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Finding Inheritance Hierarchies in Fuzzy-Valued Concept-Networks

Yih-Jen Horng and Shyi-Ming Chen

***Abstract*—In this paper, we extend the works of [2] and [4] to present a new method for finding the inheritance hierarchies in fuzzy-valued concept-networks, where the relevant values (degrees of generalization or degrees of similarity) between concepts in a fuzzy-valued concept network are represented by fuzzy numbers. The proposed method is more flexible than the ones presented in [2] and [4] due to the fact that it allows the grades of similarity and the grades of generalization between concepts to be represented by fuzzy numbers rather than crisp real values between zero and one or interval values in [0, 1].**

***Index Terms*—Fuzzy numbers, fuzzy-valued concept-networks, inheritance hierarchies, synonymous concepts.**

I. Introduction

In [4], Itzkovich and Hawkes pointed out that inheritance hierarchies provide a significant descriptive capability using only the generalization relations. They also presented a fuzzy extension of inheritance hierarchies to fuzzy concept-networks which contain not only generalization relations but also similarity relations. A fuzzy concept-network can then be used in the application of reusable software retrieval and information retrieval. In [8], Lucaralla and Morara presented a kind of concept-networks for fuzzy information retrieval. In [3] and [9], we presented knowledge-based fuzzy information retrieval techniques based on [8]. However, the fuzzy concept-networks presented in [3], [4], [8], and [9] all assume that the relevant values (degrees of graded generalization or degrees of similarity) between concepts in a fuzzy concept-network are represented by crisp real values between zero and one. In [2], we extended the work of Itzkovich and Hawkes [4] to present the concepts of interval-valued fuzzy concept-networks, where the relevant values (degrees of graded generalization or degrees of similarity) between concepts in a fuzzy concept-networks are represented by interval values in [0, 1] rather than crisp real values between zero and one. In [2], we also presented an algorithm for finding the collection of inheritance hierarchies in interval-valued fuzzy concept-networks. However, if we can allow the relevant values (degrees of generalization or degrees of similarity) between concepts to be represented by fuzzy numbers, then there is room for more flexibility.

In this paper, we extend the works of [2] and [4] to present a new method for finding the collection of inheritance hierarchies in fuzzy-valued concept networks, where the relevant values (degrees of generalization or degrees of similarity) between concepts are represented by fuzzy numbers. The proposed method is more flexible than the ones presented in [2] and [4] due to the fact that it allows the grades of similarity and the grades of generalization between concepts to be represented by fuzzy numbers rather than crisp real values between zero and one or interval values in [0, 1].

The rest of this paper is organized as follows. In Section II, we present the concepts of fuzzy-valued concept-networks. In Section III, we present an algorithm for finding the inheritance hierarchies in fuzzy-valued concept-networks. The conclusions are discussed in Section IV.

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II. Fuzzy-Valued Concept-Networks

In [4], Itzkovich and Hawkes presented a fuzzy extension of inheritance hierarchies to provide a more refined construction that facilitate the representation of relations among concepts under uncertain conditions. The extension is done in the following two steps:

Step 1: Incorporate the synonymy relation in the inheritance hierarchy, resulting in a new construction denoted as a concept-network.

Step 2: The relations on the concept-network are fuzzified to yield a new construction denoted as a fuzzy concept-network, where the relevant values between concepts are represented by real values between zero and one.

The definitions of fuzzy concept-networks are reviewed from [4] as follows.

Definition 2.1: The similarity relation $R_{\rm sim}$ over a finite set of concepts $C, C = \{c_1, c_2, \cdots, c_n\}$, is a binary fuzzy relation which satisfies all of the following properties:

1) Reflexive: $\mu_{\rm sim}(c_i, c_i) = 1$.
2) Symmetric: $\mu_{\rm sim}(c_i, c_j) = \mu_{\rm sim}(c_j, c_i)$.
3) Transitive: $\mu_{\rm sim}(c_i, c_k) \geq \vee_{Cj}\, (\mu_{\rm sim}(c_i, c_j) \wedge \mu_{\rm sim}(c_j, c_k))$.

Definition 2.2: The graded generalization relation R_g over a finite set of concepts $C, C = \{c_1, c_2, \cdots, c_n\}$, is a binary fuzzy relation which satisfies all of the following properties:

1) Reflexive: $\mu_g(c_i, c_i) = 1$.
2) Anti-symmetric: If $\mu_g(c_i, c_j) > 0$ and $\mu_g(c_j, c_i) > 0$, then $c_i = c_j$.
3) Transitive: $\mu_g(c_i, c_k) \geq \vee_{Cj}\, (\mu_g(c_i, c_j) \wedge \mu_g(c_j, c_k))$.

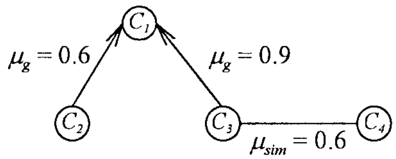

Fig. 1. A fuzzy concept network.

Definition 2.3: A fuzzy concept-network is denoted by $\mathrm{FCN}(C, R)$, where C is a finite set of concepts and R consists of two relations R_{sim} and R_g over C as defined in Definitions 2.1 and 2.2.

For example, Fig. 1 shows an example of a fuzzy concept network.

In [2], we presented the concepts of interval-valued fuzzy concept networks. In an interval-valued fuzzy concept-network, the degrees of similarity and the degrees of generalization between concepts are represented by real intervals in [0, 1]. Two intervals $[a, b]$ and $[c, d]$ are called equal if $a = c$ and $b = d$. If $[a, b] > [c, d]$ then it implies $a > c$ and $b > d$ or $a = c$ and $b > d$ or $a > c$ and $b = d$. The definitions of interval-valued fuzzy concept networks are reviewed from [2] as follows:

Definition 2.4: The interval-valued similarity relation R_{ivsim} over a finite set of concepts $C, C = \{c_1, c_2, \cdots, c_n\}$, is a binary fuzzy relation which satisfies all of the following properties.

1) Reflexive: $\mu_{\mathrm{ivsim}}(c_i, c_i) = [1, 1]$.
2) Symmetric: $\mu_{\mathrm{ivsim}}(c_i, c_j) = \mu_{\mathrm{ivsim}}(c_j, c_i)$.
3) Transitive: Let the degree of interval-valued similarity between any concepts c_x and c_y be represented by $\mu_{\mathrm{ivsim}}(c_x, c_y)$, where $\mu_{\mathrm{ivsim}}(c_x, c_y) = [S^l(c_x, c_y), S^h(c_x, c_y)]$ and $0 \le S^l(c_x, c_y) \le S^h(c_x, c_y) \le 1$. Then,

$$S^l(c_i, c_k) \ge \bigvee_{Cj} (S^l(c_i, c_j) \wedge S^l(c_j, c_k)),$$

$$S^h(c_i, c_k) \ge \bigvee_{Cj} (S^h(c_i, c_j) \wedge S^h(c_j, c_k)).$$

Definition 2.5: The interval-valued generalization relation R_{ivg} over a finite set of concepts $C, C = \{c_1, c_2, \cdots, c_n\}$, is a binary fuzzy relation which satisfies all of the following properties.

1) Reflexive: $\mu_{\mathrm{ivg}}(c_i, c_i) = [1, 1]$.
2) Anti-symmetric: If $\mu_{\mathrm{ivg}}(c_i, c_j) > [0, 0]$ and $\mu_{\mathrm{ivg}}(c_j, c_i) > [0, 0]$, then $c_i = c_j$.
3) Transitive: Let the degree of interval-valued generalization between any concepts c_x and c_y be represented by $\mu_{\mathrm{ivg}}(c_x, c_y)$, where $\mu_{\mathrm{ivg}}(c_x, c_y) = [g^l(c_x, c_y), g^h(c_x, c_y)]$, and $0 \le g^l(c_x, c_y) \le g^h(c_x, c_y) \le 1$. Then,

$$g^l(c_i, c_k) \ge \bigvee_{Cj} (g^l(c_i, c_j) \wedge g^l(c_j, c_k)),$$

$$g^h(c_i, c_k) \ge \bigvee_{Cj} (g^h(c_i, c_j) \wedge g^h(c_j, c_k)).$$

Definition 2.6: An interval-valued fuzzy concept-network is denoted by $\mathrm{IVFCN}(C, R)$, where C is a finite set of concepts and R consists of two relations R_{ivsim} and R_{ivg} over C as defined in Definitions 2.4 and 2.5.

For example, Fig. 2 shows an interval-valued fuzzy concept-network.

In the following, we present the concepts of fuzzy-valued concept-networks, where the degrees of generalizations and the degrees of similarity between concepts are represented by fuzzy numbers. A fuzzy number is a fuzzy subset in the universe of discourse of U that is both convex and normal. A fuzzy number F can be characterized by a triangular distribution parametrized by a triple (t_1, t_2, t_3) shown in Fig. 3, where $t_1 \le t_2 \le t_3$, or by a trapezoidal distribution parametrized by a quadruple (q_1, q_2, q_3, q_4) shown in Fig. 4, where $q_1 \le q_2 \le q_3 \le q_4$.

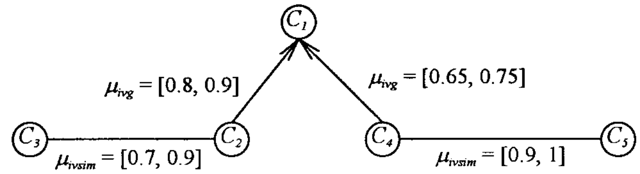

Fig. 2. An interval-valued fuzzy concept network.

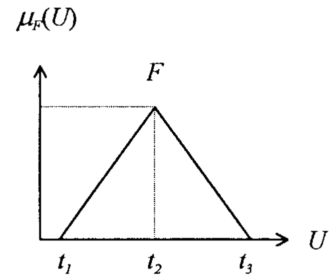

Fig. 3. A triangular fuzzy number.

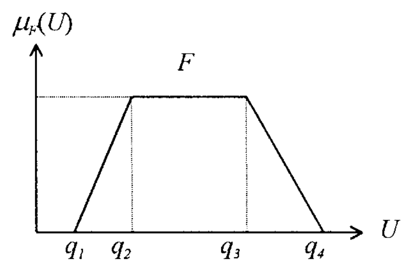

Fig. 4. A trapezoidal fuzzy number.

In the following, we introduce two kinds of fuzzy-valued concept networks. The first one allows the degrees of similarity and the degrees of generalization between concepts to be represented by triangular fuzzy numbers, whereas the second one allows the degrees of similarity and the degrees of similarity be represented by trapezoidal fuzzy numbers.

Definition 2.7: Let A and B be two triangular fuzzy numbers, where

$$A = (a_1, a_2, a_3),$$
$$B = (b_1, b_2, b_3)$$

$0 \le a_1 \le a_2 \le a_3 \le 1$, and $0 \le b_1 \le b_2 \le b_3 \le 1$. The triangular fuzzy numbers A and B are called equal (i.e., $A = B$) if and only if $a_1 = b_1, a_2 = b_2$, and $a_3 = b_3$. Otherwise, the triangular fuzzy numbers A and B are called unequal (i.e., $A \neq B$).

Definition 2.8: Let X and Y be two trapezoidal fuzzy numbers, where

$$X = (x_1, x_2, x_3, x_4)$$
$$Y = (y_1, y_2, y_3, y_4)$$

$0 \le x_1 \le x_2 \le x_3 \le x_4 \le 1$, and $0 \le y_1 \le y_2 \le y_3 \le y_4 \le 1$. The trapezoidal fuzzy numbers X and Y are called equal (i.e., $X = Y$) if and only if $x_1 = y_1, x_2 = y_2, x_3 = y_3, and\ x_4 = y_4$. Otherwise, the triangular fuzzy numbers X and Y are called unequal (i.e., $X \neq Y$).

The definitions of fuzzy-valued concept-networks using triangular fuzzy numbers to represent the grades of generalization and grades

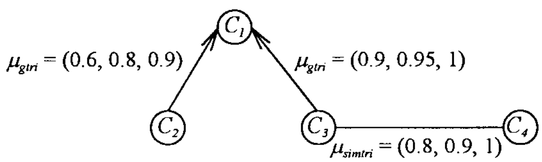

Fig. 5. A fuzzy-valued concept-network using triangular fuzzy numbers.

of similarity between concepts are presented in Definitions 2.9–2.11. The definitions of fuzzy-valued concept-networks using trapezoidal fuzzy numbers to represent the grades of generalization and grades of similarity between concepts are presented in Definitions 2.12–2.14.

Definition 2.9: The similarity relation R_{simtri} represented by triangular fuzzy numbers over a finite set of concepts $C, C = \{c_1, c_2, \cdots, c_n\}$, is a binary fuzzy relation which satisfies all of the following properties.

1) Reflexive: $\mu_{\text{simtri}}(c_i, c_i) = (1, 1, 1)$.
2) Symmetric: $\mu_{\text{simtri}}(c_i, c_j) = \mu_{\text{simtri}}(c_j, c_i)$.
3) Transitive: Let the degree of similarity between any concepts c_x and c_y be represented by $\mu_{\text{simtri}}(c_x, c_y)$, where $\mu_{\text{simtri}}(c_x, c_y) = (S_1(c_x, c_y), S_2(c_x, c_y), S_3(c_x, c_y))$, and $0 \leq S_1(c_x, c_y) \leq S_2(c_x, c_y) \leq S_3(c_x, c_y) \leq 1$. Then

$$S_1(c_i, c_k) \geq \bigvee_{Cj} (S_1(c_i, c_j) \wedge S_1(c_j, c_k))$$

$$S_2(c_i, c_k) \geq \bigvee_{Cj} (S_2(c_i, c_j) \wedge S_2(c_j, c_k))$$

$$S_3(c_i, c_k) \geq \bigvee_{Cj} (S_3(c_i, c_j) \wedge S_3(c_j, c_k)).$$

Definition 2.10: The graded generalization relation R_{gtri} represented by triangular fuzzy numbers over a finite set of concepts $C, C = \{c_1, c_2, \cdots, c_n\}$, is a binary fuzzy relation which satisfies all of the following properties.

1) Reflexive: $\mu_{\text{gtri}}(c_i, c_i) = (1, 1, 1)$.
2) Anti-symmetric: If $\mu_{\text{gtri}}(c_i, c_j) \neq (0, 0, 0)$ and $\mu_{\text{gtri}}(c_j, c_i) \neq (0, 0, 0)$, then $c_i = c_j$.
3) Transitive: Let the degree of generalization between any concepts c_x and c_y be represented by $\mu_{\text{gtri}}(c_x, c_y)$, where $\mu_{\text{gtri}}(c_x, c_y) = (g_1(c_x, c_y), g_2(c_x, c_y), g_3(c_x, c_y))$, and $0 \leq g_1(c_x, c_y) \leq g_2(c_x, c_y) \leq g_3(c_x, c_y) \leq 1$. Then

$$g_1(c_i, c_k) \geq \bigvee_{Cj} (g_1(c_i, c_j) \wedge g_1(c_j, c_k)),$$

$$g_2(c_i, c_k) \geq \bigvee_{Cj} (g_2(c_i, c_j) \wedge g_2(c_j, c_k)),$$

$$g_3(c_i, c_k) \geq \bigvee_{Cj} (g_3(c_i, c_j) \wedge g_3(c_j, c_k)).$$

Definition 2.11: A fuzzy-valued concept-network using triangular fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts is denoted by $\text{FVCNTRI}(C, R)$, where C is a finite set of concepts and R consists of two relations R_{simtri} and R_{gtri} over C as defined in Definitions 2.9 and 2.10.

For example, Fig. 5 shows a fuzzy-valued concept-network using triangular fuzzy numbers to represent the degrees of generalization and degrees of similarity between concepts.

Definition 2.12: The similarity relation R_{simtra} represented by trapezoidal fuzzy numbers over a finite set of concepts $C, C = \{c_1, c_2, \cdots, c_n\}$, is a binary fuzzy relation which satisfies all of the following properties.

1) Reflexive: $\mu_{\text{simtra}}(c_i, c_i) = (1, 1, 1, 1)$.

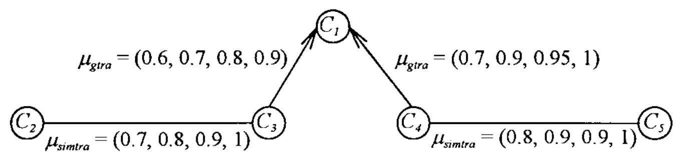

Fig. 6. A fuzzy-valued concept-network using trapezoidal fuzzy numbers.

2) Symmetric: $\mu_{\text{simtra}}(c_i, c_j) = \mu_{\text{simtra}}(c_j, c_i)$.
3) Transitive: Let the degree of similarity between any concepts c_x and c_y be represented by $\mu_{\text{simtra}}(c_x, c_y)$, where $\mu_{\text{simtra}}(c_x, c_y) = (S_1(c_x, c_y), S_2(c_x, c_y), S_3(c_x, c_y), S_4(c_x, c_y))$, and $0 \leq S_1(c_x, c_y) \leq S_2(c_x, c_y) \leq S_3(c_x, c_y) \leq S_4(c_x, c_y) \leq 1$. Then

$$S_1(c_i, c_k) \geq \bigvee_{Cj} (S_1(c_i, c_j) \wedge S_1(c_j, c_k)),$$

$$S_2(c_i, c_k) \geq \bigvee_{Cj} (S_2(c_i, c_j) \wedge S_2(c_j, c_k)),$$

$$S_3(c_i, c_k) \geq \bigvee_{Cj} (S_3(c_i, c_j) \wedge S_3(c_j, c_k)),$$

$$S_4(c_i, c_k) \geq \bigvee_{Cj} (S_4(c_i, c_j) \wedge S_4(c_j, c_k)).$$

Definition 2.13: The graded generalization relation R_{gtra} represented by trapezoidal fuzzy numbers over a finite set of concepts $C, C = \{c_1, c_2, \cdots, c_n\}$, is a binary fuzzy relation which satisfies all of the following properties.

1) Reflexive: $\mu_{\text{gtra}}(c_i, c_i) = (1, 1, 1, 1)$.
2) Anti-symmetric: If $\mu_{\text{gtra}}(c_i, c_j) \neq (0, 0, 0, 0)$ and $\mu_{\text{gtra}}(c_j, c_i) \neq (0, 0, 0, 0)$, then $c_i = c_j$.
3) Transitive: Let the degree of similarity between any concepts c_x and c_y be represented by $\mu_{\text{gtra}}(c_x, c_y)$, where $\mu_{\text{gtra}}(c_x, c_y) = (g_1(c_x, c_y), g_2(c_x, c_y), g_3(c_x, c_y), g_4(c_x, c_y))$, and $0 \leq g_1(c_x, c_y) \leq g_2(c_x, c_y) \leq g_3(c_x, c_y) \leq g_4(c_x, c_y) \leq 1$. Then

$$g_1(c_i, c_k) \geq \bigvee_{Cj} (g_1(c_i, c_j) \wedge g_1(c_j, c_k)),$$

$$g_2(c_i, c_k) \geq \bigvee_{Cj} (g_2(c_i, c_j) \wedge g_2(c_j, c_k)),$$

$$g_3(c_i, c_k) \geq \bigvee_{Cj} (g_3(c_i, c_j) \wedge g_3(c_j, c_k)),$$

$$g_4(c_i, c_k) \geq \bigvee_{Cj} (g_4(c_i, c_j) \wedge g_4(c_j, c_k)).$$

Definition 2.14: A fuzzy-valued concept-network using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts is denoted by $\text{FVCNTRA}(C, R)$, where C is a finite set of concepts and R consists of two relations R_{simtra} and R_{gtra} over C as defined in Definitions 2.12 and 2.13.

For example, Fig. 6 shows a fuzzy-valued concept-network using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts.

III. An Algorithm for Finding the Inheritance Hierarchies in Fuzzy-Valued Concept-Networks

In [4], Itzkovich and Hawkes presented an algorithm for finding the collection of inheritance hierarchies in fuzzy concept-networks, where the degrees of generalization and the degrees of similarity between concepts are represented by real values between zero and one. In [2], we have presented an algorithm for finding the collection of inheritance hierarchies in interval-valued fuzzy concept networks, where the degrees of generalization and the degrees of similarity between concepts are represented by interval values in [0, 1].

In this section, we extend the works of [2] and [4] to present an algorithm for finding the inheritance hierarchies in fuzzy-valued concept-networks, where the degrees of generalization and the degrees of similarity between concepts are represented by fuzzy numbers. Firstly, we present a method to model the fuzzy-valued concept-networks by means of concept matrices. If there are n concepts in a fuzzy concept-network, then a $n \times n$ concept matrix will be used to model the fuzzy-valued concept-network.

Case 1: If a fuzzy-valued concept-network uses triangular fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts.

If $\mu_{\text{simtri}}(c_i, c_j) = \mu_{ij}$, where $\mu_{ij} \in [0,1]$, then let $M(i,j) = M(j,i) = (\mu_{ij}, \mu_{ij}, \mu_{ij})$;

if $\mu_{\text{gtri}}(c_i, c_j) = \mu_{ij}$, where $\mu_{ij} \in [0,1]$, then let $M(i,j) = (\mu_{ij}, \mu_{ij}, \mu_{ij})$ and $M(j,i) = (0,0,0)$;

if $\mu_{\text{simtri}}(c_i, c_j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3)$, where $0 \le \mu_{ij}^1 \le \mu_{ij}^2 \le \mu_{ij}^3 \le 1$, then let $M(i,j) = M(j,i) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3)$;

if $\mu_{\text{gtri}}(c_i, c_j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3)$, where $0 \le \mu_{ij}^1 \le \mu_{ij}^2 \le \mu_{ij}^3 \le 1$, then let $M(i,j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3)$ and $M(j,i) = (0,0,0)$;

if there are no relationships between the concepts c_i and c_j, then let $M(i,j) = M(j,i) = (0,0,0)$.

Furthermore, we let $M(i,i) = (1,1,1)$, where $1 \le i \le n$, due to the fact that each concept c_i is reflexive to itself.

Example 3.1: Given a fuzzy-valued concept-network shown in Fig. 5, where c_1 is a generalization of c_2 with $\mu_{\text{gtri}}(c_2, c_1) = (0.6, 0.8, 0.9), c_1$ is also a generalization of c_3 with $\mu_{\text{gtri}}(c_3, c_1) = (0.9, 0.95, 1)$, and c_3 is similar to c_4 with $\mu_{\text{simtri}}(c_3, c_4) = \mu_{\text{simtri}}(c_4, c_3) = (0.8, 0.9, 1)$. Then, we can use a 4×4 concept matrix M to model the fuzzy-valued concept-network

$$M = \begin{bmatrix} (1,1,1) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0.6,0.8,0.9) & (1,1,1) & (0,0,0) & (0,0,0) \\ (0.9,0.95,1) & (0,0,0) & (1,1,1) & (0.8,0.9,1) \\ (0,0,0) & (0,0,0) & (0.8,0.9,1) & (1,1,1) \end{bmatrix}.$$

Case 2: If a fuzzy-valued concept-network uses trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts.

If $\mu_{\text{simtra}}(c_i, c_j) = \mu_{ij}$, where $\mu_{ij} \in [0,1]$, then let $N(i,j) = N(j,i) = (\mu_{ij}, \mu_{ij}, \mu_{ij}, \mu_{ij})$;

if $\mu_{\text{gtra}}(c_i, c_j) = \mu_{ij}$, where $\mu_{ij} \in [0,1]$, then let $N(i,j) = (\mu_{ij}, \mu_{ij}, \mu_{ij}, \mu_{ij})$ and $N(j,i) = (0,0,0,0)$;

if $\mu_{\text{simtra}}(c_i, c_j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3, \mu_{ij}^4)$, where $0 \le \mu_{ij}^1 \le \mu_{ij}^2 \le \mu_{ij}^3 \le \mu_{ij}^4 \le 1$, then let $N(i,j) = N(j,i) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3, \mu_{ij}^4)$;

if $\mu_{\text{gtra}}(c_i, c_j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3, \mu_{ij}^4)$, where $0 \le \mu_{ij}^1 \le \mu_{ij}^2 \le \mu_{Ij}^3 \le \mu_{ij}^4 \le 1$, then let $N(i,j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3, \mu_{ij}^4)$ and $N(j,i) = (0,0,0,0)$;

if there are no relationships between the concepts c_i and c_j, then let $N(i,j) = N(j,i) = (0,0,0,0)$.

Furthermore, we let $N(i,i) = (1,1,1,1)$, where $1 \le i \le n$, due to the fact that each concept c_i is reflexive to itself.

Example 3.2: Given a fuzzy-valued concept-network shown in Fig. 6, where c_1 is a generalization of c_3 with $\mu_{\text{gtra}}(c_3, c_1) = (0.6, 0.7, 0.8, 0.9), c_1$ is also a generalization of c_4 with $\mu_{\text{gtra}}(c_4, c_1) = (0.7, 0.9, 0.95, 1), c_2$ is similar to c_3 with $\mu_{\text{simtra}}(c_2, c_3) = \mu_{\text{simtra}}(c_3, c_2) = (0.7, 0.8, 0.9, 1)$, and c_4 is similar to c_5 with $\mu_{\text{simtra}}(c_4, c_5) = \mu_{\text{simtra}}(c_5, c_4) = (0.8, 0.9, 0.9, 1)$. Then, we can use a 5×5 concept matrix N to model the fuzzy-valued concept-network shown in the equation at the bottom of the page.

$$N = \begin{bmatrix} (1,1,1,1) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (1,1,1,1) & (0.7,0.8,0.9,1) & (0,0,0,0) & (0,0,0,0) \\ (0.6,0.7,0.8,0.9) & (0.7,0.8,0.9,1) & (1,1,1,1) & (0,0,0,0) & (0,0,0,0) \\ (0.8,0.9,0.9,1) & (0,0,0,0) & (0,0,0,0) & (1,1,1,1) & (0.8,0.9,0.9,1) \\ (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0.8,0.9,0.9,1) & (1,1,1,1) \end{bmatrix}$$

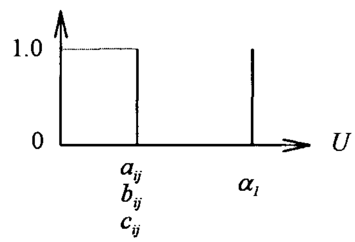

Fig. 7. Figure of Case A1.

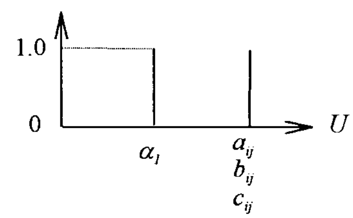

Fig. 8. Figure of Case A2.

In the following, we present a method for performing the α-cuts operations on a fuzzy-valued concept-network. Assume that a fuzzy-valued concept-network using triangular fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts has been modeled by a concept matrix M. Let P be a probability matrix derived from performing the α_1-cut operation on the concept matrix M, where α_1 is a threshold value between zero and one. Then, in a probability matrix P, the element $P(i,j)$ indicates the degree of probability that $M(i,j)$ is larger than or equal to α_1, where $M(i,j)$ is represented by a triangular fuzzy number (a_{ij}, b_{ij}, c_{ij}). The larger the value of $P(i,j)$, the more the degree of the probability that the degree of relationship (generalization relationship or similarity relationship) between the concepts c_i and c_j is larger than α_1. The value of $P(i,j)$ is decided by the following cases, where $M(i,j)$ is represented by a triangular fuzzy number parametrized by (a_{ij}, b_{ij}, c_{ij}) and $0 \le a_{ij} \le b_{ij} \le c_{ij} \le 1$.

If $a_{ij} = b_{ij} = c_{ij}$ then

Case A1: If $\alpha_1 > a_{ij}$ (see Fig. 7), then we let $P(i,j) = 0$.

Case A2: If $\alpha_1 \le a_{ij}$ (see Fig. 8), then we let $P(i,j) = 1$.

else

Case A3: If $\alpha_1 > c_{ij}$ (see Fig. 9), then we let $P(i,j) = 0$.

Case A4: If $b_{ij} \le \alpha_1 \le c_{ij}$ and $b_{ij} \ne c_{ij}$ (see Fig. 10), then we let $P(i,j) = (c_{ij} - \alpha_1)^2/(c_{ij} - b_{ij})(c_{ij} - a_{ij})$. (Since the area of the shadow triangle is $(1/2)((c_{ij} - \alpha_1)^2/(c_{ij} - b_{ij}))$ and the area of the whole triangle parametrized by (a_{ij}, b_{ij}, c_{ij}) is $\frac{1}{2}(c_{ij} - a_{ij})$, the $P(i,j)$ is equal to the proportion of the shadow triangle to the whole triangle parametrized by (a_{ij}, b_{ij}, c_{ij}) which is $(c_{ij} - \alpha_1)^2/(c_{ij} - b_{ij})(c_{ij} - a_{ij})$).

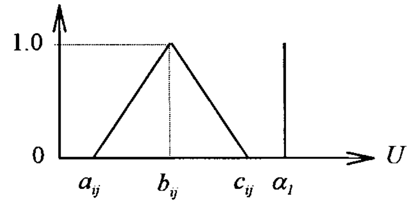

Fig. 9. Figure of Case A3.

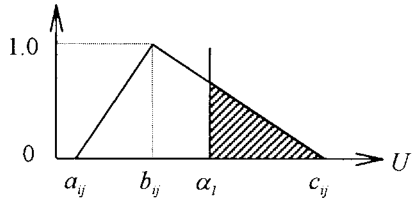

Fig. 10. Figure of Case A4.

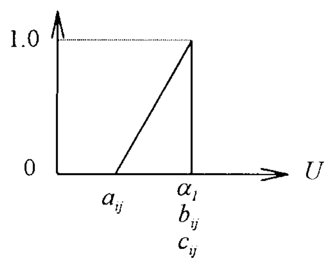

Fig. 11. Figure of Case A5.

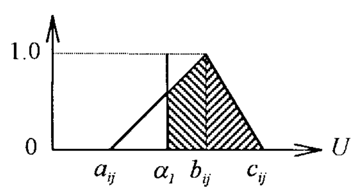

Fig. 12. Figure of Case A6.

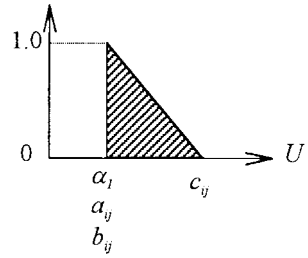

Fig. 13. Figure of Case A7.

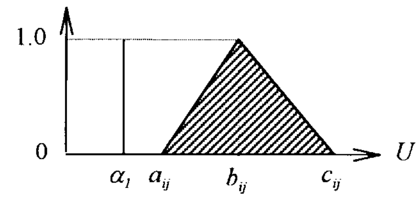

Fig. 14. Figure of Case A8.

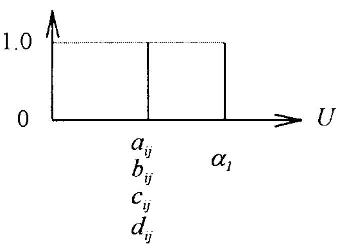

Fig. 15. Figure of Case B1.

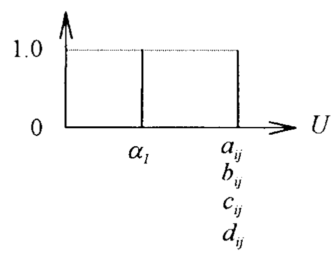

Fig. 16. Figure of Case B2.

Case A5: If $b_{ij} \le \alpha_1 \le c_{ij}$ and $b_{ij} = c_{ij}$ (see Fig. 11), then we let $P(i,j) = 0$.

Case A6: If $a_{ij} \le \alpha_1 \le b_{ij}$ and $a_{ij} \ne b_{ij}$ (see Fig. 12), then we let $P(i,j) = 1 - ((\alpha_1 - a_{ij})^2/(b_{ij} - a_{ij})(c_{ij} - a_{ij}))$. (Since the area of the empty triangle is $(1/2)((\alpha_1 - a_{ij})^2/(b_{ij} - a_{ij}))$ and the area of the whole triangle parametrized by (a_{ij}, b_{ij}, c_{ij}) is $\frac{1}{2}(c_{ij} - a_{ij})$, the $P(i,j)$ is equal to one minus the proportion of the empty triangle to the whole triangle parametrized by (a_{ij}, b_{ij}, c_{ij}) which is $1 - ((\alpha_1 - a_{ij})^2/(b_{ij} - a_{ij})(c_{ij} - a_{ij})))$.

Case A7: If $a_{ij} \le \alpha_1 \le b_{ij}$ and $a_{ij} = b_{ij}$ (see Fig. 13), then we let $P(i,j) = 1$.

Case A8: If $\alpha_1 < a_{ij}$ (see Fig. 14), then we let $P(i,j) = 1$.

Assume that a fuzzy-valued concept-network using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts has been modeled by a concept matrix N. Let Q be a probability matrix derived from performing the α_1-cut operation on the concept matrix N, where α_1 is a threshold value between zero and one. Then, in a probability matrix Q, the element $Q(i,j)$ indicates the degree of probability that $N(i,j)$ is larger than or equal to α_1, where $N(i,j)$ is represented by a trapezoidal fuzzy number $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$. The larger the value of $Q(i,j)$, the more the degree of the probability that the degree of relationship (generalization relationship or similarity relationship) between the concept c_i and c_j is larger than α_1. The value of $Q(i,j)$ is decided by the following cases, where $N(i,j)$ is represented by a fuzzy number parametrized by $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$ and $0 \le a_{ij} \le b_{ij} \le c_{ij} \le d_{ij} \le 1$.

If $a_{ij} = b_{ij} = c_{ij} = d_{ij}$ then

Case B1: If $\alpha_1 > a_{ij}$ (see Fig. 15), then we let $Q(i,j) = 0$.

Case B2: If $\alpha_1 \le a_{ij}$ (see Fig. 16), then we let $Q(i,j) = 1$.

else

Case B3: If $\alpha_1 > d_{ij}$ (see Fig. 17), then we let $Q(i,j) = 0$.

Case B4: If $c_{ij} \le \alpha_1 \le d_{ij}$ and $c_{ij} \ne d_{ij}$ (see Fig. 18), then we let $Q(i,j) = (d_{ij} - \alpha_1)^2/(d_{ij} - c_{ij})(d_{ij} + c_{ij} - b_{ij} - a_{ij})$. (Since the area of the shadow triangle is $(1/2)((d_{ij} - \alpha_1)^2/(d_{ij} - c_{ij}))$ and the area of the whole trapezoidal parametrized by $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$ is $\frac{1}{2}(d_{ij} + c_{ij} - b_{ij} - a_{ij})$, the $Q(i,j)$ is equal to the proportion

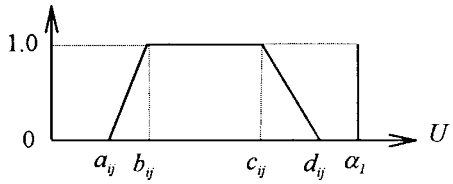

Fig. 17. Figure of Case B3.

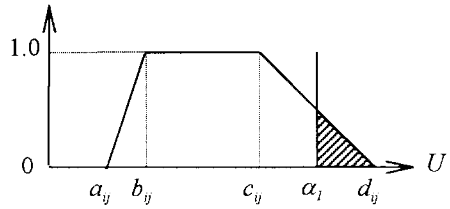

Fig. 18. Figure of Case B4.

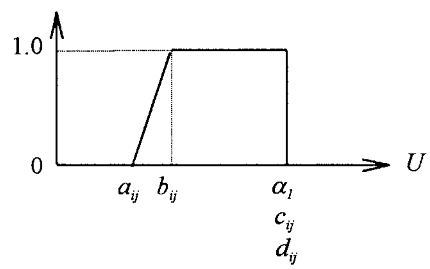

Fig. 19. Figure of Case B5.

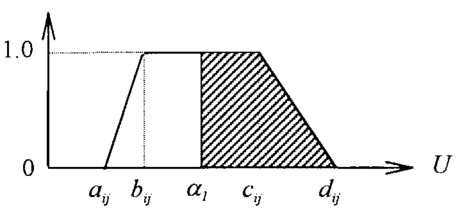

Fig. 20. Figure of Case B6.

of the shadow triangle to the whole triangle parametrized by $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$ which is $(d_{ij} - \alpha_1)^2/(d_{ij} - c_{ij})(d_{ij} + c_{ij} - b_{ij} - a_{ij}))$.

Case B5: If $c_{ij} \leq \alpha_1 \leq d_{ij}$ and $c_{ij} = d_{ij}$ (see Fig. 19), then we let $Q(i, j) = 0$.

Case B6: If $b_{ij} \leq \alpha_1 \leq c_{ij}$ (see Fig. 20), then we let $Q(i,j) = (d_{ij} + c_{ij} - 2\alpha_1)/(d_{ij} + c_{ij} - b_{ij} - a_{ij})$. (Since the area of the shadow trapezoidal is $\frac{1}{2}(d_{ij} + c_{ij} - 2\alpha_1)$ and the area of the whole trapezoidal parametrized by $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$ is $\frac{1}{2}(d_{ij} + c_{ij} - b_{ij} - a_{ij})$, the $Q(i,j)$ is equal to the proportion of the shadow trapezoidal to the whole triangle parametrized by $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$ which is $(d_{ij} + c_{ij} - 2\alpha_1)/(d_{ij} + c_{ij} - b_{ij} - a_{ij})$.

Case B7: If $a_{ij} \leq \alpha_1 \leq b_{ij}$ and $a_{ij} \neq b_{ij}$ (see Fig. 21), then we let $Q(i,j) = 1 - ((\alpha_1 - a_{ij})^2/(b_{ij} - a_{ij})(d_{ij} + c_{ij} - b_{ij} - a_{ij}))$. (Since the area of the empty triangle is $(1/2)((\alpha_1 - a_{ij})^2/(b_{ij} - a_{ij}))$ and the area of the whole trapezoidal parametrized by $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$ is $\frac{1}{2}(d_{ij} + c_{ij} - b_{ij} - a_{ij})$, the $Q(i,j)$ is equal to one minus the

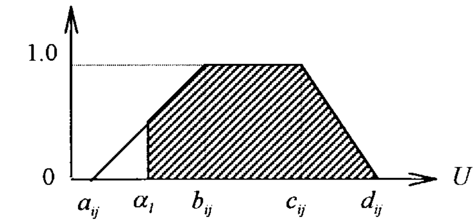

Fig. 21. Figure of Case B7.

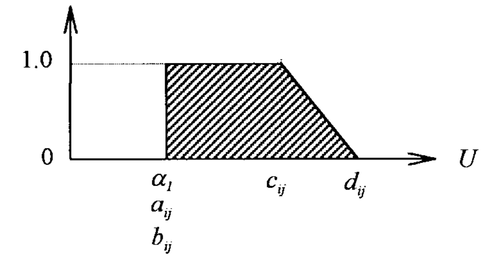

Fig. 22. Figure of Case B8.

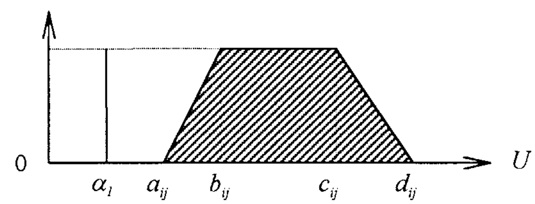

Fig. 23. Figure of Case B9.

proportion of the empty triangle to the whole triangle parametrized by $(a_{ij}, b_{ij}, c_{ij}, d_{ij})$ which is $1 - ((\alpha_1 - a_{ij})^2/(b_{ij} - a_{ij})(d_{ij} + c_{ij} - b_{ij} - a_{ij})))$.

Case B8: If $a_{ij} \leq \alpha_1 \leq b_{ij}$ and $a_{ij} = b_{ij}$ (see Fig. 22), then we let $Q(i,j) = 1$.

Case B9: If $\alpha_1 < a_{ij}$ (see Fig. 23), then we let $Q(i,j) = 1$.

Let S be a confidence matrix derived from P, and let α_2 be a threshold value between zero and one. If $P(i,j) \geq \alpha_2$, where $\alpha_2 \in [0, 1]$, then we let $S(i,j) = 1$. Otherwise, we let $S(i,j) = 0$. $S(i,j) = 1$ indicates that the degree of probability β in which the degree of relationship between the concepts c_i and c_j is larger than or equal to α_1 is larger than or equal to α_2, where $\alpha_2 \in [0, 1]$.

Let S be a confidence matrix derived from Q, and let α_2 be a threshold value between zero and one. If $Q(i,j) \geq \alpha_2$, where $\alpha_2 \in [0, 1]$, then we let $S(i,j) = 1$. Otherwise, we let $S(i,j) = 0$. $S(i,j) = 1$ indicates that the degree of probability β in which the degree of relationship between the concepts c_i and c_j is larger than or equal to α_1 is larger than or equal to α_2, where $\alpha_2 \in [0, 1]$.

In the following, we assume that a fuzzy-valued concept-network which consists of n concepts using triangular fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts has been modeled by an $n \times n$ concept matrix M, where $M(i,j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3), 0 \leq \mu_{ij}^1 \leq \mu_{ij}^2 \leq \mu_{ij}^3 \leq 1, 1 \leq i \leq n$, and $1 \leq j \leq n$. The algorithm for performing α-cuts operations on the fuzzy-valued concept-network to obtain the probability matrix

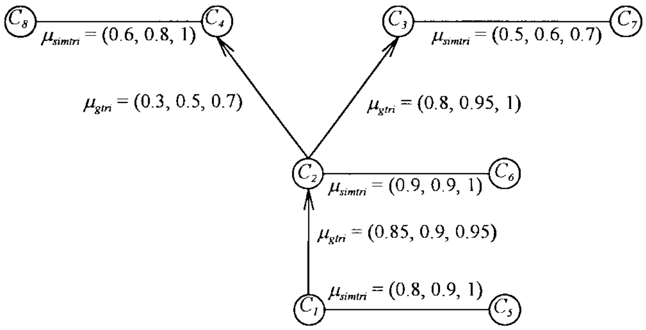

Fig. 24. Fuzzy-valued concept-network of Example 3.3.

P and the confidence matrix S is now presented as follows.

α-Cuts Operations Algorithm (Algorithm_A) for Fuzzy-Valued Concept-Networks Using Triangular Fuzzy Numbers:

for $i \leftarrow 1$ to n do
  for $j \leftarrow 1$ to n do
  begin
    if $(\mu_{ij}^1 = \mu_{ij}^2 = \mu_{ij}^3)$ then
      if $(\alpha_1 \geq \mu_{ij}^1)$ then $P(i,j) \leftarrow 0$
      else $P(i,j) \leftarrow 1$
    else
    begin
      if $(\alpha_1 \geq \mu_{ij}^3)$ then $P(i,j) \leftarrow 0$;
      if $(\mu_{ij}^2 \leq \alpha_1 \leq \mu_{ij}^3)$ and $(\mu_{ij}^2 \neq \mu_{ij}^3)$ then
        $P(i,j) \leftarrow \dfrac{(\mu_{ij}^3 - \alpha_1)^2}{(\mu_{ij}^3 - \mu_{ij}^2)(\mu_{ij}^3 - \mu_{ij}^1)}$;
      if $(\mu_{ij}^2 \leq \alpha_1 \leq \mu_{ij}^3)$ and $(\mu_{ij}^2 = \mu_{ij}^3)$
        then $P(i,j) \leftarrow 0$;
      if $(\mu_{ij}^1 \leq \alpha_1 \leq \mu_{ij}^2)$ and $(\mu_{ij}^1 \neq \mu_{ij}^2)$ then
        $P(i,j) \leftarrow 1 - \dfrac{(\alpha_1 - \mu_{ij}^1)^2}{(\mu_{ij}^3 - \mu_{ij}^1)(\mu_{ij}^2 - \mu_{ij}^1)}$;
      if $(\mu_{ij}^1 \leq \alpha_1 \leq \mu_{ij}^2)$ and $(\mu_{ij}^1 = \mu_{ij}^2)$
        then $P(i,j) \leftarrow 1$;
      if $(\alpha_1 \leq \mu_{ij}^1)$ then $P(i,j) \leftarrow 1$
    end;
    if $(P(i,j) \geq \alpha_2)$ then $S(i,j) \leftarrow 1$
    else $S(i,j) \leftarrow 0$
  end.

Example 3.3: Given a fuzzy-valued concept-network shown in Fig. 24. Assume that $\alpha_1 = 0.6$ and $\alpha_2 = 0.7$, then we can use the concept matrix M to model the fuzzy-valued concept-network shown at the bottom of the page. By performing the α-cuts operations, the probability matrix P and confidence matrix S can be obtained as follows:

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1/8 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In the following, we assume that a fuzzy-valued concept-network which consists of n concepts using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts has been modeled by an $n \times n$ concept matrix N, where $N(i,j) = (\mu_{ij}^1, \mu_{ij}^2, \mu_{ij}^3, \mu_{ij}^4), 0 \leq \mu_{ij}^1 \leq \mu_{ij}^2 \leq \mu_{ij}^3 \leq \mu_{ij}^4 \leq 1, 1, \leq i \leq n$, and $1 \leq j \leq n$. The algorithm for performing α-cuts operations on the fuzzy-valued concept-network to obtain the probability matrix Q and the confidence matrix S is shown above the matrix at the bottom of the next page.

Example 3.4: Given a fuzzy-valued concept-network shown in Fig. 25. Assume that $\alpha_1 = 0.6$ and $\alpha_2 = 0.7$, then we can use the concept matrix N to model the fuzzy-valued concept-network shown at the bottom of the page. By performing the α-cuts operations, the probability matrix Q and confidence matrix S can be obtained as follows:

$$Q = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/18 & 13/14 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In the following, we present the definition of concept classes in a fuzzy-valued concept-network based on [4]. A concept class P_i in a fuzzy-valued concept-network using triangular fuzzy numbers to

$$M = \begin{bmatrix} (1,1,1) & (0.85,0.9,0.95) & (0,0,0) & (0,0,0) & (0.8,0.9,1) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (1,1,1) & (0.8,0.95,1) & (0.3,0.5,0.7) & (0,0,0) & (0.9,0.9,1) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (1,1,1) & (0,0,0) & (0,0,0) & (0,0,0) & (0.5,0.6,0.7) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (1,1,1) & (0,0,0) & (0,0,0) & (0,0,0) & (0.6,0.8,1) \\ (0.8,0.9,1) & (0,0,0) & (0,0,0) & (0,0,0) & (1,1,1) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0.9,0.9,1) & (0,0,0) & (0,0,0) & (0,0,0) & (1,1,1) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0.5,0.6,0.7) & (0,0,0) & (0,0,0) & (0,0,0) & (1,1,1) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0.6,0.8,1) & (0,0,0) & (0,0,0) & (0,0,0) & (1,1,1) \end{bmatrix}$$

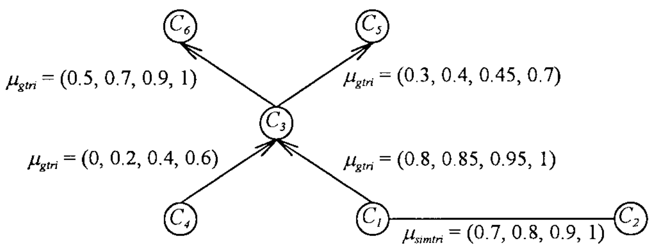

Fig. 25. Fuzzy-valued concept-network of Example 3.4.

represent the degrees of generalization and the degrees of similarity between concepts is a set of concepts, such that the set of concepts C in the fuzzy-valued concept-network is the union of each concept class, i.e., $C = \cup_i P_i$. Furthermore, after performing the α-cuts operations in the fuzzy-valued concept-network, we can define the set of synonymous concepts in each concept class.

Definition 3.1: In a concept class $P_i, \forall c_i, c_j \in P_i$, if $\mu_{\mathrm{simtri}}(c_i, c_j) > 0$, then we say that c_i and c_j are in the same set of synonymous concepts in the fuzzy-valued concept-networks using triangular fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts.

A concept class X_i in a fuzzy-valued concept-network using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts is a set of concepts, such that the set of concepts C in the fuzzy-valued concept-network is the union of each concept class, i.e., $C = \cup_i X_i$. Furthermore, after performing the α-cuts operations in the fuzzy-valued concept-network, we can define the set of synonymous concepts in each concept class.

Definition 3.2: In a concept class $X_i, \forall c_i, c_j \in X_i$, if $\mu_{\mathrm{simtra}}(c_i, c_j) > 0$, then we say that c_i and c_j are in the same set of synonymous concepts in the fuzzy-valued concept-networks using trapezoidal fuzzy numbers to represent the degrees of generalization and the degrees of similarity between concepts.

The algorithm for finding the inheritance hierarchies in a fuzzy-valued concept-network is a modification of the one we presented in [2]. The algorithm is shown at the bottom of the next page and continued on the page following that.

Example 3.5: We make the same assumptions as in Example 3.3, where the fuzzy-valued concept-network shown in Fig. 24 is modeled by the concept matrix M, and the probability matrix P

α-Cuts Operations Algorithm (Algorithm_B) for Fuzzy-Valued Concept-Networks Using Trapezoidal Fuzzy Numbers:

for $i \leftarrow 1$ to n do
  for $j \leftarrow 1$ to n do
  begin
    if $(\mu_{ij}^1 = \mu_{ij}^2 = \mu_{ij}^3 = \mu_{ij}^4)$ then
      if $(\alpha_1 \geq \mu_{ij}^1)$ then $Q(i,j) \leftarrow 0$
      else $Q(i,j) \leftarrow 1$
    else
    begin
      if $(\alpha_1 \geq \mu_{ij}^4)$ then $Q(i,j) \leftarrow 0$;
      if $(\mu_{ij}^3 \leq \alpha_1 \leq \mu_{ij}^4)$ and $(\mu_{ij}^3 = \mu_{ij}^4)$ then

$$Q(i,j) \leftarrow \frac{(\mu_{ij}^4 - \alpha_1)^2}{(\mu_{ij}^4 - \mu_{ij}^3)(\mu_{ij}^4 + \mu_{ij}^3 - \mu_{ij}^2 - \mu_{ij}^1)};$$

      if $(\mu_{ij}^3 \leq \alpha_1 \leq \mu_{ij}^4)$ and $(\mu_{ij}^3 \neq \mu_{ij}^4)$
        then $Q(i,j) \leftarrow 0$;
      if $(\mu_{ij}^2 \leq \alpha_1 \leq \mu_{ij}^3)$ then

$$Q(i,j) \leftarrow \frac{\mu_{ij}^4 + \mu_{ij}^3 - 2\alpha_1}{\mu_{ij}^4 + \mu_{ij}^3 - \mu_{ij}^2 - \mu_{ij}^1};$$

      if $(\mu_{ij}^1 \leq \alpha_1 \leq \mu_{ij}^2)$ and $(\mu_{ij}^1 \neq \mu_{ij}^2)$ then

$$Q(i,j) \leftarrow 1 - \frac{(\alpha_1 - \mu_{ij}^1)^2}{(\mu_{ij}^2 - \mu_{ij}^1)(\mu_{ij}^4 + \mu_{ij}^3 - \mu_{ij}^2 - \mu_{ij}^1)};$$

      if $(\mu'_{ij} \leq \alpha_1 \leq \mu_{ij}^2)$ and $(\mu_{ij}^1 = \mu_{ij}^2)$ then
        $Q(i,j) \leftarrow 1$;
      if $(\alpha_1 \leq \mu_{ij}^1)$ then $Q(i,j) \leftarrow 1$
    end;
    if $(Q(i,j) \geq \alpha_2)$ then $S(i,j) \leftarrow 1$
    else $S(i,j) \leftarrow 0$
  end.

$$N = \begin{bmatrix} (1,1,1,1) & (0.7,0.8,0.9,1) & (0.8,0.85,0.95,1) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0.7,0.8,0.9,1) & (1,1,1,1) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & (1,1,1,1) & (0,0,0,0) & (0.3,0.4,0.45,0.7) & (0.5,0.7,0.9,1) \\ (0,0,0,0) & (0,0,0,0) & (0,0.2,0.4,0.6) & (1,1,1,1) & (0,0,0,0) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (1,1,1,1) & (0,0,0,0) \\ (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (0,0,0,0) & (1,1,1,1) \end{bmatrix}$$

and the confidence matrix R have been obtained. By applying the inheritance hierarchy generation algorithm, we can obtain three sets of concept classes: $\{c_1, c_2, c_3, c_5, c_6\}, \{c_4, c_8\}, \{c_7\}$, and three sets of synonymous concepts: $\{c_1, c_5\}, \{c_2, c_6\}, \{c_4, c_8\}$. Assume that we are interested in the concept classes containing c_2, then after performing the algorithm, we can find the inheritance hierarchy $\{\langle c_1, c_2, c_3\rangle\}$ containing c_2, graphically as shown in Fig. 26(a). By using replacement among synonymous concepts, we can obtain the other three inheritance hierarchies: $\{\langle c_5, c_2, c_3\rangle\}, \{\langle c_1, c_6, c_3\rangle\}, \{\langle c_5, c_6, c_3\rangle\}$ as shown in Fig. 26(b)–(d), respectively.

Inheritance Hierarchy Generation Algorithm for Fuzzy-Valued Concept-Networks:

Step 1: Perform the α-cuts operations on the fuzzy-valued concept network using the α-cuts operations algorithm described previously.

(Notes: (1) If the degrees of generalization and the degree of similarity between concepts in the fuzzy-valued concept networks are represented by triangular fuzzy numbers, then we can choose Algorithm_A for performing the α-cuts operations on the fuzzy-valued concept network.

(2) If the degrees of generalization and the degree of similarity between concepts in the fuzzy-valued concept networks are represented by trapezoidal fuzzy numbers, then we can choose Algorithm_B for performing the α-cuts operations on the fuzzy-valued concept network.

(3) Because a triangular fuzzy number (a, b, c) can also be represented by a trapezoidal fuzzy number (a, b, b, c), if the degrees of generalization and the degree of similarity between concepts in the fuzzy-valued concept-networks are represented by triangular fuzzy numbers, then we also can firstly translate the triangular fuzzy numbers in the fuzzy-valued concept network into trapezoidal fuzzy numbers, and then we can choose Algorithm_B for performing the α-cuts operations on the fuzzy-valued concept-networks.)

```
Step 2: for i ← 1 to n do
            for j ← 1 to n do
              begin
                if i = j and c_i is not in any concept class then generate a new concept class, and put c_i in the
                  new generated concept class;
                if i ≠ j and S(i, j) = 1 then
                  if S(j, i) = 1 then
                    begin
                      if c_i is not in any concept class then generate a new concept class, and put c_i and c_j in the
                        new generated concept class
                      else
                        put c_j in the same concept class with c_i;
                      if c_i is not in any set of synonymous concepts then generate a new set of synonymous
                        concepts, and put c_i and c_j in the new generated set of synonymous concepts
                      else
                        put c_j in the same set of synonymous concepts with c_i
                  end
                  else
                    begin
                      if c_i is not in any concept class and c_j is not in any concept class then generate a new
                        concept class, and put c_i and c_j in the new generated concept class;
                      if c_i is in a concept class and c_j is not in any concept class then put c_j in the same
                        concept class with c_i;
                      if c_i is not in any concept class and c_j is in a concept class then put c_i in the same
                        concept class with c_j;
                      if c_i is in a concept class and c_j is in a concept class
                      then
                        begin
                          put all concepts in the concept class containing c_j in the same concept class with c_i;
                          put all fuzzy-valued generalization in the concept class containing c_j in the concept
                          class containing c_i
                        end;
                      let ⟨c_i, c_j⟩ be an fuzzy-valued generalization relation in concept class containing c_i
                    end
            end;
        find the concept class containing concept c_k;
```

```
list all fuzzy-valued generalization relations in this concept class which form an inheritance hierarchy;
for all c_i in this inheritance hierarchy do
begin
  find the set of synonymous concepts containing c_i;
  for each c_j in this set of synonymous concepts do
  begin
    substitute c_i in the fuzzy-valued generalization relation by c_j;
    list all fuzzy-valued generalization relations in this concept
      class which form a new inheritance hierarchy
  end
end.
```

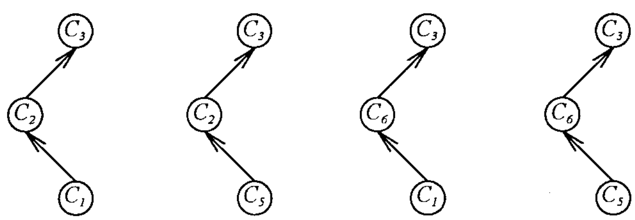

Fig. 26. Inheritance hierarchies of Example 3.5.

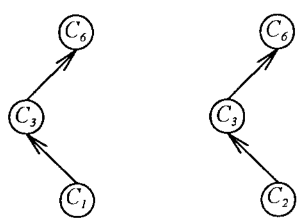

Fig. 27. Inheritance hierarchies of Example 3.6.

Example 3.6: We make the same assumptions as in Example 3.4, where the fuzzy-valued concept-network shown in Fig. 25 is modeled by the concept matrix N, and the probability matrix Q and the confidence matrix R have been obtained. By applying the inheritance hierarchy generation algorithm, we can obtain three sets of concept classes: $\{c_1, c_2, c_3, c_6\}, \{c_4\}, \{c_5\}$, and one set of synonymous concepts: $\{c_1, c_2\}$. Assume that we are interested in the concept classes containing c_3, then after performing the algorithm, we can find the inheritance hierarchy $\{\langle c_1, c_3, c_6\rangle\}$ containing c_3, graphically as shown in Fig. 27(a). By using replacement among synonymous concepts, we can obtain the other inheritance hierarchy: $\{\langle c_2, c_3, c_6\rangle\}$ as shown in Fig. 27(b).

IV. Conclusions

In this paper, we have extended the works of [2] and [4] to present the concepts of fuzzy-valued concept-networks and to present an algorithm for finding the collection of inheritance hierarchies in fuzzy-valued concept-networks where the degrees of generalization and the degrees of similarity between concepts are represented by triangular fuzzy numbers or trapezoidal fuzzy numbers. The proposed method is more flexible than the ones presented in [2] and [4] due to the fact that it allows the similarity relations and the generalization relations between concepts to be represented by triangular fuzzy numbers or trapezoidal fuzzy numbers rather than crisp real values between zero and one or interval-values in [0, 1].

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