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Integral Equations and Operator Theory

Singular Unitary Dilations

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We study the problem of determining which bounded linear operator on a Hilbert space can be dilated to a singular unitary operator. Some of the partial results we obtained are (1) every strict contraction has a diagonal unitary dilation, (2) every C_0 contraction has a singular unitary dilation, and (3) a contraction with one of its defect indices finite has a singular unitary dilation if and only if it is the direct sum of a singular unitary operator and a $C_0(N)$ contraction. Such results display a scenario which is in marked contrast to that of the classical case where we have the absolute continuity of the minimal unitary power dilation of any completely nonunitary contraction.

1. INTRODUCTION

Let A and B be bounded linear operators on the complex Hilbert spaces H and K, respectively. A is said to *dilate* to B if there is an isometry V from H to K such that $A = V^*BV$ or, equivalently, if B is unitarily equivalent to some 2×2 operator matrix

$$\begin{bmatrix} A & * \\ * & * \end{bmatrix}$$

with A in its upper left corner. In this paper, we launch the study of the problem as to which operator can be dilated to a singular unitary operator. (Recall that a unitary operator is *singular* (resp. *absolutely continuous*) if its spectral measure is singular (resp. absolutely continuous) with respect to the Lebesgue measure on the unit circle).

An early dilation result due to Halmos says that every contraction $T(\| T \| \leq 1)$ has a unitary dilation

$$\begin{bmatrix} T & (1-TT^*)^{\frac{1}{2}} \\ (1-T^*T)^{\frac{1}{2}} & -T^* \end{bmatrix}.$$

This is later generalized by Sz.-Nagy to his celebrated power dilation theorem: for every contraction T on H, there is a unitary operator U on K and an isometry V from H to K such that $T^n = V^*U^nV$ for all positive integer n. If T is further required to be *completely nonunitary* (*c.n.u.*), that is, T has no nontrivial reducing subspace on which T is unitary, and the unitary dilation U is required to be *minimal* in the sense that $K = \bigvee_{n=0}^{\infty} U^n VH$, then U is unique (up to unitary equivalence) and is absolutely continuous. This property of the minimal unitary power dilation can be exploited to yield a functional calculus for c.n.u. contractions. Its further development by Sz.-Nagy and Foias in the 60s culminates in a functional model for such contractions revealing its rich structure theory. Such developments are all chronicled in the by now classic monograph [14].

Contrary to the classical case, dilation to singular unitary operators, the other extreme of the absolutely continuous ones, seems not to have been systematically studied. Hopefully, their study here will lead to a deeper understanding of the structure of contractions. A special case of such dilations is the dilation to diagonal unitary operators. Recall that an operator is *diagonal* if it is unitarily equivalent to a diagonal matrix

$$\operatorname{diag}(\lambda_n) = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & 0 & \\ & & \cdot & \\ & 0 & \cdot & \\ & & & \cdot \end{bmatrix}.$$

In Section 2 below, we start by considering diagonal unitary dilations. One tool we use for our purpose is a result of Nakamura [12] that an operator T has its numerical range contained in a triangle Δabc if and only if T dilates to the diagonal operator $aI \oplus bI \oplus cI$. As consequences, we are able to show that every operator with numerical radius no bigger than $\frac{1}{2}$ and every c.n.u. normal contraction has a diagonal unitary dilation. These can also be proved, together with the diagonal unitary dilation for strict contractions and nilpotent contractions, via a deeper theorem of Arveson [3, Theorem 1.3.1] on the power dilation to the direct sum of copies of a Jordan block. The class of operators with a diagonal unitary dilation can be enlarged to include all algebraic contractions and even certain C_0 contractions; this will be further pursued in Section 3.

The main result in Section 3 is Theorem 3.1 in which we give a characterization of those contractions T with at least one of its defect indices finite which adimit a singular unitary dilation: T has such a dilation if and only if it is the direct sum of a singular unitary operator and a $C_0(N)$ contraction. The proof of the necessity is based on a certain generalization of Carey's finite-rank perturbation result [5], which is applicable here because of our finite defect index assumption. On the other hand, the sufficiency follows from the more general theorem that every C_0 contraction has a singular unitary dilation.

Section 4 is devoted to the related problem of the finite-rank perturbation of singular unitary operators. It turns out that the class of contractions which are such a perturbation coincides with those with finite defect indices which admit a singular unitary dilation. We end in Section 5 with two open problems.

For convenience, in the following we will only consider operators on a separable Hilbert space. The monographs [14] and [4] will be our references for the terminologies and results of the Sz.-Nagy-Foias contraction theory. Some of the theorems and the ideas of their proofs in this paper have been announced in [19].

2. DIAGONAL UNITARY DILATION

Since every contraction can be uniquely decomposed as the direct sum of a unitary operator and a c.n.u. contraction, our first result below says that in the considering of the diagonal or singular unitary dilations we may restrict ourselves to the c.n.u. ones.

PROPOSITION 2.1. *An operator T dilates to a diagonal (resp. singular) unitary operator if and only if T is a contraction, its unitary part is diagonal (resp. singular) and its c.n.u. part dilates to a diagonal (resp. singular) unitary operator.*

We omit its easy proof.

The next result gives an alternative expression for dilating to a diagonal unitary operator in terms of an (infinite) C^*-convex combination. It is an easy consequence of [16, Lemma 3.2]. We include the proof here for completeness.

PROPOSITION 2.2. *An operator T dilates to a diagonal unitary operator if and only if $T = \sum_n \lambda_n T_n$ in the strong operator topology (SOT), where $|\lambda_n| = 1$ for all n and the T_n's are positive operators satisfying $\sum_n T_n = 1$ in SOT. In this case, all the T_n's may be taken to be of rank one.*

PROOF. If T on H dilates to $D = \mathrm{diag}(\lambda_n)$ on ℓ^2, say, $T = V^*DV$ for some isometry V from H to ℓ^2, then, letting $T_n = V^*P_nV$, where P_n denotes the rank-one (orthogonal) projection

$$P_n(x_0, x_1, \cdots, x_n, \cdots) = (0, 0, \cdots, x_n, 0, \cdots)$$

on ℓ^2, we obtain $T = \sum_n \lambda_n T_n$ in SOT. On the other hand, $V^*V = 1$ implies that $\sum_n T_n = 1$ in SOT.

For the converse, if λ_n and T_n satisfy the asserted conditions, then the operator $V = [T_0^{\frac{1}{2}}\ T_1^{\frac{1}{2}}\ \cdots]^t$ is an isometry from H to $\sum_n \oplus H$ satisfying $T = V^*DV$, where D is the diagonal unitary operator

$$D(\sum_n \oplus x_n) = \sum_n \oplus \lambda_n x_n \quad \text{for} \quad \sum_n \oplus x_n \quad \text{in} \quad \sum_n \oplus H.$$

This shows that T dilates to D as asserted. ∎

A special case of the preceding proposition when there are only three $\lambda_n' s$ can be further characterized in terms of the numerical range. This was given in [20, Proposition 2.5]; the equivalence of (a) and (b) is due to Nakamura [12]. Recall that the *numerical range* $W(T)$ of an operator T on H is the set $\{\langle Tx, x\rangle : x \in H, \| x \| = 1\}$ and the *numerical radius* $w(T)$ of T is $\sup\{|\langle Tx, x\rangle| : x \in H, \| x \| = 1\}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in H. For their properties, readers may consult [11, Chapter 22].

PROPOSITION 2.3. *For any operator T, the following conditions are equivalent:*

(a) *$W(T)$ is contained inside a triangle Δabc;*

(b) *$T = aT_1 + bT_2 + cT_3$ for some positive operators T_1, T_2 and T_3 with $T_1 + T_2 + T_3 = 1$;*

(c) *T can be dilated to a normal operator with spectrum consisting of a, b and c.*

Two consequences of this proposition are the following.

COROLLARY 2.4. *If the numerical radius of an operator T is at most $\frac{1}{2}$, then T dilates to a diagonal unitary operator with three points in its spectrum.*

PROOF. Since the hypothesis implies that $W(T)$ is contained in any equilateral triangle inscribed in the unit circle, the conclusion follows from Proposition 2.3. ∎

This corollary was pointed out to the first author by D. Farenick and was noted before in [20].

COROLLARY 2.5. *A normal operator dilates to a diagonal (resp. singular) unitary operator if and only if it is a contraction whose unitary part is diagonal (resp. singular).*

PROOF. In view of Proposition 2.1, we need only check that every c.n.u. normal contraction T dilates to a diagonal unitary operator. By the spectral theorem, T can be written as the direct sum $\sum_n \oplus T_n$, where each T_n has its spectrum contained in a triangle whose three vertices are rational points on the unit circle. By Proposition 2.3, each T_n dilates to a diagonal unitary operator, say, U_n, and therefore T dilates to the singular unitary $\sum_n \oplus U_n$. ∎

Another powerful result which can be enlisted for our purpose is a deep theorem of Arveson [3, Theorem 1.3.1]. For any $n \geq 1$, let J_n denote the $n \times n$ nilpotent Jordan block

$$J_n = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix}.$$

THEOREM 2.6. *Let T be a contraction and $2 \leq n < \infty$. Then the following are equivalent:*

(a) $T^j, j = 1, \cdots, n-1$, *dilates simultaneously to* $J_n^j \oplus J_n^j \oplus \cdots$;

(b) $1 + 2\mathrm{Re}\sum_{j=1}^{n-1} \lambda^j T^j \geq 0$ *holds for all* $\lambda, |\lambda| = 1$;

(c) $2\mathrm{Re}[(1-\lambda T)^* \lambda^n T^n] \leq 1 - T^*T$ *holds for all* $\lambda, |\lambda| = 1$.

Note that Corollary 2.4 can be proved via Theorem 2.6. Indeed, it is easily seen that $w(T) \leq \frac{1}{2}$ implies that T is a contraction and is equivalent to the condition that $1 + 2\mathrm{Re}(\lambda T) \geq 0$ for all $\lambda, |\lambda| = 1$. Hence the equivalence of (a) and (b) yields that T dilates to $J_2 \oplus J_2 \oplus \cdots$, which implies that T dilates to the diagonal unitary operator

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus \cdots .$$

PROPOSITION 2.7. *Every strict contraction has a diagonal unitary dilation.*

PROOF. If T is a strict contraction ($\| T \| < 1$), then, letting ε be such that $0 < \varepsilon < 1 - T^*T$, for any vector x and any $\lambda, |\lambda| = 1$, we have

$$\begin{aligned} & \langle 2\mathrm{Re}((1-\lambda T)^* \lambda^n T^n)x, x\rangle = 2\mathrm{Re}\langle (1-\lambda T)^* \lambda^n T^n x, x\rangle \\ \leq \quad & 2 \| (1-\lambda T)^* \lambda^n T^n \| \cdot \| x \|^2 \leq 2(1+ \| T \|) \| T \|^n \| x \|^2 \\ \leq \quad & \varepsilon \| x \|^2 \end{aligned}$$

for sufficiently large n. Since $\varepsilon \| x \|^2 \leq \langle (1 - T^*T)x, x\rangle$, this verifies condition (c) in Theorem 2.6. Hence T dilates to $J_n \oplus J_n \oplus \cdots$ and therefore it dilates to the singular unitary operator

$$\begin{bmatrix} 0 & & & 1 \\ 1 & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & & & 1 \\ 1 & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & 1 & 0 \end{bmatrix} \oplus \cdots . \blacksquare$$

We remark that Corollary 2.5 can also be proved via Proposition 2.7. Indeed, if T is a c.n.u. normal contraction, then by the spectral theorem, $T = \sum_n \oplus T_n$, where each

T_n is a strict contraction. Thus Proposition 2.7 implies that T dilates to a diagonal unitary operator.

Another consequence of Theorem 2.6 is that every nilpotent contraction has a diagonal unitary dilation. For if the contraction T is such that $T^n = 0$, then (c) of Theorem 2.6 obviously holds and hence T dilates to $J_n \oplus J_n \oplus \cdots$ which implies our assertion as before. However, this particular result can be proved in a more elementary fashion by following the line of arguments in [11, Solution 152], which will yield that a contraction T satisfies $T^n = 0$ if and only if it power dilates to $\underbrace{J_n \oplus \cdots \oplus J_n}_{k}$, where k=dim $\overline{\text{ran}(1 - T^*T)^{\frac{1}{2}}}$ and, in this case, k is the smallest such number. This can be further generalized to the result that every algebraic contraction has a diagonal unitary dilation. Indeed, in [20, Theorem 1.4] it was proved that every algebraic contraction T can be power dilated to an oprator of the form $T_1 \oplus T_1 \oplus \cdots$, where T_1 is a cyclic contraction on a finite-dimensional space with rank$(1 - T_1^*T_1) \leq 1$ and with the same minimal polynomial as that of T. Hence T dilates to $U \oplus U \oplus \cdots$, where U is the Halmos dilation of T_1. Since U acts on a finite-dimensional space, $U \oplus U \oplus \cdots$ is a diagonal unitary dilation of T. We will further pursue this at the end of the next section by showing that even some C_0 contractions have diagonal unitary dilations.

3. SINGULAR UNITARY DILATION

For an arbitrary operator T, let d_T =dim $\overline{\text{ran}(1 - T^*T)^{\frac{1}{2}}}$ and d_{T^*}=dim $\overline{\text{ran}(1 - TT^*)^{\frac{1}{2}}}$ denote its *defect indices.* A contraction T is *of class* C_0 if it is c.n.u. and there is some nonzero function f in H^∞ such that $f(T) = 0$. In this case, there is an inner function ϕ, called the *minimal function* of T which satisfies $\phi(T) = 0$ and divides any f in H^∞ with $f(T) = 0$. Note that the defect indices of a C_0 contraction must equal. A C_0 contraction with defect indices at most $N(< \infty)$ is said to be *of class* $C_0(N)$. The main theorem of this section is the following.

THEOREM 3.1. *Let T be a contraction on H with at least one defect index finite. Then T dilates to a singular unitary operator U if and only if it is a direct sum of a singular unitary operator and a $C_0(N)$ contraction. Moreover, in this case, the singular unitary operator U can be chosen to act on a space K containing H with* $\dim(K \ominus H) = d_T$.

To prove this, we start with some lemmas which illustrate more clearly what the hypothesis of finite defect index means. The first lemma is in [10, Lemma 4].

LEMMA 3.2 *Let T be an operator from H_1 to H_2. Then* d_T+dim ker $T^* = d_{T^*}$+dim ker T.

LEMMA 3.3 *Let T be an operator on H. Then $d_T = d_{T^*} < \infty$ (resp.*

$d_T < d_{T^*}$) if and only if T is the sum of a unitary operator (resp. nonunitary isometry) and a finite-rank operator (with rank equal to d_T). In this case, T is Fredholm (resp. left Fredholm) with ind $T = 0$ *(resp.* ind $T = d_T - d_{T^*}$*).*

For the Fredholm theory, readers may consult [8, Chapter XI].

PROOF. If $d_T = d_{T^*} < \infty$, then dim ker T =dim ker T^* by Lemma 3.2. Hence T has the polar decomposition $T = U(T^*T)^{\frac{1}{2}}$ with U unitary. Since d_T =rank$(1 - T^*T)$ is finite, we have $T^*T = 1+F$, where rank $F = d_T < \infty$. Hence rank$((T^*T)^{\frac{1}{2}} - 1)$ =rank$((1+F)^{\frac{1}{2}} - 1) = d_T < \infty$ and thus $T = U + U((T^*T)^{\frac{1}{2}} - 1)$ expresses T as the sum of a unitary operator and a finite-rank operator.

To prove the converse, assume that $T = U + F$, where U is unitary and F has finite rank. Then $T^*T = (U^* + F^*)(U + F) = 1 + (U^*F + F^*U + F^*F)$ and hence $d_T < \infty$. Similarly, we have $d_{T^*} < \infty$. On the other hand, our assumption implies that T is a Fredholm operator with ind $T = 0$. Hence dim ker T =dim ker $T^* < \infty$. We conclude from Lemma 3.2 that $d_T = d_{T^*} < \infty$. Similar arguments work for the case $d_T < d_{T^*}$. (Note that in a separable space, the condition $d_T < d_{T^*}$ implicitly implies that $d_T < \infty$.) ∎

The assertion for the case $d_T < d_{T^*}$ in the preceding lemma appeared before in [2, Theorem 4.2].

Now we can bring into play the perturbation result. Recall that every isometry V has a canonical decomposition $V = U_s \oplus U_a \oplus S$, where U_s (resp. U_a) is a singular (resp. absolutely continuous) unitary operator, and S is a unilateral shift (with some multiplicity). In generalizing the celebrated Rosenblum-Kato perturbation theorem (for unitary operators), Carey proved that if V_1 and V_2 are isometries with $V_1 - V_2$ of finite rank, then the absolutely continuous unitary parts of V_1 and V_2 are unitarily equivalent (cf. [5]). Our next result can be seen as a generalization of this although its proof also depends on it.

PROPOSITION 3.4. *Let T be a contraction. If $T = V_1 + F$, where V_1 is an isometry (resp. nonunitary coisometry) and F is of finite rank, and T dilates to another isometry V_2, then the absolutely continuous unitary part of V_1 (resp. the simple bilateral shift) is a direct summand of V_2.*

PROOF. First assume that V_1 is an isometry. Let $V_1 = U \oplus S$, where U is unitary and S is a unilateral shift, and

$$F = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}$$

with respect to the same decomposition of the underlying space. Since $U + F_1$ also dilates to V_2, we may assume without loss of generality that V_1 is itself unitary.

Let $V_2 = \begin{bmatrix} V_1+F & A \\ B & C \end{bmatrix}$. Since V_2 is an isometry, we have

$$V_2^*V_2 = \begin{bmatrix} V_1^*+F^* & B^* \\ A^* & C^* \end{bmatrix} \begin{bmatrix} V_1+F & A \\ B & C \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence (1) $V_1^*V_1 + V_1^*F + F^*V_1 + F^*F + B^*B = 1$, (2) $A^*V_1 + A^*F + C^*B = 0$ and (3) $A^*A+C^*C = 1$. (1) implies that $B^*B = -V_1^*F - F^*V_1 - F^*F$ is finite-rank and hence rank $B < \infty$. This, together with (2), yields that $A^*V_1 = -A^*F - C^*B$ is finite-rank. Hence the same is true for $A^* = A^*V_1V_1^*$ or A. Since

$$V_2 = \begin{bmatrix} V_1 & 0 \\ 0 & C \end{bmatrix} + \begin{bmatrix} F & A \\ B & 0 \end{bmatrix},$$

by the Fredholm theory C is left Fredholm with

$$\text{ind } C = \text{ind } V_1 + \text{ind } C = \text{ind} \begin{bmatrix} V_1 & 0 \\ 0 & C \end{bmatrix} = \text{ind } V_2 \leq 0.$$

Thus C has the polar decomposition $C = V_3(C^*C)^{\frac{1}{2}}$ with V_3 isometry. Now (3) implies that C^*C is the sum of the identity and a finite-rank operator. The same is true for $(C^*C)^{\frac{1}{2}}$. Thus C is the sum of V_3 and a finite-rank operator. Consequently, the difference of the isometries V_2 and $V_1 \oplus V_3$ is of finite rank. Carey's result then implies that their absolutely continuous unitary parts are unitarily equivalent. In particular, this imples that the absolutely continuous unitary part of V_1 is a direct summand of V_2 as asserted.

The proof for V_1 a nonunitary coisometry is analogous to the one above. Here we only give a brief sketch. As before, we may assume that V_1 is the simple backward shift, that is, the adjoint of the unilateral shift of multiplicity one. Then we deduce that B and A are both of finite rank. By the Fredholm theory and the polar decomposition, we have $C = V_3(C^*C)^{\frac{1}{2}}$, where V_3 is a nonunitary isometry. As before, C is the sum of V_3 and a finite-rank operator, and hence the difference of V_2 and $V_1 \oplus V_3$ is of finite rank. Since the simple unilateral shift is a direct summand of V_3 and since the former is a rank-one perturbation of a simple bilateral shift W, we infer that V_2 and an isometry of the form $W \oplus V_4$ differ by a finite-rank operator. Carey's result then implies that W is a direct summand of the absolutely continuous unitary part of V_2, completing the proof. ∎

PROPOSITION 3.5. *Any c.n.u. C_{11} contraction with finite defect indices and any contraction with unequal defect indices admit no singular unitary dilation.*

A contraction T is *of class* C_{11} if $T^nx \not\to 0$ and $T^{*n}x \not\to 0$ in norm for any nonzero vector x.

PROOF. Let T be a c.n.u. C_{11} contraction which dilates to a singular unitary operator V. Then T is quasisimilar to an absolutely continuous unitary operator U (cf.[14, Proposition II.3.5.]). By [1], there is a nonzero invariant subspace K for T such that $T_1 = T|K$ is similar to a direct summand of U, say, U_1. In particular, T_1 is a c.n.u. C_{11} contraction with finite (and equal) defect indices and U_1 is an absolutely continuous unitary operator. Lemma 3.3 implies that $T_1 = U_2 + F$, where U_2 is unitary and F is finite-rank. We infer from [7, Lemma] that U_1 is unitarily equivalent to the absolutely continuous part U_a of U_2. Since $T_1 = U_2 + F$ dilates to the singular unitary V, Proposition 3.4 implies that U_a is a direct summand of V, which is impossible. This shows that T cannot have any singular unitary dilation.

On the other hand, if T is a contraction with $d_T \neq d_{T^*}$, then, without loss of generality, we may assume that $d_T > d_{T^*}$. Hence Lemma 3.3 implies that T is the sum of a nonunitary coisometry and a finite-rank operator. Our assertion for T again follows from Proposition 3.4. ∎

Now we are ready for the proof of the

NECESSITY OF THEOREM 3.1. Let T be a c.n.u. contraction with $d_T < \infty$ which admits a singular unitary dilation. We have to show that T is of class $C_0(N)$. Let $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ be the triangulation of type $\begin{bmatrix} C_{.1} & * \\ 0 & C_{.0} \end{bmatrix}$ (cf. [14, Theorem II. 4.1]). Since $d_T < \infty, T_1$ and T_2 have the triangulations $T_1 = \begin{bmatrix} T_3 & * \\ 0 & T_4 \end{bmatrix}$ and $T_2 = \begin{bmatrix} T_5 & * \\ 0 & T_6 \end{bmatrix}$ of type $\begin{bmatrix} C_{01} & * \\ 0 & C_{11} \end{bmatrix}$ and $\begin{bmatrix} C_{00} & * \\ 0 & C_{10} \end{bmatrix}$, respectively (cf. [18, Lemma 3.2]). We can deduce that $d_{T_1}, d_{T_2} \leq d_T < \infty$, and also $d_{T_3^*} < d_{T_3} \leq d_{T_1} < \infty, d_{T_4} = d_{T_4^*} \leq d_{T_1} < \infty, d_{T_5} = d_{T_5^*} \leq d_{T_2} < \infty$ and $d_{T_6} < d_{T_6^*}$ (cf. [14, Proposition VII. 3.6]). Hence T_3, T_4 and T_6 have no singular unitary dilation by Proposition 3.5. This implies that in the above triangulations, these three operators are absent. Therefore, $T = T_5$ is of class $C_0(N)$. ∎

To prove the sufficiency part of Theorem 3.1, we need show that every $C_0(N)$ contraction has a singular unitary dilation. Actually, more is true: every C_0 contraction has such a dilation. We now embark on its proof. The next proposition should be known among experts; we include its proof here for completeness. In the finite-dimensional case, it is a special case of [15, Theorem 2].

PROPOSITION 3.6. *If T is a contraction on H with equal defect indices, then T has a unitary dilation on a space K which contains H with* $\dim(K \ominus H) = d_T$. *Moreover, in this case, d_T is the smallest such dimension.*

PROOF. Let W be a unitary operator from $\overline{\text{ran}(1-T^*T)^{\frac{1}{2}}}$ onto $\overline{\text{ran}(1-TT^*)^{\frac{1}{2}}}$, and let

$$U = \begin{bmatrix} T & (1-TT^*)^{\frac{1}{2}}W \\ (1-T^*T)^{\frac{1}{2}} & -T^*W \end{bmatrix}$$

on $K = H \oplus \overline{\text{ran}(1-T^*T)^{\frac{1}{2}}}$. Here we need the fact that $T^*(1-TT^*)^{\frac{1}{2}} = (1-T^*T)^{\frac{1}{2}}T^*$ to ensure that $-T^*W$ is an operator on $\overline{\text{ran}(1-T^*T)^{\frac{1}{2}}}$. It is routine to check that U is indeed unitary on K.

On the other hand, if

$$U' = \begin{bmatrix} T & A \\ B & C \end{bmatrix} \quad \text{on} \quad K' = H \oplus H'$$

is another unitary dilation of T, then from

$$U'^*U' = \begin{bmatrix} T^* & B^* \\ A^* & C^* \end{bmatrix} \begin{bmatrix} T & A \\ B & C \end{bmatrix} = 1$$

we obtain that $T^*T + B^*B = 1$ and hence d_T =dim $\overline{\text{ran}(1-T^*T)}$ = dim $\overline{\text{ran}B^*B} \leq$ dim H'. ∎

PROPOSITION 3.7. *If T is a $C_0(N)$ contraction and $T = U + F$, where U is unitary and F is of finite rank, then U is singular.*

PROOF. Let $V = \begin{bmatrix} T & 0 \\ T_1 & S \end{bmatrix}$ be the minimal isometric power dilation of T (cf. [14, Theorem I.4.1]). Since V is an isometry, a simple computation yields that $T^*T + T_1^*T_1 = 1$. Hence rank $T_1^*T_1$=rank$(1-T^*T) = d_T < \infty$, and therefore rank $T_1 < \infty$. Thus $V - (U \oplus S) = \begin{bmatrix} F & 0 \\ T_1 & 0 \end{bmatrix}$ is of finite rank. Carey's result implies that the absolutely continuous unitary parts of the isometries V and $U \oplus S$ are unitarily equivalent. Since V itself is a unilateral shift (cf. [14, Therem VI.3.1]), we infer that U must be singular. ∎

The next result is our promised unitary dilation for C_0 contractions.

PROPOSITION 3.8 *Every C_0 contraction on H has a singular unitary dilation on a space $K(\supseteq H)$ with* dim $(K \ominus H) = d_T$.

PROOF. Let T be a $C_0(N)$ contraction on H with $d_T = d_{T^*} = n < \infty$. By Proposition 3.6, T has a unitary dilation $U = \begin{bmatrix} T & A \\ B & C \end{bmatrix}$ on $K = H \oplus \mathbb{C}^n$. If $T' = \begin{bmatrix} T & 0 \\ B & 0 \end{bmatrix}$, then it is easily seen that $d_{T'} = n$ and $T'^m \to 0$ nd $T'^{*m} \to 0$ in the strong operator topology. Hence T' is also a $C_0(N)$ contraction. Since $T' = U + F$, where $F = \begin{bmatrix} 0 & -A \\ 0 & -C \end{bmatrix}$ is finite-rank, Proposition 3.7 implies that U is singular. Hence T has the singular unitary dilation U with the asserted property.

More generally, if T is a C_0 contraction with minimal function ϕ, then it was proved in [13, Lemma 4] that T power dilates to $S(\phi) \oplus S(\phi) \oplus \cdots$, where $S(\phi)$ is the operator on $H(\phi) = H^2 \ominus \phi H^2$ defined by $S(\phi)f = P(zf(z))$, P being the (orthogonal) projection from H^2 onto $H(\phi)$. Since $S(\phi)$ has defect indices equal to 1, as shown above it dilates to a singular unitary operator, say, U_1. Then T dilates to the singular unitary $U_1 \oplus U_1 \oplus \cdots$. ∎

To conclude this section, we derive a condition under which a C_0 contraction has a diagonal unitary dilation. A complete characterization of such C_0 contractions seems difficult to come by.

PROPOSITION 3.9. *If T is a C_0 contraction whose minimal function ϕ is a Blaschke product with the property that the closure of the zeros of ϕ is a countable set, then T has a diagonal unitary dilation.*

PROOF. Since T dilates to $S(\phi) \oplus S(\phi) \oplus \cdots$, we need only prove our assertion for $S(\phi)$. By Proposition 3.6, $S(\phi)$ has a unitary dilation $U = \begin{bmatrix} S(\phi) & A \\ B & C \end{bmatrix}$ on $K = H(\phi) \oplus \mathbf{C}$. We check that U is diagonal unitary. If $T' = \begin{bmatrix} S(\phi) & 0 \\ B & 0 \end{bmatrix}$, then, as in the proof of Proposition 3.8, T' is a $C_0(N)$ contraction with $d_{T'} = 1$. Hence T' is unitarily equivalent to $S(\psi)$ for some inner function ψ. Since $\psi(T') = \begin{bmatrix} \psi(S(\phi)) & 0 \\ * & \psi(0) \end{bmatrix} = 0$, we have $\psi(S(\phi)) = 0$ and $\psi(0) = 0$, and thus both ϕ and $\xi, \xi(z) = z$, are divisors of ψ. We infer that $\psi(z) = z\phi(z)$. Hence ψ is also a Blaschke product with the closure of its zeros a countable set. Since U is a rank-one perturbation of T', we may apply [6, Theorem 7.1] to $S(\psi)$ to deduce that U is a diagonal unitary operator. This completes the proof. ∎

COROLLARY 3.10. *Every algebraic contraction has a diagonal unitary dilation.*

PROOF. This follows from Proposition 3.9 since every algebraic contraction is the direct sum of a unitary operator with finitely many points in its spectrum and a C_0 contraction whose minimal function is a Blaschke product with finitely many zeros. ∎

4. FINITE-RANK PERTURBATION

From the proofs of Propositions 3.7 and 3.8, it can be observed that there is an intimate relation between finite-rank perturbations and dilations by a finite-dimensional space. This will be made more transparent by the next result, the main theorem of this section. It shows that in the situation considered here, they are actually equivalent.

THEOREM 4.1. *A contraction T is the sum of a singular unitary operator U and a finite-rank operator F if and only if it is the direct sum of a singular unitary operator U_1 and a $C_0(N)$ contraction T_1. In this case, the multiplicities of U and U_1 differ at most by* rank F *and* $d_{T_1} \leq$ rank F *holds. Moreover, we may choose F to have rank equal to d_T.*

Recall that the *multiplicity* $\mu(T)$ of an operator T on H is the minimal cardinality of a subset X of H for which H is the closed linear span of the vectors $T^n x$ with $x \in X$ and $n = 0, 1, 2, \cdots$. T is *cyclic* if $\mu(T) = 1$.

An immediate corollary of the preceding theorem is

COROLLARY 4.2. *A contraction T is the rank-one perturbation of a singular unitary operator if and only if it is the direct sum of a singular unitary operator and an operator of the form $S(\phi)$, where ϕ is an inner function.*

This generalizes Clark's result [6] that every $S(\phi)$ is the rank-one perturbation of a singular unitary operator. (Actually, he did more than this by constructing explicitly all rank-one perturbations of $S(\phi)$ which are unitary.)

The following two lemmas are in the domain of the general contraction theory. Their proofs are given here for completeness. Recall that any contraction T can be decomposed uniquely as the direct sum $T = U_s \oplus U_a \oplus T_0$, where U_s and U_a are singular and absolutely continuous unitary and T_0 is a c.n.u. contraction. T is said to be absolutely continuous if in this decomposition U_s is absent. For an operator T, Lat T denotes the lattice of its invariant subspaces.

LEMMA 4.3. *If $T = U_s \oplus U_a \oplus T_0$ on $H = H_s \oplus H_a \oplus H_0$ is a contraction decomposed as above, then* Lat $T =$Lat $U_s \oplus$Lat $(U_a \oplus T_0)$.

PROOF. Let $M \in$Lat T. We decompose the contraction $T|M$ as $T|M = T_1 \oplus T_2 \oplus T_3$ on $M = M_1 \oplus M_2 \oplus M_3$, where T_1 and T_2 are singular and absolutely continuous unitary operators and T_3 is a c.n.u. contraction. Since M_1 and $M_2 \oplus M_3$ are invariant subspaces for T, to prove that M is in Lat $U_s \oplus$ Lat$(U_a \oplus T_0)$ we need only show that $M_1 \subseteq H_s$ and $M_2 \oplus M_3 \subseteq H_a \oplus H_0$.

If U on $K(\supseteq H_0)$ is the minimal unitary power dilation of T_0, then $W = U_s \oplus U_a \oplus U$ is the minimal unitary power dilation of T. Hence, in particular, W is a unitary power dilation of $T_2 \oplus T_3$. There exists a reducing subspace $L(\supseteq M_2 \oplus M_3)$ for W such that $W|L$ is the minimal unitary power dilation of $T_2 \oplus T_3$. Since T_2 is absolutely continuous unitary and T_3 is c.n.u., $W|L$ must be absolutely continuous unitary. On the other hand, since Lat $W =$ Lat $U_s \oplus$ Lat $(U_a \oplus U)$ (cf. [9, Lemma 1]), we have $L \subseteq H_a \oplus K$. Therefore, $M_2 \oplus M_3 \subseteq L \cap H \subseteq (H_a \oplus K) \cap (H_s \oplus H_a \oplus H_0) = H_a \oplus H_0$. An even simpler argument along this line applied to M_1 yields that $M_1 \subseteq H_s$. This shows that Lat $T \subseteq$ Lat $U_s \oplus$ Lat $(U_a \oplus T_0)$. Since the converse inclusion is trivial, the proof is completed. ∎

LEMMA 4.4. *If* $T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ *on* $H = H_1 \oplus H_2$ *is a contraction, then the singular unitary part of* T *is the direct sum of the singular unitary parts of* T_1 *and* T_2.

PROOF. Let $T = U_s \oplus T'$ on $H = H_s \oplus H'$, where U_s is singular unitary and T' is an absolutely continuous contraction. Since Lat $T =$ Lat $U_s \oplus$ Lat T' by Lemma 4.3, we have $H_1 = K \oplus K'$, where $K \in$ Lat U_s and $K' \in$ Lat T'. Then K is an invariant subspace for T_1 and $U_1 \equiv T_1|K_1 = T|K = U_s|K$ is singular unitary since every invariant subspace for a singular unitary operator is actually reducing (cf. [17, Lemma 3]). Hence $T_1 = U_1 \oplus T_1'$ on $H_1 = K \oplus K'$, where T_1' is absolutely continuous. Similarly, applying the above arguments to $T^* = \begin{bmatrix} T_1^* & 0 \\ * & T_2^* \end{bmatrix}$ yields that $T_2^* = U_2^* \oplus T_2'^*$ on $H_2 = M \oplus M'$ for some singular unitary U_2 and absolutely continuous T_2'. It follows that

$$T = \begin{bmatrix} U_1 & 0 & 0 & 0 \\ 0 & T_1' & 0 & X \\ 0 & 0 & U_2 & 0 \\ 0 & 0 & 0 & T_2' \end{bmatrix} \quad \text{on} \quad H = K \oplus K' \oplus M \oplus M'$$

with $U_1 \oplus U_2$ singular unitary and $\begin{bmatrix} T_1' & X \\ 0 & T_2' \end{bmatrix}$ absolutely continuous. This implies that $K \oplus M \subseteq H_s$ and $K' \oplus M' \subseteq H'$. Since $(K \oplus M) \oplus (K' \oplus M') = H = H_s \oplus H'$, we must have $K \oplus M = H_s$ and $K' \oplus M' = H'$. It follows that $U_1 \oplus U_2 = U_s$ as asserted. ∎

The next lemma yields the relation between the multiplicities of the singular unitary parts of contractions which differ by a finite-rank operator.

LEMMA 4.5. *Let* T_1 *and* T_2 *be contractions on* H *with* rank $(T_1 - T_2) < \infty$. *Then the multiplicities of the singular unitary parts* U_{1s} and U_{2s} *of* T_1 *and* T_2 *satisfy* $\mu(U_{1s}) \leq \mu(U_{2s})+\text{rank}(T_1 - T_2)$ *and* $\mu(U_{2s}) \leq \mu(U_{1s})+\text{rank}(T_1 - T_2)$.

PROOF. Let $F = T_1 - T_2$ and k =rank F. Assume that ran F is spanned by the vectors $x_1, \cdots, x_k$. If K is the closed subspace of H spanned by $T_1^m x_n, m \geq 0$ and $1 \leq n \leq k$, then K is invariant for both T_1 and T_2. Hence we have the triangulations

$$T_1 = \begin{bmatrix} T_{11} & * \\ 0 & T_{12} \end{bmatrix}, \quad T_2 = \begin{bmatrix} T_{21} & * \\ 0 & T_{22} \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} F_1 & F_2 \\ 0 & 0 \end{bmatrix}$$

on $H = K \oplus K^\perp$. If U_{ij} is the singular unitary part of $T_{ij}, i, j = 1, 2$, then $U_{1s} = U_{11} \oplus U_{12}$ and $U_{2s} = U_{21} \oplus U_{22}$ by Lemma 4.4. We have $\mu(U_{i1}) \leq \mu(T_{i1}) \leq k$ for $i = 1, 2$. On the other hand, from $T_1 - T_2 = F$ we have $T_{12} = T_{22}$ and hence $U_{12} = U_{22}$. Finally. $\mu(U_{1s}) \leq \mu(U_{11}) + \mu(U_{12}) \leq k + \mu(U_{22}) \leq k + \mu(U_{2s})$ and similarly for the other inequality. ∎

We are now ready for the

PROOF OF THEOREM 4.1. Assume first that $T = U + F$, where U is singular unitary and F is of finite rank. Let $T = U_1 \oplus T_1$ on $H = K \oplus L$, where U_1 is unitary and T_1 is c.n.u.. Let $R = \overline{\text{ran}(1 - T_1^*T_1)^{\frac{1}{2}}}$ and let

$$V_1 = U_1 \oplus \begin{bmatrix} T_1 & & & \\ (1-T_1^*T_1)^{\frac{1}{2}} & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \\ & & & \ddots \end{bmatrix} \quad \text{on} \quad K \oplus (L \oplus R \oplus R \oplus \cdots)$$

be the minimal isometric power dilation of T (cf. [14, pp.17-18]). If

$$V = U \oplus \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{bmatrix} \quad \text{on} \quad (K \oplus L) \oplus (R \oplus R \oplus \cdots)$$

and $E = V_1 - V$, then we claim that rank E =rank F. Since

$$E = \begin{bmatrix} & F & 0 & \cdots \\ 0 & (1-T_1^*T_1)^{\frac{1}{2}} & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix},$$

we obviously have rank E $\geq$rank F. To prove the reverse inequality, let

$$E' = \begin{bmatrix} F & 0 \\ (1-T^*T)^{\frac{1}{2}} & 0 \end{bmatrix} \quad \text{on} \quad H \oplus H.$$

The rank E =rank E'. Since $1 - T^*T = 1 - (U^* + F^*)(U + F) = -U^*F - F^*U - F^*F \geq 0$, we have $-U^*F - F^*U \geq F^*F$ and hence ran F^*=ranF^*F $\subseteq$ran$(-U^*F - F^*U)$. Let $-U^*F = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix}$ on H =ran$(U^*F^*)^*\oplus$ker U^*F. Since

$$-U^*F - F^*U = \begin{bmatrix} A + A^* & B^* \\ B & 0 \end{bmatrix} \geq F^*F \geq 0,$$

we infer that $B = 0$ and thus

$$-U^*F - F^*U = \begin{bmatrix} A + A^* & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that ranF^* $\subseteq$ran$(-U^*F - F^*U) \subseteq$ ran$(U^*F)^*$. Therefore

$$F^* = \begin{bmatrix} F_1 & F_2 \\ 0 & 0 \end{bmatrix} \quad \text{on} \quad H = \text{ran}(U^*F)^* \oplus \ker U^*F.$$

We also have $\mathrm{ran}(1-T^*T) = \mathrm{ran}(-U^*F - F^*U - F^*F) \subseteq \mathrm{ran}(-U^*F - F^*U) + \mathrm{ran}\, F^*F \subseteq \mathrm{ran}(U^*F)^*$. Let $(1-T^*T)^{\frac{1}{2}} = \begin{bmatrix} F_3 & 0 \\ 0 & 0 \end{bmatrix}$ on $H = \mathrm{ran}(U^*F)^* \oplus \ker U^*F$. Then

$$E' = \begin{bmatrix} F_1^* & 0 & 0 & 0 \\ F_2^* & 0 & 0 & 0 \\ F_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and thus $\mathrm{rank}\, E = \mathrm{rank}\, E' \leq \mathrm{rank}(U^*F)^* = \mathrm{rank}\, F$, completing the proof of our claim. In particular, we have $d_{T_1} = \mathrm{rank}\, (1-T_1^*T_1)^{\frac{1}{2}} \leq \mathrm{rank}\, E = \mathrm{rank}\, F$.

Carey's result implies that the absolutely continuous unitary parts of V and V_1 are unitarily equivalent. Since U, the unitary part of V, is singular, the same is true for the unitary part of V_1. This implies that U_1 and the unitary part of the minimal isometric power dilation V_2 of T_1 are both singular. Hence V_2 must be a unilateral shift (cf. [14, Theorem II. 6.4]). This latter condition dictates that T_1 satisfy $T_1^{*n} \to 0$ in the strong operator topology. A similar argument applied to T^* yields that $T_1^n \to 0$ in the strong operator topology. Since $d_{T_1} \leq \mathrm{rank}\, F < \infty$ as noted above, we conclude that T_1 is a $C_0(N)$ contraction. Finally, since $\mathrm{rank}(V - V_1) = \mathrm{rank}\, F$, the assertion on the multiplicities of U and U_1 follows from Lemma 4.5.

For the converse, we need only consider for $C_0(N)$ contractions. Let T be such a contraction. Since $d_T = d_{T^*} < \infty$, by Lemma 3.3, T is the sum of a unitary operator U and a finite-rank operator F with rank $F = d_T$. Proposition 3.7 implies that U is singular. This completes the proof. ∎

5. OPEN PROBLEMS

Although we obtained various necessary and/or sufficient conditions, the main problem addressed in this paper, which contraction has a singular (resp. diagonal) unitary dilation, remains open. It may turn out that this general problem admits no tractable answer. Some special cases of it are still worth exploring. Here are two.

QUESTION 5.1. *Is it true that no c.n.u. C_{11} contraction has a singular unitary dilation?*

Proposition 3.5 says that this is indeed the case for those c.n.u. C_{11} contractions with finite defect indices. The problem can be reduced to considering only those which are similar to a cyclic unitary operator. Indeed, since every c.n.u. C_{11} contraction T is quasisimilar to a unitary operator, [1] implies that there is an invariant subspace K for T such that $T|K$ is similar to some unitary operator. Using the spectral theorem, we may even assume that the unitary operator is cyclic.

QUESTION 5.2. *For which (nonconstant) inner function ϕ does $S(\phi)$ have a diagonal unitary dilation?*

A slightly more general condition on ϕ than the one in Proposition 3.9 was give in [6, Theorem 7.1]. Its complete characterization will yield, as did in our Section 3, a necessary and sufficient condition for a contraction with at least one defect index finite to have a diagonal unitary dilation.

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REFERENCES

[1] C. Apostol, *Operators quasisimilar to a normal operator,* Proc. Amer. Math. Soc., 53 (1975), 104-106.

[2] C. Apostol, C. Foias and L. Zsidó, *Some results on non-quasitriangular operators*, Indiana Univ. Math. J., 22 (1973), 1151-1161.

[3] W. Arveson, *Subalgebras of C^*-algebras* II, Acta Math., 128 (1972), 271-308.

[4] H. Bercovici, *Operator theory and arithmetic in H^∞*, Amer Math. Soc., Providence, 1988.

[5] R.W. Carey, *Trace class perturbations of isometries and unitary dilations,* Proc. Amer. Math. Soc., 45(1974), 229-234.

[6] D.N. Clark, *One dimensional perturbations of restricted shifts*, J. D'analyse Math., 25 (1972), 169-191.

[7] D.N. Clark, *Perturbation and similarity of Toeplitz operators*, Operator Theory: Adv. Appl., 48 (1990), 235-243.

[8] J.B. Conway, *A course in functional analysis*, 2nd ed., Springer-Verlag, New York, 1990.

[9] J.A. Deddens, *Every isometry is reflexive,* Proc. Amer. Math. Soc., 28 (1971), 509-512.

[10] F. Gilfeather, *Seminormal operators*, Michigan Math., J., 18 (1971), 235-242.

[11] P.R. Halmos, *A Hilbert space problem book*, 2nd ed., Springer-Verlag, New York, 1982.

[12] Y. Nakamura, *Numerical range and norm*, Math. Japonica, 27 (1982), 149-150.

[13] T. Nakazi and K. Takahashi, *Two-dimensional representations of uniform algebras*, Proc. Amer. Math. Soc., 123 (1995), 2777-2784.

[14] B. Sz.-Nagy and C. Foias, *Harmonic analysis of operators on Hilbert space,* North Holland, Amsterdam, 1970.

[15] R.C. Thompson and C.-C. T. Kuo, *Doubly stochastic, unitary, unimodular, and complex orthogonal power embeddings*, Acta Sci. Math. (Szeged), 44 (1982), 345-357.

[16] S.-H. Tso and P.Y. Wu, *Matricial ranges of quadratic operators*, Rocky Mountain J. Math., to appear.

[17] J. Wermer, *On invariant subspaces of normal operators,* Proc. Amer. Math. Soc., 3(1952), 270-277.

[18] P.Y. Wu, *Toward a characterization of reflexive contractions*, J. Operator Theory, 13 (1985), 73-86.

[19] P.Y. Wu, *Diagonal unitary dilations of contractions*, Proc. Intern. Math. Conf. '94, Y. Fong, W.J. Huang, Y.-J. Lee & N.-C. Wong, eds., World Scientific, Singapore, 1996, pp.219-226.

[20] P.Y. Wu, *Unitary dilations and numerical ranges*, J. Operator Theory, 38 (1997), 25-42.

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