ELASTIC SOLUTIONS FOR A TRANSVERSELY ISOTROPIC HALF-SPACE SUBJECTED TO BURIED ASYMMETRIC-LOADS

C. D. WANG AND J. J. LIAO*

Department of Civil Engineering, National Chiao-Tung University, Hsinchu 30050, Taiwan, R.O.C.

SUMMARY

Elastic closed-form solutions for the displacements and stresses in a transversely isotropic half-space subjected to various buried loading types are presented. The loading types include finite line loads and asymmetric loads (such as uniform and linearly varying rectangular loads, or trapezoidal loads). The planes of transverse isotropy are assumed to be parallel to its horizontal surface. These solutions are directly obtained from integrating the point load solutions in a transversely isotropic half-space, which were derived using the principle of superposition, Fourier and Hankel transformation techniques. The solutions for the displacements and stresses in transversely isotropic half-spaces subjected to linearly variable loads on a rectangular region are never mentioned in literature. These exact solutions indicate that the displacements and stresses are influenced by several factors, such as the buried depth, the loading types, and the degree and type of rock anisotropy. Two illustrative examples, a vertical uniform and a vertical linearly varying rectangular load acting on the surface of transversely isotropic rock masses, are presented to show the effect of various parameters on the vertical surface displacement and vertical stress. The results indicate that the displacement and stress distributions accounted for rock anisotropy are quite different for those calculated from isotropic solutions. Copyright © 1999 John Wiley & Sons, Ltd.

Key words: transversely isotropic half-space; buried asymmetric-loads; finite line load; uniform rectangular load; linearly varying rectangular load; rock anisotropy

INTRODUCTION

In general, the magnitude and distribution of the displacements and stresses in rock are predicted by using exact solutions that model rock as a linearly elastic, homogeneous and isotropic continuum. However, for rock masses cut by discontinuities, such as cleavages, foliations, stratifications, schistosities, joints, these analytical solutions should account for anisotropy. Anisotropic rocks are often modelled as orthotropic or transversely isotropic materials from the practical considerations in engineering analysis. In this paper, an elastic problem for a transversely isotropic medium is relevant.

A point load solution is the basis of complex loading problems. Solutions of the displacements and stresses due to a concentrated force for transversely isotropic half-spaces have been presented

*Correspondence to: J. J. Liao, Department of Civil Engineering, National Chiao-Tung University, Hsinchu 30050, Taiwan, R.O.C.

by several investigators.[1-4] Nevertheless, the types of external loads of a half-space should be more complex than a point load in most engineering cases. In fact, the complex external loads are generally simplified as a finite line load, a rectangular load or a linearly varying rectangular load, etc., for engineering analysis. Hence, the closed-form solutions for the displacements and stresses in a half-space subjected to various loads are needed for engineering design.

Closed-form solutions for the displacements and stresses in an elastic isotropic half-space induced by various loading types have been proposed by many investigators. Corresponding to the isotropic solutions, the literature contributed to loading problems of transversely isotropic media are very limited. Some studies presented the elastic solutions for the displacements, strains or stresses in a transversely isotropic half-space subjected to infinite line loads,[5-7] circular loads,[5, 8-14] parabolic loads,[12, 15-17] ring loads,[18, 19] elliptical loads,[20, 21] infinite linearly varying rectangular loads,[22, 23] and other related problems.[24-26] A summary of the existing solutions is given in Table I. Table I indicates the type of loading, the loaded direction, and the results presented in the solutions. Although, the loading surface, the loading type and the orientation of planes of transversely isotropy with respect to the loading surface in these solutions are complex or variable, they are almost limited to axisymmetric or plane problems. Recently, Lin et al.[7] presented the closed-form solutions for displacements and stresses in a transversely isotropic half-space subjected to various loading types using a series of potential functions, which suggested by Green and Zerna[27] and Pan and Chou.[28] In their solutions, the loads can be a point load, an infinite line load, and a uniform load over a rectangular area, etc. Nevertheless, the solutions can not be extended to solve the non-uniform loading problems. Hence, using Hankel and Fourier transforms with respect to r and θ in a cylindrical co-ordinate system, respectively, the authors rederived and presented the exact solutions for the displacements and stresses in a transversely isotropic half-space subjected to a point load with components in r, θ, z directions.[4] Then, in this paper, we extend the point load solutions to present a series of exact solutions for the displacements and stresses in the half-space subjected to buried asymmetric-loads by direct integrations. The asymmetric loads include finite line loads, uniform rectangular loads, and linearly varying rectangular loads. These solutions indicate that both of the displacements and stresses in a transversely isotropic half-space are affected by the buried depth, the loading types, and the degree and type of rock anisotropy. Two illustrative examples are given to investigate the effect of rock anisotropy on the displacement and stress in the medium acting by a uniform and a linearly varying vertical rectangular load on its horizontal surface, respectively.

POINT LOAD SOLUTIONS (CARTESIAN CO-ORDINATE SYSTEM)

In this paper, the solutions for the displacements and stresses in a transversely isotropic half-space subjected to buried asymmetric-loads are directly integrated from the point load solutions in a Cartesian co-ordinate system. The approaches[4] for solving the displacements and stresses subjected to a static point load with components (P_r, P_θ, P_z) in a cylindrical co-ordinate, which are expressed as the form of body forces, are shown in Figure 1. Figure 1 depicts that a half-space is composed of two infinite spaces, one acting a point load in its interior and the other being free loading, and zero stress conditions on the $z = 0$ plane. The Hankel and Fourier transforms with respect to r and θ are employed for solving this problem, respectively. Hence, the solutions can be derived from the governing equations for an infinite space (including the general solutions (I) and homogeneous solutions (II)) by satisfying the free traction on the surface of the half-space. Therefore, the analytical solutions for the displacements and stresses subjected to

Table I. Existing solutions for transversely isotropic media subjected to regular loads

| Type of loading	Author	Loaded direction	Solutions
Infinite line loads	Anon[5]	Vertical	Stresses
	Urena et al.[6]	Vertical Horizontal	All displacements and stresses All stresses
	Lin et al.[7]	Vertical Horizontal	All displacements and stresses All displacements and stresses
Circular loads	Anon[5]	Vertical	Vertical surface displacement at the center and edge, and vertical stress on load axis
	Barden[8]	Vertical	Vertical stress on load axis
	Gerrard and Harrison[9]	Vertical	All displacements, strains, stresses
	Nayak[10]	Vertical	Vertical surface displacement (incompressible conditon)
	Hooper[11]	Vertical	Vertical surface displacement
	Misra and Sen[12]	Vertical	Vertical surface displacement at the center and edge, and vertical stress beneath the center of load
	Chowdhury[13]	Buried, vertical	All surface displacements
	Hanson and Puja[14]	3-D	All displacements and stresses
Parabolic loads	Quinlan[15]	Vertical	Vertical surface displacement and vertical stress on load axis
	Misra and Sen[12]	Vertical	Vertical surface displacement at the center and edge, and vertical stress beneath the center of load
	Gazetas[16]	Vertical	All surface displacements, and all stresses beneath the center of load
	Gazetas[17]	Vertical	All displacements and stresses
Ring loads	Hasegawa and Watanabe[18]	Vertical	Displacements
	Hanson and Wang[19]	Buried, 3-D	All displacements and stresses
Elliptical loads	Sveklo[20]	Vertical	Horizontal displacement
	Gladwell[21]	Vertical	Vertical surface displacement
Uniform rectangular loads	Lin et al.[7]	Vertical	Vertical surface displacement and all stresses
		Horizontal	All stresses
Infinite linearly varying rectangular loads	Piquer et al.[22]	Vertical Horizontal	All stresses All stresses
	Moroto and Hasegawa[23]	Vertical	Vertical stress (Poisson's ratios = 0)
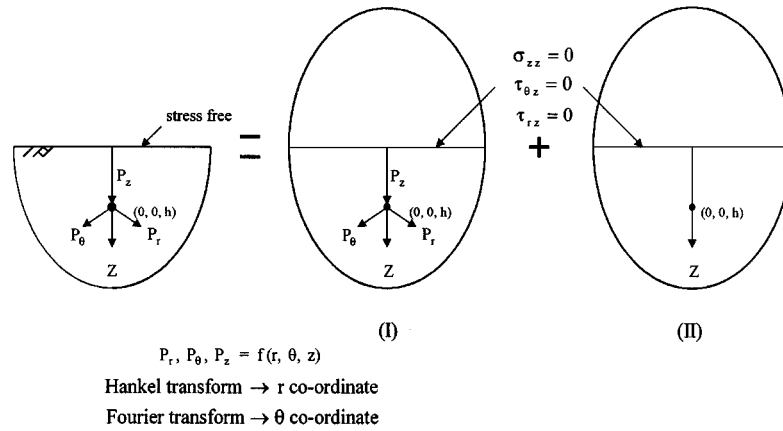
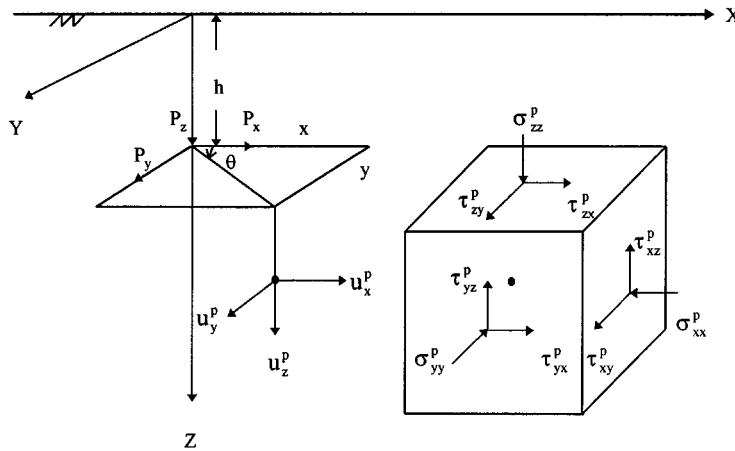Figure 1. The approach for solving a point load problem

Figure 2. A point load (P_x, P_y, P_z) acting in the interior of a semi-infinite space

a point load (P_x, P_y, P_z) acting at $z = h$ (from the surface, as shown in Figure 2) in the interior of a transversely isotropic half-space are transformed as follows:

$$u_x^p = \frac{P_x}{4\pi} \left[\frac{k}{m_1} p_{d11} - \frac{k}{m_2} p_{d12} - T_1 p_{d1a} + T_2 p_{d1b} + T_3 p_{d1c} - T_4 p_{d1d} + \frac{1}{u_3 A_{44}} (p_{d23} + p_{d2e}) \right]$$

$$+ \frac{P_y}{4\pi} \left[-\frac{k}{m_1} p_{d31} + \frac{k}{m_2} p_{d32} + T_1 p_{d3a} - T_2 p_{d3b} - T_3 p_{d3c} + T_4 p_{d3d} + \frac{1}{u_3 A_{44}} (p_{d33} + p_{d3e}) \right]$$

$$- \frac{P_z}{4\pi} \left[k(p_{d41} - p_{d42}) + m_1 (T_1 p_{d4a} - T_3 p_{d4c}) - m_2 (T_2 p_{d4b} - T_4 p_{d4d}) \right] \tag{1}$$

$$u_y^{\mathrm{p}} = \frac{P_x}{4\pi}\left[-\frac{k}{m_1}p_{\mathrm{d}31} + \frac{k}{m_2}p_{\mathrm{d}32} + T_1 p_{\mathrm{d}3a} - T_2 p_{\mathrm{d}3b} - T_3 p_{\mathrm{d}3c} + T_4 p_{\mathrm{d}3d} + \frac{1}{u_3 A_{44}}(p_{\mathrm{d}33} + p_{\mathrm{d}3e})\right]$$

$$+ \frac{P_y}{4\pi}\left[\frac{k}{m_1}p_{\mathrm{d}21} - \frac{k}{m_2}p_{\mathrm{d}22} - T_1 p_{\mathrm{d}2a} + T_2 p_{\mathrm{d}2b} + T_3 p_{\mathrm{d}2c} - T_4 p_{\mathrm{d}2d} + \frac{1}{u_3 A_{44}}(p_{\mathrm{d}13} + p_{\mathrm{d}1e})\right]$$

$$- \frac{P_z}{4\pi}\left[k(p_{\mathrm{d}51} - p_{\mathrm{d}52}) + m_1(T_1 p_{\mathrm{d}5a} - T_3 p_{\mathrm{d}5c}) - m_2(T_2 p_{\mathrm{d}5b} - T_4 p_{\mathrm{d}5d})\right] \tag{2}$$

$$u_z^{\mathrm{p}} = \frac{P_x}{4\pi}\left[-k(p_{\mathrm{d}41} - p_{\mathrm{d}42}) + m_1(T_1 p_{\mathrm{d}4a} - T_2 p_{\mathrm{d}4b}) - m_2(T_3 p_{\mathrm{d}4c} - T_4 p_{\mathrm{d}4d})\right]$$

$$+ \frac{P_y}{4\pi}\left[-k(p_{\mathrm{d}51} - p_{\mathrm{d}52}) + m_1(T_1 p_{\mathrm{d}5a} - T_2 p_{\mathrm{d}5b}) - m_2(T_3 p_{\mathrm{d}5c} - T_4 p_{\mathrm{d}5d})\right]$$

$$- \frac{P_z}{4\pi}\left[m_1(k p_{\mathrm{d}61} + T_1 m_1 p_{\mathrm{d}6a} - T_2 m_2 p_{\mathrm{d}6b}) - m_2(k p_{\mathrm{d}62} + T_3 m_1 p_{\mathrm{d}6c} - T_4 m_2 p_{\mathrm{d}6d})\right] \tag{3}$$

$$\sigma_{xx}^{\mathrm{p}} = \frac{P_x}{4\pi}\left[(A_{11} - u_1 m_1 A_{13})\left(\frac{k}{m_1}p_{\mathrm{s}11} - T_1 p_{\mathrm{s}1a} + T_2 p_{\mathrm{s}1b}\right)\right.$$

$$- (A_{11} - u_2 m_2 A_{13})\left(\frac{k}{m_2}p_{\mathrm{s}12} - T_3 p_{\mathrm{s}1c} + T_4 p_{\mathrm{s}1d}\right)$$

$$\left. - 2A_{66}\left(\frac{k}{m_1}p_{\mathrm{s}71} - \frac{k}{m_2}p_{\mathrm{s}72} - T_1 p_{\mathrm{s}7a} + T_2 p_{\mathrm{s}7b} + T_3 p_{\mathrm{s}7c} - T_4 p_{\mathrm{s}7d}\right) + 2u_3(p_{\mathrm{s}73} + p_{\mathrm{s}7e})\right]$$

$$+ \frac{P_y}{4\pi}\left[(A_{11} - u_1 m_1 A_{13} - 2A_{66})\left(\frac{k}{m_1}p_{\mathrm{s}21} - T_1 p_{\mathrm{s}2a} + T_2 p_{\mathrm{s}2b}\right)\right.$$

$$- (A_{11} - u_2 m_2 A_{13} - 2A_{66})\left(\frac{k}{m_2}p_{\mathrm{s}22} - T_3 p_{\mathrm{s}2c} + T_4 p_{\mathrm{s}2d}\right)$$

$$\left. + 2A_{66}\left(\frac{k}{m_1}p_{\mathrm{s}81} - \frac{k}{m_2}p_{\mathrm{s}82} - T_1 p_{\mathrm{s}8a} + T_2 p_{\mathrm{s}8b} + T_3 p_{\mathrm{s}8c} - T_4 p_{\mathrm{s}8d}\right) - 2u_3(p_{\mathrm{s}83} + p_{\mathrm{s}8e})\right]$$

$$+ \frac{P_z}{4\pi}\left\{(A_{11} - u_1 m_1 A_{13} - 2A_{66})(k p_{\mathrm{s}31} + T_1 m_1 p_{\mathrm{s}3a} - T_2 m_2 p_{\mathrm{s}3b})\right.$$

$$- (A_{11} - u_2 m_2 A_{13} - 2A_{66})(k p_{\mathrm{s}32} + T_3 m_1 p_{\mathrm{s}3c} - T_4 m_2 p_{\mathrm{s}3d})$$

$$\left. + 2A_{66}\left[k(p_{\mathrm{s}51} - p_{\mathrm{s}52}) + m_1(T_1 p_{\mathrm{s}5a} - T_3 p_{\mathrm{s}5c}) - m_2(T_2 p_{\mathrm{s}5b} - T_4 p_{\mathrm{s}5d})\right]\right\} \tag{4}$$

$$\sigma_{yy}^{\mathrm{p}} = \frac{P_x}{4\pi}\left[(A_{11} - u_1 m_1 A_{13} - 2A_{66})\left(\frac{k}{m_1}p_{\mathrm{s}11} - T_1 p_{\mathrm{s}1a} + T_2 p_{\mathrm{s}1b}\right)\right.$$

$$- (A_{11} - u_2 m_2 A_{13} - 2A_{66})\left(\frac{k}{m_2}p_{\mathrm{s}12} - T_3 p_{\mathrm{s}1c} + T_4 p_{\mathrm{s}1d}\right)$$

$$\left. + 2A_{66}\left(\frac{k}{m_1}p_{\mathrm{s}71} - \frac{k}{m_2}p_{\mathrm{s}72} - T_1 p_{\mathrm{s}7a} + T_2 p_{\mathrm{s}7b} + T_3 p_{\mathrm{s}7c} - T_4 p_{\mathrm{s}7d}\right) - 2u_3(p_{\mathrm{s}73} + p_{\mathrm{s}7e})\right]$$

$$+ \frac{P_y}{4\pi}\left[(A_{11} - u_1 m_1 A_{13})\left(\frac{k}{m_1}p_{\mathrm{s}21} - T_1 p_{\mathrm{s}2a} + T_2 p_{\mathrm{s}2b}\right)\right.$$

$$- (A_{11} - u_2 m_2 A_{13})\left(\frac{k}{m_2}p_{\mathrm{s}22} - T_3 p_{\mathrm{s}2c} + T_4 p_{\mathrm{s}2d}\right)$$

$$\left. - 2A_{66}\left(\frac{k}{m_1}p_{\mathrm{s}81} - \frac{k}{m_2}p_{\mathrm{s}82} - T_1 p_{\mathrm{s}8a} + T_2 p_{\mathrm{s}8b} + T_3 p_{\mathrm{s}8c} - T_4 p_{\mathrm{s}8d}\right) + 2u_3(p_{\mathrm{s}83} + p_{\mathrm{s}8e})\right]$$

$$+ \frac{P_z}{4\pi}\{(A_{11} - u_1 m_1 A_{13} - 2A_{66})(k p_{\mathrm{s}31} + T_1 m_1 p_{\mathrm{s}3a} - T_2 m_2 p_{\mathrm{s}3b})$$

$$- (A_{11} - u_2 m_2 A_{13} - 2A_{66})(k p_{\mathrm{s}32} + T_3 m_1 p_{\mathrm{s}3c} - T_4 m_2 p_{\mathrm{s}3d})$$

$$+ 2A_{66}[k(p_{\mathrm{s}61} - p_{\mathrm{s}62}) + m_1(T_1 p_{\mathrm{s}6a} - T_3 p_{\mathrm{s}6c}) - m_2(T_2 p_{\mathrm{s}6b} - T_4 p_{\mathrm{s}6d})]\} \qquad (5)$$

$$\sigma_{zz}^{\mathrm{p}} = \frac{P_x}{4\pi}\left[(A_{13} - u_1 m_1 A_{33})\left(\frac{k}{m_1}p_{\mathrm{s}11} - T_1 p_{\mathrm{s}1a} + T_2 p_{\mathrm{s}1b}\right)\right.$$

$$\left. - (A_{13} - u_2 m_2 A_{33})\left(\frac{k}{m_2}p_{\mathrm{s}12} - T_3 p_{\mathrm{s}1c} + T_4 p_{\mathrm{s}1d}\right)\right]$$

$$+ \frac{P_y}{4\pi}\left[(A_{13} - u_1 m_1 A_{33})\left(\frac{k}{m_1}p_{\mathrm{s}21} - T_1 p_{\mathrm{s}2a} + T_2 p_{\mathrm{s}2b}\right)\right.$$

$$\left. - (A_{13} - u_2 m_2 A_{33})\left(\frac{k}{m_2}p_{\mathrm{s}22} - T_3 p_{\mathrm{s}2c} + T_4 p_{\mathrm{s}2d}\right)\right]$$

$$+ \frac{P_z}{4\pi}[(A_{13} - u_1 m_1 A_{33})(k p_{\mathrm{s}31} + T_1 m_1 p_{\mathrm{s}3a} - T_2 m_2 p_{\mathrm{s}3b})$$

$$- (A_{13} - u_2 m_2 A_{33})(k p_{\mathrm{s}32} + T_3 m_1 p_{\mathrm{s}3c} - T_4 m_2 p_{\mathrm{s}3d})] \qquad (6)$$

$$\tau_{xy}^{p} = \frac{P_x}{4\pi}\left[2A_{66}\left(\frac{k}{m_1}p_{s81} - \frac{k}{m_2}p_{s82} - T_1p_{s8a} + T_2p_{s8b} + T_3p_{s8c} - T_4p_{s8d}\right)\right.$$

$$\left. - u_3(2p_{s83} + 2p_{s8e} - p_{s23} - p_{s2e})\right]$$

$$+ \frac{P_y}{4\pi}\left[2A_{66}\left(\frac{k}{m_1}p_{s71} - \frac{k}{m_2}p_{s72} - T_1p_{s7a} + T_2p_{s7b} + T_3p_{s7c} - T_4p_{s7d}\right)\right.$$

$$\left. - u_3(2p_{s73} + 2p_{s7e} - p_{s13} - p_{s1e})\right]$$

$$- \frac{P_z}{2\pi}A_{66}\left[k(p_{s41} - p_{s42}) + m_1(T_1p_{s4a} - T_3p_{s4c}) - m_2(T_2p_{s4b} - T_4p_{s4d})\right] \qquad (7)$$

$$\tau_{yz}^{p} = -\frac{P_x}{4\pi}\left[(u_1 + m_1)A_{44}\left(\frac{k}{m_1}p_{s41} - T_1p_{s4a} + T_2p_{s4b}\right)\right.$$

$$\left. - (u_2 + m_2)A_{44}\left(\frac{k}{m_2}p_{s42} - T_3p_{s4c} + T_4p_{s4d}\right) - (p_{s43} + p_{s4e})\right]$$

$$+ \frac{P_y}{4\pi}\left[(u_1 + m_1)A_{44}\left(\frac{k}{m_1}p_{s61} - T_1p_{s6a} + T_2p_{s6b}\right)\right.$$

$$\left. - (u_2 + m_2)A_{44}\left(\frac{k}{m_2}p_{s62} - T_3p_{s6c} + T_4p_{s6d}\right) + (p_{s53} + p_{s5e})\right]$$

$$- \frac{P_z}{4\pi}A_{44}\left[(u_1 + m_1)(kp_{s21} + T_1m_1p_{s2a} - T_2m_2p_{s2b})\right.$$

$$\left. - (u_2 + m_2)(kp_{s22} + T_3m_1p_{s2c} - T_4m_2p_{s2d})\right] \qquad (8)$$

$$\tau_{xz}^{p} = \frac{P_x}{4\pi}\left[(u_1 + m_1)A_{44}\left(\frac{k}{m_1}p_{s51} - T_1p_{s5a} + T_2p_{s5b}\right)\right.$$

$$\left. - (u_2 + m_2)A_{44}\left(\frac{k}{m_2}p_{s52} - T_3p_{s5c} + T_4p_{s5d}\right) + (p_{s63} + p_{s6e})\right]$$

$$- \frac{P_y}{4\pi}\left[(u_1 + m_1)A_{44}\left(\frac{k}{m_1}p_{s41} - T_1p_{s4a} + T_2p_{s4b}\right)\right.$$

$$\left. - (u_2 + m_2)A_{44}\left(\frac{k}{m_2}p_{s42} - T_3p_{s4c} + T_4p_{s4d}\right) - (p_{s43} + p_{s4e})\right]$$

$$- \frac{P_z}{4\pi}A_{44}\left[(u_1 + m_1)(kp_{s11} + T_1m_1p_{s1a} - T_2m_2p_{s1b})\right.$$

$$\left. - (u_2 + m_2)(kp_{s12} + T_3m_1p_{s1c} - T_4m_2p_{s1d})\right] \qquad (9)$$

where

(a) A_{ij} $(i, j = 1–6)$ are the elastic moduli[29] or elasticity constants[30] of the medium, and can be expressed in terms of five independent elastic constants for a transversely isotropic material as

$$A_{11} = \frac{E\left(1 - \dfrac{E}{E'}v'^2\right)}{(1 + v)\left(1 - v - \dfrac{2E}{E'}v'^2\right)}, \quad A_{13} = \frac{Ev'}{1 - v - \dfrac{2E}{E'}v'^2}, \quad A_{33} = \frac{E'(1 - v)}{1 - v - \dfrac{2E}{E'}v'^2}, \quad A_{44} = G'.$$

$$A_{66} = \frac{E}{2(1 + v)} \tag{10}$$

whereas E and E' are Young's moduli in the plane of transverse isotropy and in direction normal to it, respectively; v and v' are Poisson's ratios characterizing the lateral strain response in the plane of transverse isotropy to a stress acting parallel or normal to it, respectively; G' is the shear modulus in planes normal to the plane of transverse isotropy.

(b) $u_3 = \sqrt{A_{66}/A_{44}}$, u_1 and u_2 are the roots of the following characteristic equation:

$$u^4 - su^2 + q = 0 \tag{11}$$

whereas

$$s = \frac{A_{11}A_{33} - A_{13}(A_{13} + 2A_{44})}{A_{33}A_{44}}, \quad q = \frac{A_{11}}{A_{33}}.$$

Since the strain energy is assumed to be positive definite in the medium, the values of elastic constants are restricted.[31,32] Hence, there is three categories of the characteristic roots, u_1 and u_2 as follows:

Case 1. $u_{1,2} = \pm\sqrt{\{\frac{1}{2}[s \pm \sqrt{(s^2 - 4q)}]\}}$ are two real distinct roots when $s^2 - 4q > 0$,

Case 2. $u_{1,2} = \pm\sqrt{s/2}, \pm\sqrt{s/2}$ are double equal real roots when $s^2 - 4q = 0$,

Case 3. $u_1 = \frac{1}{2}\sqrt{(s + 2\sqrt{q})} - \mathrm{i}\frac{1}{2}\sqrt{(-s + 2\sqrt{q})} = \gamma - \mathrm{i}\delta$, $u_2 = \gamma + \mathrm{i}\delta$ are two complex conjugate roots (where γ cannot be equal to zero[1]) when $s^2 - 4q < 0$.

(c) $m_j = \dfrac{(A_{13} + A_{44})u_j}{A_{33}u_j^2 - A_{44}} = \dfrac{A_{11} - A_{44}u_j^2}{(A_{13} + A_{44})u_j}$ $(j = 1, 2)$, $\quad k = \dfrac{(A_{13} + A_{44})}{A_{33}A_{44}(u_1^2 - u_2^2)}$, $\quad T_1 = \dfrac{k}{m_1}\dfrac{u_1 + u_2}{u_2 - u_1}$

$$T_2 = \frac{k}{m_2}\frac{2u_1(u_2 + m_2)}{(u_2 - u_1)(u_1 + m_1)}, \quad T_3 = \frac{k}{m_1}\frac{2u_2(u_1 + m_1)}{(u_2 - u_1)(u_2 + m_2)}, \quad T_4 = \frac{k}{m_2}\frac{u_1 + u_2}{u_2 - u_1}$$

$$p_{\mathrm{d}1i} = \frac{1}{R_i + z_i} - \frac{x^2}{R_i(R_i + z_i)^2}, \quad p_{\mathrm{d}2i} = \frac{1}{R_i + z_i} - \frac{y^2}{R_i(R_i + z_i)^2}, \quad p_{\mathrm{d}3i} = \frac{xy}{R_i(R_i + z_i)^2}$$

$$p_{\mathrm{d}4i} = \frac{x}{R_i(R_i + z_i)}, \quad p_{\mathrm{d}5i} = \frac{y}{R_i(R_i + z_i)}, \quad p_{\mathrm{d}6i} = \frac{1}{R_i}, \quad p_{\mathrm{s}1i} = \frac{x}{R_i^3}, \quad p_{\mathrm{s}2i} = \frac{y}{R_i^3}, \quad p_{\mathrm{s}3i} = \frac{z_i}{R_i^3}$$

$$p_{\mathrm{s}4i} = \frac{xy(2R_i + z_i)}{R_i^3(R_i + z_i)^2}, \quad p_{\mathrm{s}5i} = \frac{1}{R_i(R_i + z_i)} - \frac{x^2(2R_i + z_i)}{R_i^3(R_i + z_i)^2}, \quad p_{\mathrm{s}6i} = \frac{1}{R_i(R_i + z_i)} - \frac{y^2(2R_i + z_i)}{R_i^3(R_i + z_i)^2}$$

$$p_{\mathrm{s}7i} = \frac{x}{R_i^3} - \frac{3x}{R_i(R_i + z_i)^2} + \frac{x^3(3R_i + z_i)}{R_i^3(R_i + z_i)^3}, \quad p_{\mathrm{s}8i} = \frac{y}{R_i^3} - \frac{3y}{R_i(R_i + z_i)^2} + \frac{y^3(3R_i + z_i)}{R_i^3(R_i + z_i)^3}.$$

(d) $R_i = \sqrt{x^2 + y^2 + z_i^2}$ $(i = 1, 2, 3, a, b, c, d, e)$, $z_1 = u_1(z - h)$, $z_2 = u_2(z - h)$, $z_3 = u_3(z - h)$, $z_a = u_1(z + h)$, $z_b = u_1 z + u_2 h$, $z_c = u_1 h + u_2 z$, $z_d = u_2(z + h)$, $z_e = u_3(z + h)$. If the buried depth $h = 0$, in other words, the load is applied at the surface.

BURIED ASYMMETRIC-LOADING SOLUTIONS

In the field of geotechnical engineering, continuous-wall foundations, retaining walls, spread or strip footing foundations, gravity dams, and embankment structures are frequently built for engineering safety. The external loads transferred from such structures to soils or rocks are complex and usually modeled as various loading types, such as finite line loads, uniform rectangular loads, linearly varying rectangular loads or any combination of these loads. Hence, elastic solutions for the displacements and stresses in the transversely isotropic media induced by these types of loads are needed.

   In this paper, we derive the asymmetric loading solutions from directly integrating the point load solutions. In the case of point load solutions, we define p_{d1i}–p_{d6i} in equations (1)–(3) and p_{s1i}–p_{s8i} in equations (4)–(9) as the elementary functions for the displacements and stresses, respectively. Then, the solutions for the displacements and stresses in a transversely isotropic half-space subjected to various loading types are directly integrated from the elementary functions of the point load solutions in a Cartesian co-ordinate system. The closed-form solutions for the displacements and stresses subjected to finite line loads, uniform rectangular loads, and linearly varying rectangular loads are presented below.

Finite line loads

   A transversely isotropic half-space subjected to a perfectly flexible line load over the length w at the buried depth of h, as demonstrated in Figure 3, is considered as follows. Taking an infinitesimal element $d\eta$ along the Y-axis, a line load could be divided into a finite number of elementary forces with $dP_j = \bar{P}_j\, d\eta$ $(j = x, y, z)$ $(\bar{P}_j$, forces per unit length). Replacing y by $(y - \eta)$ in the elementary functions p_{d1i}–p_{d6i} and p_{s1i}–p_{s8i}, and integrating η between the limits 0 and w[33] as follows:

$$\begin{bmatrix} U \\ \sigma \end{bmatrix}^1 = \int_0^w \begin{bmatrix} U \\ \sigma \end{bmatrix}^p d\eta \tag{12}$$

where $[U] = [u_x, u_y, u_z]^T$, $[\sigma] = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yz}, \tau_{zx}]^T$ (superscript T denotes that the transpose matrix), and the superscripts 1 and p express the displacement and stress components that are induced by a line load and a point load, respectively. By mathematics operations, the explicit solutions of the displacements and stresses in a half-space can be regrouped as the forms of equations (1)–(9). It means that the exact solutions of this case are the same as equations (1)–(9) except that the displacement elementary functions p_{d1i}–p_{d6i} and stress elementary functions p_{s1i}–p_{s8i} are replaced by the displacement integral functions L_{d1i}–L_{d6i} for u_x^1, u_y^1, u_z^1 and stress integral functions L_{s1i}–L_{s8i} for $\sigma_{xx}^1, \sigma_{yy}^1, \sigma_{zz}^1, \tau_{xy}^1, \tau_{yz}^1, \tau_{xz}^1$, respectively. Similarly, the solutions for various loading types given below also can be expressed as the forms of equations (1)–(9), except for the integral functions. Hence, only the displacement and stress integral functions will be presented in the following cases. For the case of finite line loads, the displacement and stress
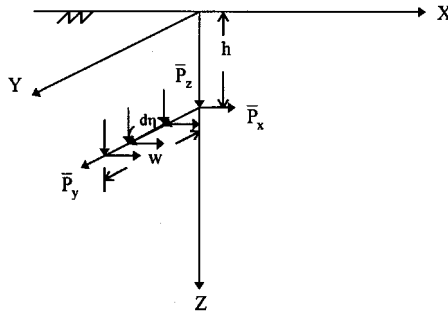
Figure 3. The case of finite line loads over the length w at the buried depth h

integral functions are given as follows:

$$L_{d1i} = -\frac{y}{R_i + z_i} + \frac{y^*}{R_{y^*i} + z_i} + \ln\left|\frac{R_i + y}{R_{y^*i} + y^*}\right| \tag{13}$$

$$L_{d2i} = \frac{y}{R_i + z_i} - \frac{y^*}{R_{y^*i} + z_i} \tag{14}$$

$$L_{d3i} = -\frac{x}{R_i + z_i} + \frac{x}{R_{y^*i} + z_i} \tag{15}$$

$$L_{d4i} = -\tan^{-1}\frac{x}{y} + \tan^{-1}\frac{x}{y^*} - \tan^{-1}\frac{yz_i}{xR_i} + \tan^{-1}\frac{y^*z_i}{xR_{y^*i}} \tag{16}$$

$$L_{d5i} = \ln\left|\frac{R_i + z_i}{R_{y^*i} + z_i}\right| \tag{17}$$

$$L_{d6i} = L_{d1i} + L_{d2i} \tag{18}$$

$$L_{s1i} = \frac{x}{x^2 + z_i^2}\left(\frac{y}{R_i} - \frac{y^*}{R_{y^*i}}\right) \tag{19}$$

$$L_{s2i} = -\frac{1}{R_i} + \frac{1}{R_{y^*i}} \tag{20}$$

$$L_{s3i} = \frac{z_i}{x}L_{s1i} \tag{21}$$

$$L_{s4i} = -x\left[\frac{1}{R_i(R_i + z_i)} - \frac{1}{R_{y^*i}(R_{y^*i} + z_i)}\right] \tag{22}$$

$$L_{s5i} = L_{s3i} - \frac{y}{R_i(R_i + z_i)} + \frac{y^*}{R_{y^*i}(R_{y^*i} + z_i)}\right] \tag{23}$$

$$L_{s6i} = L_{s3i} - L_{s5i} \tag{24}$$

$$L_{s7i} = x \left[\frac{y}{R_i(R_i + z_i)^2} - \frac{y^*}{R_{y*i}(R_{y*i} + z_i)^2} \right] \tag{25}$$

$$L_{s8i} = -\frac{1}{R_i + z_i} + \frac{1}{R_{y*i} + z_i} + \frac{x^2}{R_i(R_i + z_i)^2} - \frac{x^2}{R_{y*i}(R_{y*i} + z_i)^2} \tag{26}$$

where $y^* = y - w$, $R_{y*i} = \sqrt{x^2 + y^{*2} + z_i^2}$.

The presented formulations of the displacements and stresses are in agreement with Urena et al.[6] and Lin et al.,[7] when the loads applied at the surface ($h = 0$) and in the state of plane strain. If the half-space is isotropic, the closed-form solutions are similar to several isotropic solutions in literature.[34−36]

Uniform rectangular loads

A uniform load, $\hat{P}_j$ ($j = x, y, z$) (forces per unit area) distributed on a rectangle with length l and width w at the buried depth of h as shown in Figure 4 is considered. For solving the displacements and stresses in the half-space induced by this load, an elementary force $\hat{P}_j \, d\eta \, d\varsigma$ acting on an elementary surface $d\eta \, d\zeta$ is extracted from the rectangle. Replacing the concentrated force P_j by $\hat{P}_j \, d\zeta \, d\eta$, y by $(y - \eta)$ and x by $(x - \zeta)$ in equations (1)–(9), the solutions of the displacements and stresses for the elementary force acting in the half-space are obtained. Then, the complete solutions can be obtained by integrating the solutions induced by the elementary force with η from 0 to w and ζ from 0 to l,[33] respectively, as follows:

$$\begin{bmatrix} U \\ \sigma \end{bmatrix}^r = \int_0^l \int_0^w \begin{bmatrix} U \\ \sigma \end{bmatrix}^p d\eta \, d\varsigma \tag{27}$$

where the superscript r denotes the displacement and stress components that are induced by a uniform rectangular load. The displacement integral functions r_{d1i}–r_{d6i} for u_x^r, u_y^r, u_z^r and stress
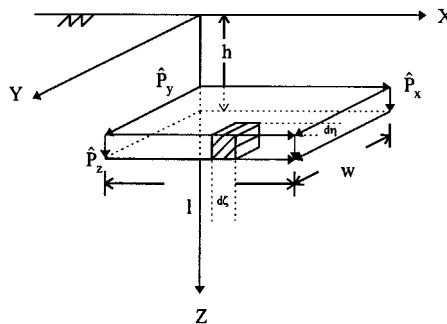

Figure 4. The case of uniform rectangular loads with $l*w$ area at the buried depth h

integral functions $r_{s1i} - r_{s8i}$ for $\sigma_{xx}^r, \sigma_{yy}^r, \sigma_{zz}^r, \tau_{xy}^r, \tau_{yz}^r, \tau_{xz}^r$ are derived and listed in the following:

$$r_{d1i} = -x \ln \left| \frac{R_{y^*i} + y^*}{R_i + y} \right| + x^* \ln \left| \frac{R_{x^*y^*i} + y^*}{R_{x^*i} + y} \right|$$

$$+ z_i \left[\tan^{-1} \frac{x^2 + z_i(R_i + z_i)}{xy} - \tan^{-1} \frac{x^{*2} + z_i(R_{x^*i} + z_i)}{x^*y} \right.$$

$$\left. - \tan^{-1} \frac{x^2 + z_i(R_{y^*i} + z_i)}{xy^*} + \tan^{-1} \frac{x^{*2} + z_i(R_{x^*y^*i} + z_i)}{x^*y^*} \right] \qquad (28)$$

$$r_{d2i} = -y \ln \left| \frac{R_{x^*i} + x^*}{R_i + x} \right| + y^* \ln \left| \frac{R_{x^*y^*i} + x^*}{R_{y^*i} + x} \right|$$

$$+ z_i \left[\tan^{-1} \frac{y^2 + z_i(R_i + z_i)}{xy} - \tan^{-1} \frac{y^{*2} + z_i(R_{y^*i} + z_i)}{xy^*} \right.$$

$$\left. - \tan^{-1} \frac{y^2 + z_i(R_{x^*i} + z_i)}{x^*y} + \tan^{-1} \frac{y^{*2} + z_i(R_{x^*y^*i} + z_i)}{x^*y^*} \right] \qquad (29)$$

$$r_{d3i} = -z_i \left(\ln \left| \frac{R_{x^*i} + z_i}{R_i + z_i} \right| - \ln \left| \frac{R_{x^*y^*i} + z_i}{R_{y^*i} + z_i} \right| \right) - R_i + R_{x^*i} + R_{y^*i} - R_{x^*y^*i} \qquad (30)$$

$$r_{d4i} = -x \left[\tan^{-1} \frac{x^2 + z_i(R_i + z_i)}{xy} - \tan^{-1} \frac{x^2 + z_i(R_{y^*i} + z_i)}{xy^*} \right]$$

$$+ x^* \left[\tan^{-1} \frac{x^{*2} + z_i(R_{x^*i} + z_i)}{x^*y} - \tan^{-1} \frac{x^{*2} + z_i(R_{x^*y^*i} + z_i)}{x^*y^*} \right]$$

$$- y \ln \left| \frac{R_{x^*i} + z_i}{R_i + z_i} \right| + y^* \ln \left| \frac{R_{x^*y^*i} + z_i}{R_{y^*i} + z_i} \right| - z_i \left(\ln \left| \frac{R_{y^*i} + y^*}{R_i + y} \right| - \ln \left| \frac{R_{x^*y^*i} + y^*}{R_{x^*i} + y} \right| \right) \qquad (31)$$

$$r_{d5i} = -x \ln \left| \frac{R_{y^*i} + z_i}{R_i + z_i} \right| + x^* \ln \left| \frac{R_{x^*y^*i} + z_i}{R_{x^*i} + z_i} \right| - z_i \left(\ln \left| \frac{R_{x^*i} + x^*}{R_i + x} \right| - \ln \left| \frac{R_{x^*y^*i} + x^*}{R_{y^*i} + x} \right| \right)$$

$$- y \left[\tan^{-1} \frac{y^2 + z_i(R_i + z_i)}{xy} - \tan^{-1} \frac{y^2 + z_i(R_{x^*i} + z_i)}{x^*y} \right]$$

$$+ y^* \left[\tan^{-1} \frac{y^{*2} + z_i(R_{y^*i} + z_i)}{xy^*} - \tan^{-1} \frac{y^{*2} + z_i(R_{x^*y^*i} + z_i)}{x^*y^*} \right] \qquad (32)$$

$$r_{d6i} = r_{d1i} + r_{d2i} \qquad (33)$$

$$r_{s1i} = \ln \left| \frac{R_{y^*i} + y^*}{R_i + y} \right| - \ln \left| \frac{R_{x^*y^*i} + y^*}{R_{x^*i} + y} \right| \qquad (34)$$

$$r_{s2i} = \ln \left| \frac{R_{x^*i} + x^*}{R_i + x} \right| - \ln \left| \frac{R_{x^*y^*i} + x^*}{R_{y^*i} + x} \right| \qquad (35)$$

$$r_{s3i} = \tan^{-1}\frac{xy}{z_iR_i} - \tan^{-1}\frac{x^*y}{z_iR_{x^*i}} - \tan^{-1}\frac{xy^*}{z_iR_{y^*i}} + \tan^{-1}\frac{x^*y^*}{z_iR_{x^*y^*i}} \tag{36}$$

$$r_{s4i} = \ln\left|\frac{R_{y^*i} + z_i}{R_i + z_i}\right| - \ln\left|\frac{R_{x^*y^*i} + z_i}{R_{x^*i} + z_i}\right| \tag{37}$$

$$r_{s5i} = -\tan^{-1}\frac{x^2 + z_i(R_i + z_i)}{xy} + \tan^{-1}\frac{x^{*2} + z_i(R_{x^*i} + z_i)}{x^*y}$$

$$+ \tan^{-1}\frac{x^2 + z_i(R_{y^*i} + z_i)}{xy^*} - \tan^{-1}\frac{x^{*2} + z_i(R_{x^*y^*i} + z_i)}{x^*y^*} \tag{38}$$

$$r_{s6i} = r_{s3i} - r_{s5i} \tag{39}$$

$$r_{s7i} = -y\left(\frac{1}{R_i + z_i} - \frac{1}{R_{x^*i} + z_i}\right) + y^*\left(\frac{1}{R_{y^*i} + z_i} - \frac{1}{R_{x^*y^*i} + z_i}\right) \tag{40}$$

$$r_{s8i} = -x\left(\frac{1}{R_i + z_i} - \frac{1}{R_{y^*i} + z_i}\right) + x^*\left(\frac{1}{R_{x^*i} + z_i} - \frac{1}{R_{x^*y^*i} + z_i}\right) \tag{41}$$

where $x^* = x - l$, $R_{x^*i} = \sqrt{x^{*2} + y^2 + z_i^2}$, $R_{x^*y^*i} = \sqrt{x^{*2} + y^{*2} + z_i^2}$.

Comparing with the transversely isotropic solutions of Lin *et al.*,[7] and various isotropic solutions,[37-49] it can be found that the closed-form solutions are more general than those in existing literature.

Linearly varying rectangular loads

For the case of subjected loads with non-uniform distributions, we use a non-uniform load with triangular distribution on a rectangle (Figure 5) to present the results. Figure 5 depicts that the load is linearly varied in the X-direction on a rectangle with sides l and w. The elementary force acting on a small rectangle can be expressed as $\tilde{P}_j\zeta\,\mathrm{d}\zeta\,\mathrm{d}\eta/l$ ($j = x, y, z$) ($\tilde{P}_j$ are the maximum forces per unit area). Similarly, by the same way as the case of a uniform rectangular load, the solutions of the displacements and stresses for this case can be obtained by directly integrating[33] as follows:

$$\begin{bmatrix} U \\ \sigma \end{bmatrix}^t = \int_0^1\int_0^w\begin{bmatrix} U \\ \sigma \end{bmatrix}^p \frac{\varsigma}{l}\,\mathrm{d}\eta\,\mathrm{d}\varsigma \tag{42}$$

where the superscript t expresses the displacement and stress components that are induced by a linearly varying rectangular load.
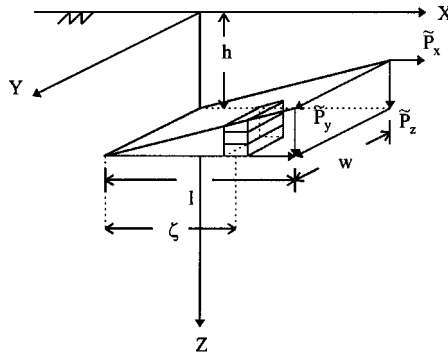
Figure 5. The case of linearly varying rectangular loads

By overcoming several integrating techniques, the displacement and stress integral functions, $t_{d1i} - t_{d6i}$ for u_x^t, u_y^t, u_z^t and $t_{s1i}-t_{s8i}$ for σ_{xx}^t, σ_{yy}^t, σ_{zz}^t, τ_{xy}^t, τ_{yz}^t, τ_{xz}^t are presented as

$$t_{d1i} = \left[xr_{d1i} + \frac{(x^2 + z_i^2)}{2}\ln\left|\frac{R_{y*i} + y^*}{R_i + y}\right| - \frac{(x^{*2} + z_i^2)}{2}\ln\left|\frac{R_{x*y*i} + y^*}{R_{x*i} + y}\right| \right.$$

$$\left. + \frac{y}{2}(R_i - R_{x*i}) - \frac{y^*}{2}(R_{y*i} - R_{x*y*i}) + z_i\left(y\ln\left|\frac{R_{x*i} + z_i}{R_i + z_i}\right| - y^*\ln\left|\frac{R_{x*y*i} + z_i}{R_{y*i} + z_i}\right|\right)\right]\bigg/ l$$

$$(43)$$

$$t_{d2i} = \left[xr_{d2i} - y(R_i - R_{x*i}) + y^*(R_{y*i} - R_{x*y*i}) - z_i\left(y\ln\left|\frac{R_{x*i} + z_i}{R_i + z_i}\right| - y^*\ln\left|\frac{R_{x*y*i} + z_i}{R_{y*i} + z_i}\right|\right)\right]\bigg/ l$$

$$(44)$$

$$t_{d3i} = \left\{ xr_{d3i} + \frac{x}{2}(R_i - R_{y*i}) - \frac{x^*}{2}(R_{x*i} - R_{x*y*i})\right.$$

$$+ \frac{(y^2 - z_i^2)}{2}\ln\left|\frac{R_{x*i} + x^*}{R_i + x}\right| - \frac{(y^{*2} - z_i^2)}{2}\ln\left|\frac{R_{x*y*i} + x^*}{R_{y*i} + x}\right|$$

$$- yz_i\left[\tan^{-1}\frac{y^2 + z_i(R_i + z_i)}{xy} - \tan^{-1}\frac{y^2 + z_i(R_{x*i} + z_i)}{x^*y}\right]$$

$$\left. + y^*z_i\left[\tan^{-1}\frac{y^{*2} + z_i(R_{y*i} + z_i)}{xy^*} - \tan^{-1}\frac{y^{*2} + z_i(R_{x*y*i} + z_i)}{x^*y^*}\right]\right\}\bigg/ l \qquad (45)$$

$$t_{d4i} = \left\{ xr_{d4i} - \frac{z_i^2}{2} r_{s3i} - \frac{lw}{2} - z_i \left(y \ln \left| \frac{R_{x^*i} + x^*}{R_i + x} \right| - y^* \ln \left| \frac{R_{x^*y^*i} + x^*}{R_{y^*i} + x} \right| \right) \right.$$

$$+ \frac{x^2}{2} \left[\tan^{-1} \frac{x^2 + z_i(R_i + z_i)}{xy} - \tan^{-1} \frac{x^2 + z_i(R_{y^*i} + z_i)}{xy^*} \right]$$

$$- \frac{x^{*2}}{2} \left[\tan^{-1} \frac{x^{*2} + z_i(R_{x^*i} + z_i)}{x^*y} - \tan^{-1} \frac{x^{*2} + z_i(R_{x^*y^*i} + z_i)}{x^*y^*} \right]$$

$$- \frac{y^2}{2} \left[\tan^{-1} \frac{y^2 + z_i(R_i + z_i)}{xy} - \tan^{-1} \frac{y^2 + z_i(R_{x^*i} + z_i)}{x^*y} \right]$$

$$\left. + \frac{y^{*2}}{2} \left[\tan^{-1} \frac{y^{*2} + z_i(R_{y^*i} + z_i)}{xy^*} - \tan^{-1} \frac{y^{*2} + z_i(R_{x^*y^*i} + z_i)}{x^*y^*} \right] \right\} \Big/ l \qquad (46)$$

$$t_{d5i} = \left[xr_{d5i} + \frac{x^2}{2} \left| \frac{R_{y^*i} + z_i}{R_i + z_i} \right| - \frac{x^{*2}}{2} \ln \left| \frac{R_{x^*y^*i} + z_i}{R_{x^*i} + z_i} \right| + \frac{y^2}{2} \ln \left| \frac{R_{x^*i} + z_i}{R_i + z_i} \right| - \frac{y^{*2}}{2} \ln \left| \frac{R_{x^*y^*i} + z_i}{R_{y^*i} + z_i} \right| \right.$$

$$\left. - \frac{z_i}{2} (R_i - R_{x^*i} - R_{y^*i} + R_{x^*y^*i}) \right] / l \qquad (47)$$

$$t_{d6i} = t_{d1i} + t_{d2i} \qquad (48)$$

$$t_{s1i} = (-r_{d2i} + xr_{s1i} + z_i r_{s5i})/l \qquad (49)$$

$$t_{s2i} = (-r_{d3i} + xr_{s2i} - z_i r_{s4i})/l \qquad (50)$$

$$t_{s3i} = (xr_{s3i} - z_i r_{s1i})/l \qquad (51)$$

$$t_{s4i} = \left(-r_{d5i} + xr_{s4i} - x \ln \left| \frac{R_{y^*i} + z_i}{R_i + z_i} \right| + x^* \ln \left| \frac{R_{x^*y^*i} + z_i}{R_{x^*i} + z_i} \right| \right) \Big/ l \qquad (52)$$

$$t_{s5i} = \left(xr_{s5i} - z_i r_{s1i} - y \ln \left| \frac{R_{x^*i} + z_i}{R_i + z_i} \right| + y^* \ln \left| \frac{R_{x^*y^*i} + z_i}{R_{y^*i} + z_i} \right| \right) \Big/ l \qquad (53)$$

$$t_{s6i} = t_{s3i} - t_{s5i} \qquad (54)$$

$$t_{s7i} = \left[-r_{d2i} + xr_{s7i} + y \left(\frac{x}{R_i + z_i} - \frac{x^*}{R_{x^*i} + z_i} \right) - y^* \left(\frac{x}{R_{y^*i} + z_i} - \frac{x^*}{R_{x^*y^*i} + z_i} \right) \right] \Big/ l \qquad (55)$$

$$t_{s8i} = \left[xr_{s8i} - z_i r_{s4i} - y^2 \left(\frac{1}{R_i + z_i} - \frac{1}{R_{x^*i} + z_i} \right) + y^{*2} \left(\frac{1}{R_{y^*i} + z_i} - \frac{1}{R_{x^*y^*i} + z_i} \right) \right] \Big/ l \qquad (56)$$

The transversely isotropic solutions of Piquer et al.[22] and isotropic solutions[42,50] indicate that the displacements or stresses in a half-space subjected to a linearly varying rectangular load are limited to plane problems. Only a few isotropic solutions[36,51,52] provided the displacements or stresses at some specific points in a finite plane. Hence, those solutions are the special cases of this study.

Another practical problem is a load with trapezoidal distribution, which models the load caused by the body force of an embankment. Using the principle of superposition, the displacements and stresses at any point below this transversely isotropic footing can be calculated.

ILLUSTRATIVE EXAMPLES

In this section, a series of parametric study is conducted to verify the solutions derived and investigate the effect of the loading types, degree of rock anisotropy on the displacements and stresses. Illustrative examples include a vertical uniform and a vertical non-uniform load with triangular distribution acting on a rectangle region, as depicted in Figures 6 and 7, respectively. Several types of isotropic and transversely isotropic rocks are considered to constitute the foundation materials. Their elastic properties are listed in Table II with E/E' and G/G' ranging between 1 and 3 and v/v' varying between 0.75–1.5. The values adopted in Table II of E and v are 50 GPa and 0.25, respectively. The chosen domains of variation are based on the suggestions of Gerrard[53] and Amadei et al.[54] The loads act on the horizontal surface ($h = 0$) of a transversely isotropic half-space for both examples. The degree of anisotropy including the ratio E/E', v/v' and G/G' is accounted for investigating its effect on the displacements and stresses.

A FORTRAN program based on equations (1)–(9) for various loading types was written to calculate the displacements and stresses. In this program, all the components of displacement and stress at any point in the half-space can be calculated. In this study, only the vertical surface displacement and vertical stress at/below the right corner of the loaded area was presented. Figures 6 and 7 show the results for both examples. The normalized vertical surface displacement $(u_z^r/l\hat{P}_z, u_z^t/l\tilde{P}_z)$ at the corner induced by a uniform rectangular load and a linearly varying rectangular load vs. the non-dimensional ratio of the loaded side (w/l) is given in Figures 6(a) and 7(a), respectively. Knowing the loading types and magnitudes, the dimensions of loaded area, and rock types, the vertical surface displacement at that point can be estimated from the figures. The others in Figures 6 and 7 show that the induced vertical stress below the point for different rock types and dimensions of the loaded area. The relation of two non-dimensional factors, l/z vs. $\sigma_{zz}^r/\hat{P}_z$ and l/z vs. $\sigma_{zz}^t/\tilde{P}_z$ is reported in Figures 6(b)–6(d) and Figures 7(b)–7(d), respectively. Also, the vertical stress at a given depth below the corner can be obtained from these figures. In these figures, the other non-dimensional factor w/z is adopted for investigating the influence of loading region on the vertical stress. The loads can be assumed as a strip load (plane strain condition) when the ratio of w/z approaching to infinity. Based on the results reported in Figures 6 and 7, the effect of degree of rock anisotropy and the loading types on the displacements and stresses induced by surface loads is investigated below.

Figures 6(a) and 7(a) indicate that for a given shape, the vertical displacement increases with the increase of E/E' with $v/v' = G/G' = 1$ (Rocks 1–3), v/v' with $E/E' = G/G' = 1$ (Rocks 1, 4, and 5), and G/G' with $E/E' = v/v' = 1$ (Rocks 1, 6 and 7). Especially, the increases of the ratio of E/E' and G/G' do have a great influence on the vertical displacement. It reflects the fact that the vertical surface displacement increases with the increase of deformability in the direction parallel to the applied load. Figures 6(b) and 7(b) present the induced vertical stress for Rocks 1–3 with variable non-dimensional factors (m, n). The results indicate that for a given depth and loaded region, the magnitude of the vertical stress decreases with the increase of E/E' ($v/v' = G/G' = 1$). Figures 6(c) and 7(c) report the effect of v/v' with $E/E' = G/G' = 1$ (Rocks 1, 4, and 5) on the vertical stress. The figures show that the induced stress is little affected by the value of v/v'. Comparing with Figures 6(d) and 7(d), it can be seen that the non-dimensional vertical stress increases with the increase of
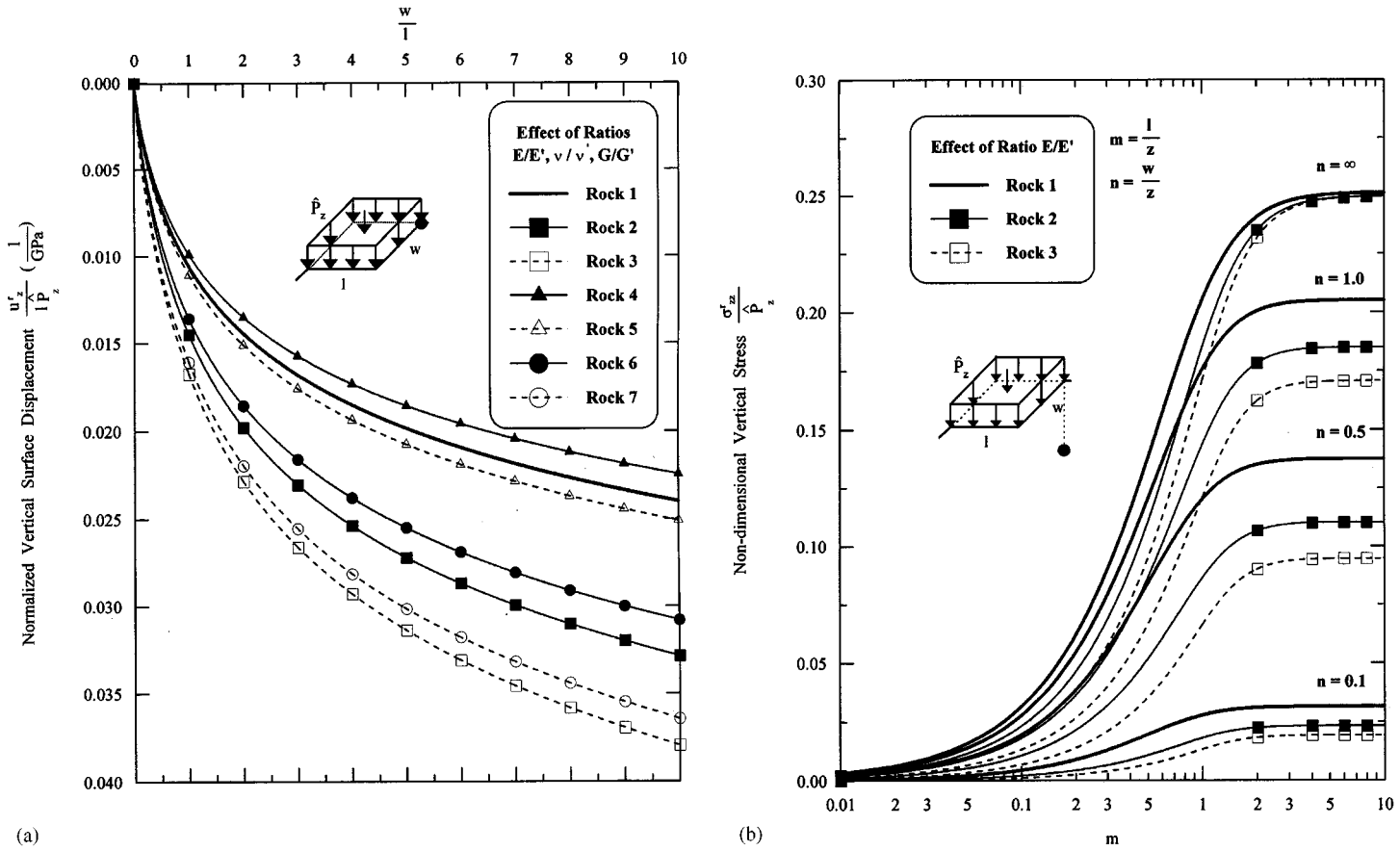
(a)

(b)

Figure 6. Effect of rock anisotropy on vertical surface displacement and vertical stress induced by a vertical uniform rectangular load $\hat{P}_z$: (a) vertical surface displacement for all rocks; (b) vertical stress for Rocks 1, 2, 3 with $E/E' = 1, 2, 3$, and $v/v' = G/G' = 1$, respectively; (c) vertical stress for Rocks 1, 4, 5 with $v/v' = 1$, 0.75,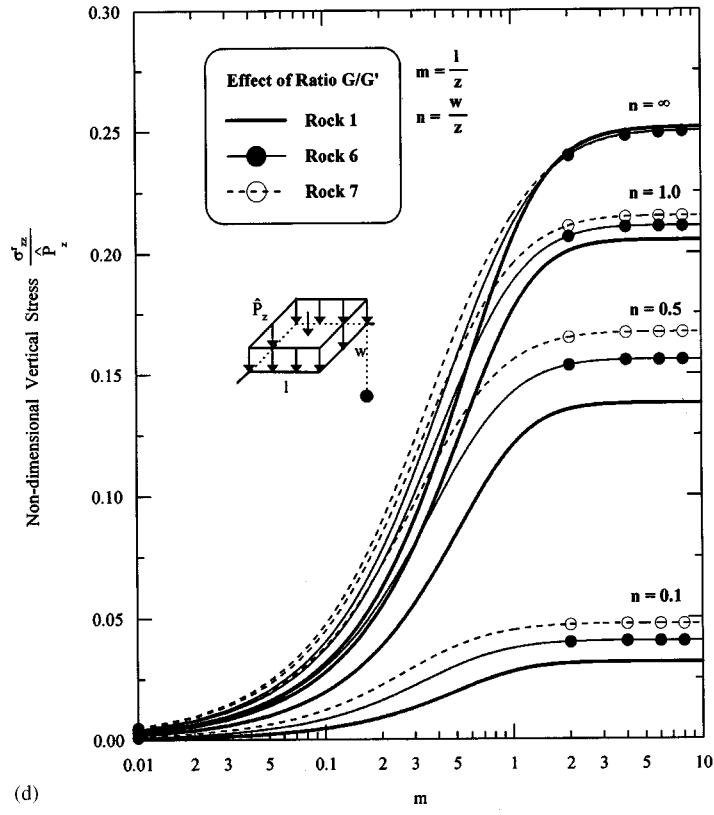 1.5, and $E/E' = G/G' = 1$, respectively; (d) vertical stress for Rocks 1, 6, 7 with $G/G' = 1, 2, 3$, and $E/E' = v/v' = 1$, respectively
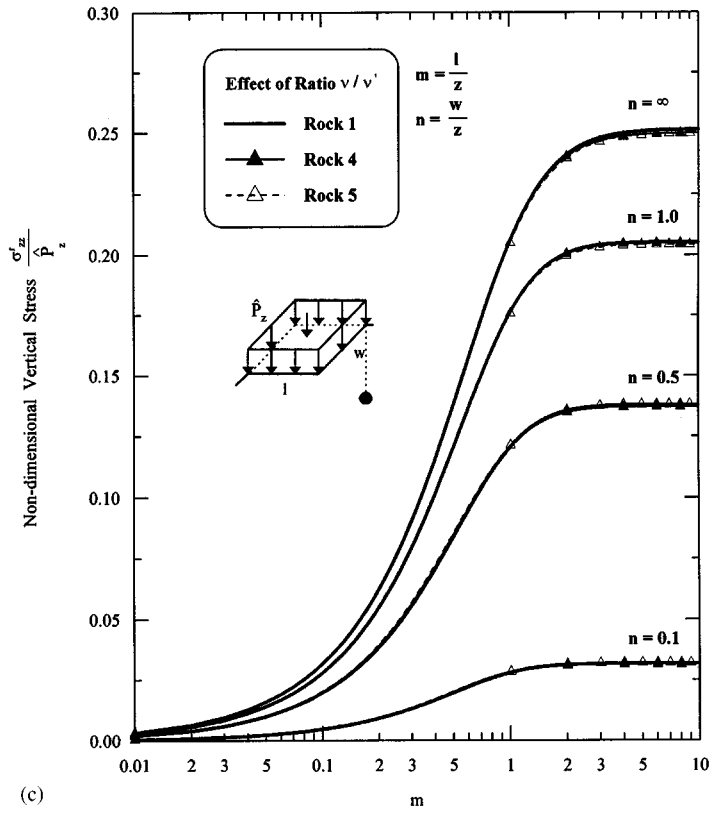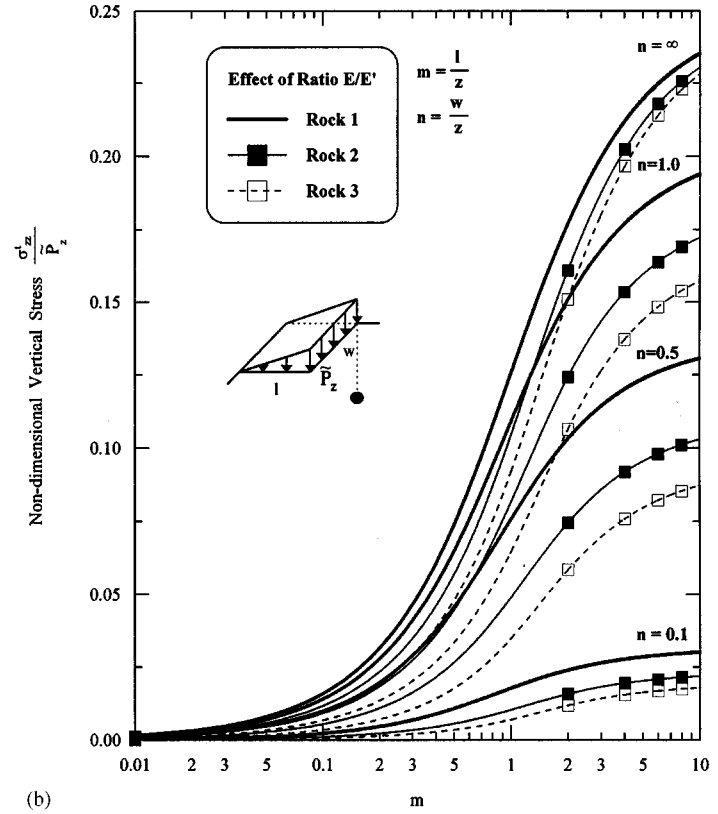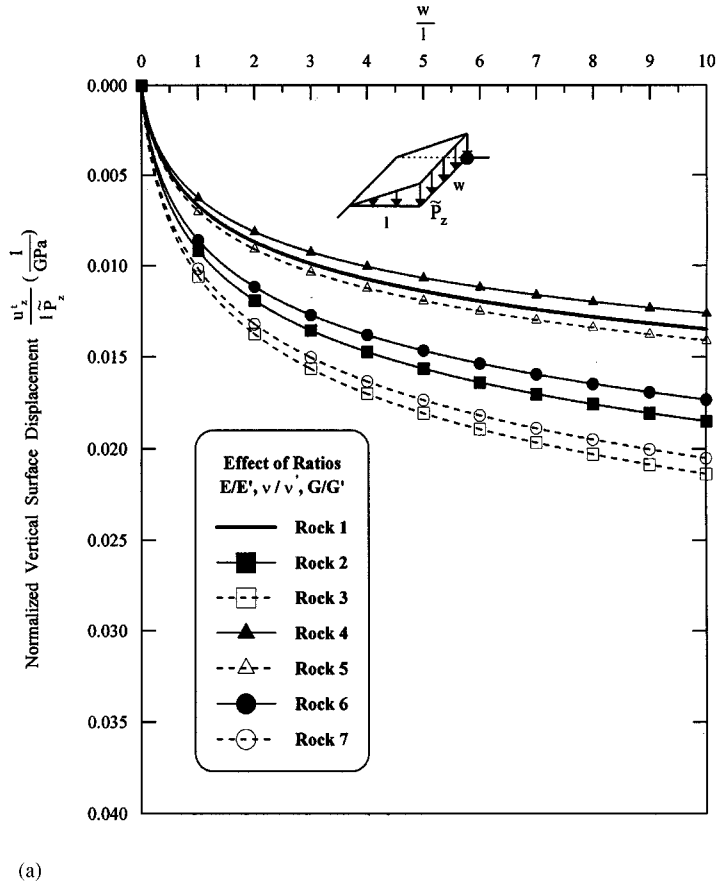
(c)

(d)

Figure 6. Continued

(a)

(b)

Figure 7. Effect of rock anisotropy on vertical surface displacement and vertical stress induced by a vertical linearly varying rectangular load $\tilde{P}_z$: (a) vertical surface displacement for all rocks; (b) vertical stress for Rocks 1, 2, 3 with $E/E' = 1, 2, 3$, and $v/v' = G/G' = 1$, respectively; (c) vertical stress for Rocks 1, 4, 5 with $v/v' = 1$, 0·75, 1·5, and $E/E' = G/G' = 1$, respectively; (d) vertical stress for Rocks 1, 6, 7 with $G/G' = 1, 2, 3$, and $E/E' = v/v' = 1$, respectively
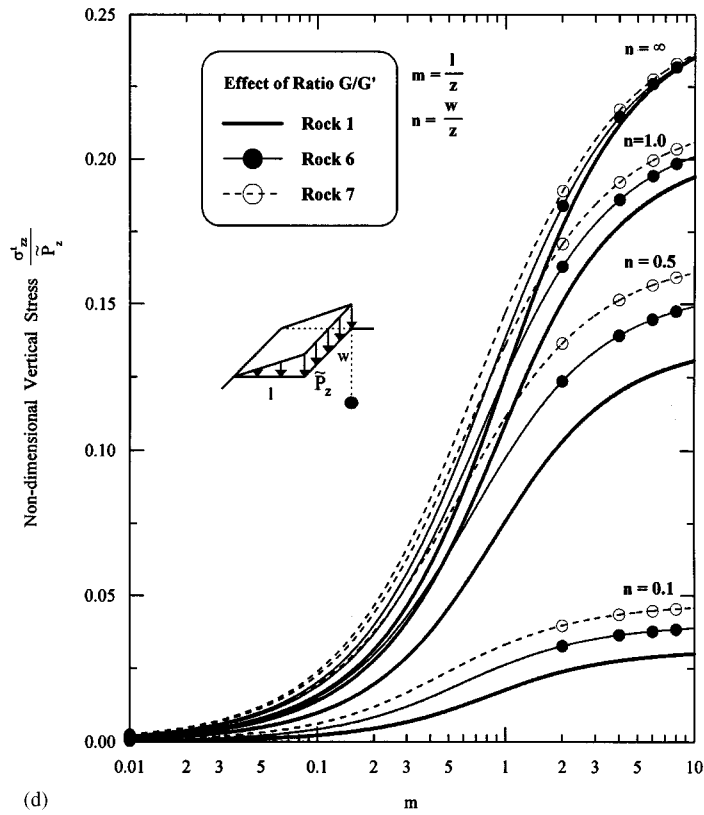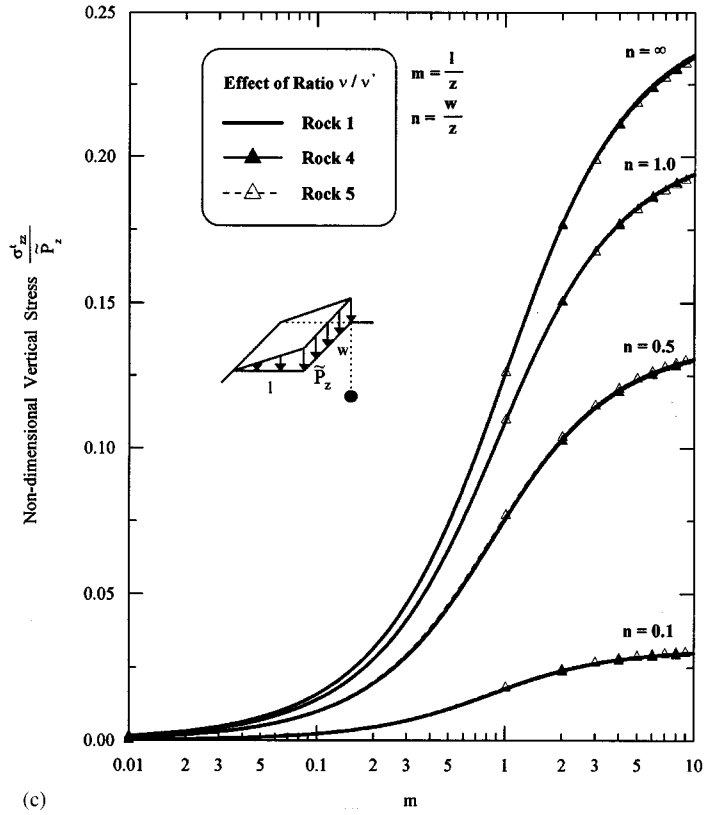
Figure 7. Continued

Table II. Elastic properties and root types for different rocks

Rock type	E/E'	v/v'	G/G'	Root type
Rock 1. Isotropic	1·0	1·0	1·0	Equal
Rock 2. Transversely isotropic	2·0	1·0	1·0	Complex
Rock 3. Transversely isotropic	3·0	1·0	1·0	Complex
Rock 4. Transversely isotropic	1·0	0·75	1·0	Complex
Rock 5. Transversely isotropic	1·0	1·5	1·0	Distinct
Rock 6. Transversely isotropic	1·0	1·0	2·0	Distinct
Rock 7. Transversely isotropic	1·0	1·0	3·0	Distinct

G/G' (Rocks 6 and 7). The results of this study indicate that the displacements and stresses induced by surface loads strongly depend on the type and degree of material anisotropy.

From Figures 6 and 7, the effect of the loading type on the displacement and stress is explicit. The trend of these figures for both loading types is similar. It also can be found that for a given load, the vertical surface displacement induced by a linearly varying rectangular load is only one-half of that induced by a uniform load (Figures 6(a) and 7(a)). However, comparing Figure 6(b)–6(d) with Figures 7(b)–7(d), the influence of the loading type on the vertical stress is not clear. In Figures 6(a) and 7(a), the vertical surface displacement increases with the increase of the ratio w/l for all the constituted rocks. It implicates that the displacement calculated from plane strain solution is larger than that obtained from three-dimensional solution. The similar results can also be found in Figures 6(b)–(d) and Figures 7(b)–(d) for the induced vertical stress.

Employing the two examples, the results show that the displacement and stress in the medium subjected to various loading types (on the surface or in the interior) are easy and correct to be calculated by the presented solutions. Also, the results indicate that the displacements and stresses are affected by the effect of rock anisotropy. Hence, the traditional isotropic or plane strain solutions are not suitable for estimating the displacements and stresses in a transversely isotropic medium subjected to a finite load.

CONCLUSIONS AND DISCUSSIONS

Using the Fourier and Hankel transforms, the point load solutions in a Cartesian co-ordinate system for the displacements and stresses in a transversely isotropic half-space were rederived and expressed in terms of several elementary functions. Integrating of these elementary functions, the point load solutions can be extended to derive the solutions of the displacements and stresses in a transversely isotropic half-space subjected to various buried loading types. The loading types include finite line loads, uniform rectangular loads, and linearly varying rectangular loads. These solutions indicate that the displacements and stresses are influenced by several factors. Factors include the buried depth, the loading types and the degree and type of material anisotropy. These closed-form solutions are the same as some isotropic solutions when the medium is isotropic, and are also in agreement with a few anisotropic solutions when loads applied at the surface or plane strain conditions assumed.

Based on the results of parametric studies, the following conclusions are made: (1) The vertical surface displacement under a surface load increases with the increase of deformability in the direction parallel to the applied load. (2) The vertical stress for transversely isotropic rocks

subjected to a uniform or linearly varying rectangular load decreases with the increase of E/E' ($v/v' = G/G' = 1$), and increases with the increase of G/G' ($E/E' = v/v' = 1$), but is little affected by the value of v/v' ($E/E' = G/G' = 1$). (3) The displacements and stresses calculated from plane strain solutions are larger than that obtained from these three-dimensional solutions.

In engineering practices, an elastic half-space is usually subjected to an arbitrarily shaped load. The loaded area can be divided into many regularly shaped areas, such as triangles. However, no solutions of the displacement/stress due to such loaded areas have been proposed for a transversely isotropic medium. The point load solutions presented in this paper can also be extended to solve the displacements and stresses for three-dimensional uniform, linear, or quadratic pressures acting on a triangular region in the interior of a transversely isotropic half-space. The results will be presented in the forthcoming papers.

APPENDIX

Notation

$A_{ij}(i, j = 1–6)$	elastic moduli or elasticity constants
$d\eta$, $d\zeta$	infinitesimal element along Y- or X-axis, respectively
E, E', v, v', G'	elastic constants of a transversely isotropic rock
h	the buried depth, as seen in Figures 1–5
i	complex number ($= \sqrt{-1}$)
$k, m_1, m_2, T_1, T_2, T_3, T_4$	coefficients
l, w	length along X-axis and width along Y-axis, respectively
$L_{d1i}–L_{d6i}, L_{s1i}–L_{s8i}$	integral functions for the displacements and stresses induced by finite line loads, respectively
$p_{d1i}–p_{d6i}, p_{s1i}–p_{s8i}$	elementary functions for the displacements and stresses induced by a point load, respectively
$P_j(j = x, y, z)$	a point load (force)
$\bar{P}_j(j = x, y, z)$	finite line loads (forces per unit length)
$\hat{P}_j(j = x, y, z)$	uniform rectangular loads (forces per unit area)
$\tilde{P}_j(j = x, y, z)$	linearly varying rectangular loads (maximum forces per unit area)
q, s	coefficients (see equation (11))
r, θ, z	a cylindrical co-ordinate system
$r_{d1i}–r_{d6i}, r_{s1i}–r_{s8i}$	integral functions for the displacements and stresses induced by uniform rectangular loads, respectively
$t_{d1i}–t_{d6i}, t_{s1i}–t_{s8i}$	integral functions for the displacements and stresses induced by linearly varying rectangular loads, respectively
u_1, u_2, u_3	roots of the characteristic equation
u_x^l, u_y^l, u_z^l	displacements induced by finite line roads
u_x^p, u_y^p, u_z^p	displacements induced by a point load
u_x^r, u_y^r, u_z^r	displacements induced by uniform rectangular loads

u_x^t, u_y^t, u_z^t	displacements induced by linearly varying rectangular loads
U	displacement components
X, Y, Z	a Cartesian co-ordinate system

Greek letters

σ	stress components
$\sigma_{xx}^1, \sigma_{yy}^1, \sigma_{zz}^1, \tau_{xy}^1, \tau_{yz}^1, \tau_{xz}^1$	stress induced by finite line loads
$\sigma_{xx}^p, \sigma_{yy}^p, \sigma_{zz}^p, \tau_{xy}^p, \tau_{yz}^p, \tau_{xz}^p$	stress induced by a point load
$\sigma_{xx}^r, \sigma_{yy}^r, \sigma_{zz}^r, \tau_{xy}^r, \tau_{yz}^r, \tau_{xz}^r$	stress induced by uniform rectangular loads
$\sigma_{xx}^t, \sigma_{yy}^t, \sigma_{zz}^t, \tau_{xy}^t, \tau_{yz}^t, \tau_{xz}^t$	stress induced by linearly varying rectangular loads

Superscripts

l	displacements and stresses induced by finite line loads
p	displacements and stresses induced by a point load
r	displacements and stresses induced by uniform rectangular loads
t	displacements and stresses induced by linearly varying rectangular loads
T	transpose matrix

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