_{x_0^2+2}, \ldots, y_a, y_{a+1}, y_{a+2}, \ldots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \ldots, y_n),\\
v^2&=(a, y_1, \ldots, y_{x_0^2}, y_{x_0^2+1}, y_{x_0^2+2}, \ldots, y_a, \overline{x^1_{a+1}}, x^2_{a+2}, \ldots, x^2_{y_0-1}, x^2_{y_0}, x^2_{y_0+1}, \ldots, x^2_n),\\
v^3&=(0, x_1^2, \ldots, x^2_{x_0^2}, x^2_{x_0^2+1}, x^2_{x_0^2+2}, \ldots, x^2_a, \overline{x^1_{a+1}}, x^2_{a+2}, \ldots, x^2_{y_0-1}, x^2_{y_0}, x^2_{y_0+1}, \ldots, x^2_n),\\
v^4&=(a+1, x_1^2, \ldots, x^2_{x_0^2}, \overline{y_{x_0^2+1}}, x^2_{x_0^2+2}, \ldots, x^2_a, \overline{x^1_{a+1}}, x^2_{a+2}, \ldots, x^2_{y_0-1}, x^2_{y_0}, x^2_{y_0+1}, \ldots, x^2_n),\\
x^2&=(x_0^2, x_1^2, \ldots, x^2_{x_0^2}, x^2_{x_0^2+1}, x^2_{x_0^2+2}, \ldots, x^2_a, x^2_{a+1}, x^2_{a+2}, \ldots, x^2_{y_0-1}, x^2_{y_0}, x^2_{y_0+1}, \ldots, x^2_n).
\end{aligned}$$

Note that the total length of W' is $(n-y_0)+(n-a)+a+(a+1)+(a+1-x_0^2)=2n+2+2a-y_0-x_0^2 \leqslant 2n+1$. Moreover, between levels n and y_0, vertices in W and W' differ at $(a+1)$th coordinate; between levels y_0 and 0, vertices in W and W' differ at y_0th, $(a+1)$th, or (x_0^2+1)th coordinate. So, W and W' are vertex-disjoint.

Case 3.2.2: $a<x_0^1$. The y–x^1 walk is

$$W=P_{y_0,n}(y,u)Q_{n,x_0^1}(u,x^1),$$

where

$$\begin{aligned}
y&=(y_0, y_1, \ldots, y_a, y_{a+1}, y_{a+2}, \ldots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \ldots, y_n),\\
u&=(n, y_1, \ldots, y_a, y_{a+1}, y_{a+2}, \ldots, y_{y_0-1}, y_{y_0}, y_{y_0+1}, \ldots, y_n),\\
x^1&=(x_0^1, y_1, \ldots, y_{x_0^1}, x^2_{x_0^1+1}, x^2_{x_0^1+2}, \ldots, x^2_{y_0-1}, x^2_{y_0}, x^2_{y_0+1}, \ldots, x^2_n).
\end{aligned}$$

Note that the total length of W is $(n-y_0)+(n-x_0^1)=2n-y_0-x_0^1 \leqslant 2n-2$. The y–x^2 walk is

$$W' = Q_{y_0,a}(y,v^1)P_{a,n}(v^1,v^2)Q_{n,0}(v^2,v^3)P_{0,x_0^2}(v^3,x^2),$$

where

$$y=(y_0,y_1,\ldots,y_a,y_{a+1},y_{a+2},\ldots,y_{y_0-1},y_{y_0},y_{y_0+1},\ldots,y_n),$$

$$v^1=(a,y_1,\ldots,y_a,\overline{y_{a+1}},y_{a+2},\ldots,y_{y_0-1},\overline{x_{y_0}^1},y_{y_0+1},\ldots,y_n),$$

$$v^2=(n,y_1,\ldots,y_a,\overline{y_{a+1}},y_{a+2},\ldots,y_{y_0-1},\overline{x_{y_0}^1},y_{y_0+1},\ldots,y_n),$$

$$v^3=(0,x_1^2,\ldots,x_a^2,x_{a+1}^2,x_{a+2}^2,\ldots,x_{y_0-1}^2,x_{y_0}^2,x_{y_0+1}^2,\ldots,x_n^2),$$

$$x^2=(x_0^2,x_1^2,\ldots,x_a^2,x_{a+1}^2,x_{a+2}^2,\ldots,x_{y_0-1}^2,x_{y_0}^2,x_{y_0+1}^2,\ldots,x_n^2).$$

Note that the total length of W' is $(y_0-a)+(n-a)+n+x_0^2=2n-2a+y_0+x_0^2 \leqslant 2n+2$. Moreover, between levels n and y_0, vertices in W and W' differ at $(a+1)$th coordinate; between levels y_0 and 0, vertices in W and W' differ at y_0th or $(a+1)$th coordinate. So, W and W' are vertex-disjoint. □

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