*Appl. Statist.* (1993)
**42**, *No.* 1, *pp.* 55–62

# Some Process Capability Indices are More Reliable than One Might Think

By SAMUEL KOTZ†

*University of Maryland, College Park, USA*

WEN LEA PEARN

*National Chiao Tung University, Hsin Chu, Republic of China*

and N. L. JOHNSON

*University of North Carolina, Chapel Hill, USA*

[Received January 1991. Revised August 1991]

SUMMARY

In this paper we obtain formulae for exact expected values and standard deviations of estimators of certain process capability indices discussed by Bissell. In particular, we show that for the index $C_{pk}$ Bissell's formula gives values for the standard deviation which are too high especially when the actual population mean value is close to (or equal to) the average of the upper and lower specification limits.

*Keywords*: Capability indices; Non-central $\chi^2$-distribution; Standard error; Stirling approximation

## 1. Introduction

Process capability indices (PCIs) (whose purpose is to provide a numerical statement of the extent to which the output of a process satisfies a preassigned specification) have received substantial attention in statistical and quality control publications in recent years. Most prominently, Kane (1986) provides a thorough discussion and lucid comparison of five basic capability indices ($C_\text{p}$, CPU, CPL, $k$ and $C_{\text{p}k}$) which were developed in British and European, American and Japanese quality control branches of industrial and engineering institutions with special attention to the Japanese CPU and $C_\text{p}$ indices popularized by Sullivan (1984). Recently, Chan *et al.* (1988) proposed and investigated some distributional properties of a new measure of process capability $C_{\text{pm}}$ to take into account the proximity to target as well as the process variation in the assessment of process performance based on a Bayesian approach. Porter and Oakland (1990) advocate the construction of confidence intervals for the basic indices $C_\text{p}$ and $C_{\text{p}k}$. Most of the investigations depend heavily on the underlying assumption of normal variability, although an attempt to extend the results available for non-normal distributions using the Pearson family of probability curves has recently been

†*Address for correspondence*: College of Business and Management, University of Maryland, College Park, MD 20742, USA.



0035–9254/93/42055 $2.00

made by Clements (1989). Finally Bissell (1990) obtained simple but efficient approximate formulae for the variances of several PCIs. Among the PCIs considered by Bissell are

$$C_{\mathrm{p}} = \frac{\mathrm{USL} - \mathrm{LSL}}{6\sigma} \tag{1}$$

and

$$C_{\mathrm{p}k} = \min\left(\frac{\mathrm{USL} - \mu}{3\sigma}, \frac{\mu - \mathrm{LSL}}{3\sigma}\right), \tag{2}$$

where USL and LSL denote the upper and lower specification limits respectively, $\mu$ denotes the process mean and $\sigma$ the process standard deviation. Estimators of these PCIs can be obtained by replacing $\mu$ and $\sigma$ by the estimators $\hat{\mu}$ and $\hat{\sigma}$ respectively.

## 2. Distribution

On the basis of a normal distribution of measured characteristics. Bissell considered two types of estimator of $\sigma$:

(a) the sample standard deviation $S = \{(n-1)^{-1}\Sigma(X_i - \bar{X})^2\}^{1/2}$ where $\bar{X} = n^{-1}\Sigma_{i=1}^{n} X_i$ and

(b) the range (or mean range of subsamples) multiplied by an appropriate unbiasing factor.

In case (a) $S^2$ is distributed as $(n-1)^{-1}\sigma^2$ times a $\chi^2$-variable with $n-1$ degrees of freedom—symbolically, $S^2 \sim (n-1)^{-1}\chi^2_{n-1}\sigma^2$; in case (b) $\hat{\sigma}^2$ is distributed approximately as $f^{-1}\chi^2_f\sigma^2$ where $f$ is an appropriate constant depending on $n$.

The process mean $\mu$ is estimated by $\bar{X}$. On the assumption of normality, $\bar{X}$ and $S^2$, or $\hat{\sigma}^2$, are mutually independent, even if based (as is usual) on the same sample.

The numerator of $C_{\mathrm{p}k}$ can be written as $d - |\mu - \mu_0|$ where $d = \frac{1}{2}(\mathrm{USL} - \mathrm{LSL})$ and $\mu_0 = \frac{1}{2}(\mathrm{USL} + \mathrm{LSL})$. Hence we consider the estimator

$$\hat{C}_{\mathrm{p}k} = \frac{d - |\bar{X} - \mu_0|}{3\hat{\sigma}} = \left(\frac{d}{\sigma} - \frac{1}{\sqrt{n}}\frac{|\bar{X} - \mu_0|\sqrt{n}}{\sigma}\right) \Big/ \frac{3\chi_f}{\sqrt{f}}. \tag{3}$$

On the assumption of a normal distribution, $\chi_f$ and $|\bar{X} - \mu_0|\sqrt{n}/\sigma$ are independently distributed. The statistic $|\bar{X} - \mu_0|\sqrt{n}/\sigma$ has a folded normal distribution as defined by Leone *et al.* (1961). From the results of this paper (see also Johnson and Kotz (1970)) we have

$$E\left(\frac{|\bar{X} - \mu_0|\sqrt{n}}{\sigma}\right) = \sqrt{\left(\frac{2}{\pi}\right)} \exp\left\{-\frac{n(\mu - \mu_0)^2}{2\sigma^2}\right\} + \frac{|\mu - \mu_0|\sqrt{n}}{\sigma}\left\{1 - 2\Phi\left(-\frac{|\mu - \mu_0|\sqrt{n}}{\sigma}\right)\right\}, \tag{4a}$$

where

$$\Phi(u) = \sqrt{(2\pi)^{-1}} \int_{-\infty}^{u} \exp(-\tfrac{1}{2}t^2)\,\mathrm{d}t$$

and

$$E\left\{\left(\frac{\sqrt{n}}{\sigma}|\bar{X}-\mu_0|\right)^2\right\} = 1 + \frac{n(\mu-\mu_0)^2}{\sigma^2}. \tag{4b}$$

The distribution of $\hat{C}_{pk}$ depends on the parameters $d/\sigma$ and $|\mu-\mu_0|\sqrt{n}$. The $r$th moment about 0 of $\hat{C}_{pk}$ is

$$E(\hat{C}^r_{pk}) = \left(\frac{\sqrt{f}}{3}\right)^r E(\chi_f^{-r}) \sum_{j=0}^{r} (-1)^j \binom{r}{j} \left(\frac{d}{\sigma}\right)^{r-j} \left(\frac{1}{\sqrt{n}}\right)^j E\left\{\left(\frac{|\bar{X}-\mu_0|\sqrt{n}}{\sigma}\right)^j\right\},$$

whence

$$E(\hat{C}_{pk}) = \frac{1}{3}\sqrt{\left(\frac{f}{2}\right)} - \left[\frac{d}{\sigma} - \sqrt{\left(\frac{2}{\pi n}\right)} \exp\left\{-\frac{n(\mu-\mu_0)^2}{2\sigma^2}\right\} - \frac{|\mu-\mu_0|}{\sigma}\left\{1-2\Phi\left(-\frac{|\mu-\mu_0|\sqrt{n}}{\sigma}\right)\right\}\right] \Gamma\left(\frac{f-1}{2}\right) \Big/ \Gamma\left(\frac{f}{2}\right) \tag{5a}$$

and

$$\operatorname{var}(\hat{C}_{pk}) = \frac{f}{9(f-2)}\left(\left(\frac{d}{\sigma}\right)^2 - 2\left(\frac{d}{\sigma}\right)\left[\sqrt{\left(\frac{2}{\pi n}\right)} \exp\left\{-\frac{n(\mu-\mu_0)^2}{2\sigma^2}\right\} + \frac{|\mu-\mu_0|}{\sigma}\left\{1-2\Phi\left(-\frac{|\mu-\mu_0|\sqrt{n}}{\sigma}\right) + \frac{(\mu-\mu_0)^2}{\sigma^2} + \frac{1}{n}\right\} - E(\hat{C}_{pk})^2\right]\right). \tag{5b}$$

Expressions (5a) and (5b) are equivalent to those obtained by Zhang *et al.* (1990) by using a different method, without obtaining the actual distribution of $\hat{C}_{pk}$. Dr Zhang has told us that his numerical calculations coincide with ours.

If we use

$$\hat{\sigma} = S = \left\{(n-1)^{-1} \sum_{j=1}^{n} (X_j - \bar{X})^2\right\}^{1/2}$$

then $f = n-1$. Some numerical values of $E(\hat{C}_{pk})$ and $\operatorname{var}(\hat{C}_{pk})$ are presented in Table 1. We urge the reader to examine the column corresponding to $\mu = \mu_0$ most carefully. Corresponding values of $C_{pk}$ are presented in Table 2.

$\hat{C}_{pk}$ is a biased estimator of $C_{pk}$. The bias arises from two sources:

(a) $$E(\hat{\sigma}^{-1}) = \sqrt{\left(\frac{f}{2}\right)} \Gamma\left(\frac{f-1}{2}\right)\sigma^{-1} \Big/ \Gamma\left(\frac{f}{2}\right) \neq \sigma^{-1}$$

(this bias is positive);

(b) $$E\left(\frac{|\bar{X}-\mu_0|\sqrt{n}}{\sigma}\right) > \frac{|\mu-\mu_0|\sqrt{n}}{\sigma}$$

(this leads to negative bias since $|\bar{X}-\mu_0|\sqrt{n}/\sigma$ has a negative sign in the numerator of $\hat{C}_{pk}$).

The resultant bias is positive for all cases shown in Table 1 for which $\mu \neq \mu_0$. When $\mu = \mu_0$ the bias is positive for $n = 10$ but becomes negative for larger $n$. (For $d/\sigma$ values

TABLE 1
*Moments of $\hat{C}_{pk}$*†

| $d/\sigma$ | Results for the following values of $(\mu-\mu_0)/\sigma$: 0.0 EV | 0.0 Var | 0.5 EV | 0.5 Var | 1.0 EV | 1.0 Var | 1.5 EV | 1.5 Var | 2.0 EV | 2.0 Var |
|---|---|---|---|---|---|---|---|---|---|---|
| $n=10$ | | | | | | | | | | |
| 2 | 0.637 | 0.035 | 0.542 | 0.034 | 0.365 | 0.024 | 0.182 | 0.017 | 0.000 | 0.014 |
| 3 | 1.002 | 0.079 | 0.906 | 0.073 | 0.729 | 0.054 | 0.547 | 0.036 | 0.365 | 0.024 |
| 4 | 1.367 | 0.143 | 1.271 | 0.131 | 1.094 | 0.103 | 0.912 | 0.076 | 0.729 | 0.054 |
| 5 | 1.732 | 0.226 | 1.636 | 0.209 | 1.459 | 0.171 | 1.277 | 0.135 | 1.094 | 0.103 |
| 6 | 2.096 | 0.329 | 2.001 | 0.307 | 1.824 | 0.260 | 1.641 | 0.213 | 1.459 | 0.171 |
| $n=20$ | | | | | | | | | | |
| 2 | 0.633 | 0.014 | 0.520 | 0.014 | 0.347 | 0.010 | 0.174 | 0.007 | 0.000 | 0.006 |
| 3 | 0.980 | 0.031 | 0.867 | 0.028 | 0.695 | 0.021 | 0.521 | 0.014 | 0.347 | 0.010 |
| 4 | 1.327 | 0.055 | 1.215 | 0.050 | 1.042 | 0.039 | 0.868 | 0.029 | 0.695 | 0.021 |
| 5 | 1.674 | 0.086 | 1.562 | 0.079 | 1.389 | 0.064 | 1.215 | 0.050 | 1.042 | 0.039 |
| 6 | 2.022 | 0.124 | 1.909 | 0.115 | 1.736 | 0.096 | 1.563 | 0.079 | 1.389 | 0.064 |
| $n=30$ | | | | | | | | | | |
| 2 | 0.635 | 0.009 | 0.513 | 0.009 | 0.342 | 0.006 | 0.171 | 0.005 | 0.000 | 0.004 |
| 3 | 0.977 | 0.019 | 0.856 | 0.018 | 0.685 | 0.013 | 0.513 | 0.009 | 0.342 | 0.006 |
| 4 | 1.319 | 0.034 | 1.198 | 0.031 | 1.027 | 0.024 | 0.856 | 0.018 | 0.685 | 0.013 |
| 5 | 1.662 | 0.053 | 1.540 | 0.048 | 1.369 | 0.039 | 1.198 | 0.031 | 1.027 | 0.024 |
| 6 | 2.004 | 0.076 | 1.882 | 0.070 | 1.711 | 0.059 | 1.540 | 0.048 | 1.369 | 0.039 |
| $n=40$ | | | | | | | | | | |
| 2 | 0.637 | 0.007 | 0.510 | 0.006 | 0.340 | 0.004 | 0.170 | 0.003 | 0.000 | 0.003 |
| 3 | 0.977 | 0.014 | 0.850 | 0.013 | 0.680 | 0.009 | 0.510 | 0.006 | 0.340 | 0.004 |
| 4 | 1.317 | 0.025 | 1.190 | 0.022 | 1.020 | 0.017 | 0.850 | 0.013 | 0.680 | 0.009 |
| 5 | 1.657 | 0.038 | 1.530 | 0.035 | 1.360 | 0.028 | 1.190 | 0.022 | 1.020 | 0.017 |
| 6 | 1.997 | 0.055 | 1.870 | 0.050 | 1.700 | 0.042 | 1.530 | 0.035 | 1.360 | 0.028 |
| $n=50$ | | | | | | | | | | |
| 2 | 0.639 | 0.005 | 0.508 | 0.005 | 0.339 | 0.004 | 0.169 | 0.003 | 0.000 | 0.002 |
| 3 | 0.977 | 0.011 | 0.846 | 0.010 | 0.677 | 0.007 | 0.508 | 0.005 | 0.339 | 0.004 |
| 4 | 1.316 | 0.019 | 1.185 | 0.017 | 1.016 | 0.013 | 0.846 | 0.010 | 0.677 | 0.007 |
| 5 | 1.655 | 0.030 | 1.523 | 0.027 | 1.354 | 0.022 | 1.185 | 0.017 | 1.016 | 0.013 |
| 6 | 1.993 | 0.043 | 1.862 | 0.039 | 1.693 | 0.033 | 1.523 | 0.027 | 1.354 | 0.022 |

†$f=n-1$; EV, expected value; Var, variance.

TABLE 2
*Values of $C_{pk}$*

| $d/\sigma$ | Results for the following values of $(\mu-\mu_0)/\sigma$: 0.0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|
| 2.0 | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{6}$ | 0 |
| 3.0 | 1 | $\frac{5}{6}$ | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{3}$ |
| 4.0 | $1\frac{1}{3}$ | $1\frac{1}{6}$ | 1 | $\frac{5}{6}$ | $\frac{2}{3}$ |
| 5.0 | $1\frac{2}{3}$ | $1\frac{1}{2}$ | $1\frac{1}{3}$ | $1\frac{1}{6}$ | 1 |
| 6.0 | 2 | $1\frac{5}{6}$ | $1\frac{2}{3}$ | $1\frac{1}{2}$ | $1\frac{1}{3}$ |

TABLE 3
*Values of $E(\hat{C}_{pk})$ for $\mu=\mu_0$ and $d/\sigma=3$ corresponding to $C_{pk}=1$ for a series of increasing values of n*

| *Sample size n* | *$E(C_{pk})$* | *Sample size n* | *$E(C_{pk})$* |
|---|---|---|---|
| 10 | 1.002 | 600 | 0.990 |
| 20 | 0.980 | 2200 | 0.995 |
| 30 | 0.977 | 3200 | 0.996 |
| 60 | 0.978 | 5400 | 0.997 |
| 80 | 0.980 | 10800 | 0.998 |
| 100 | 0.981 | 30500 | 0.999 |
| 200 | 0.985 | 79500 | 1.000 |
| 400 | 0.989 | | |

of 3.0 and 4.0 it is negative for all $n \geqslant 20$, for $d/\sigma = 5.0$ for $n \geqslant 30$, and for $d/\sigma = 6.0$ for $n \geqslant 40$.) Ultimately, as $n \to \infty$ the bias tends to 0.

This is explored in more detail in Table 3 which presents the values of $E(\hat{C}_{pk})$ for $(\mu - \mu_0)/\sigma = 0$ and $d/\sigma = 3$ (in this case the 'theoretical' value of $C_{pk}$ is 1). An explicit formula in this case for $E(\hat{C}_{pk})$ is easily seen to be

$$E(\hat{C}_{pk}) = \left\{1 - \frac{1}{3}\sqrt{\left(\frac{2}{\pi n}\right)}\right\}\sqrt{\left(\frac{n-1}{2}\right)}\,\Gamma\left(\frac{n-2}{2}\right)\Big/\Gamma\left(\frac{n-1}{2}\right). \tag{6}$$

For our calculations we have used this exact formula together with an approximate formula in which the ratio of the gamma functions was approximated via the Stirling formula by

$$\Gamma\left(\frac{n-2}{2}\right)\Big/\Gamma\left(\frac{n-1}{2}\right) \doteq 1\Big/\sqrt{\left(\frac{n-2}{2}\right)}\left\{1 - \frac{1}{4(n-2)} + \frac{1}{32(n-2)^2} + \frac{5}{128(n-2)^3}\right\}.$$

The values of $E(\hat{C}_{pk})$ calculated by using these two formulae coincide (up to the fourth decimal place) for the values of $n$ presented in Table 3, which indicates the high accuracy of the Stirling approximation used.

## 3. Comments on Bissell's Modification

Although Bissell (1990) defines $C_{pk}$ as in equation (2), in the later part of this paper he uses the estimator

$$\hat{C}'_{pk} = \frac{\text{USL} - \bar{X}}{3\hat{\sigma}} \tag{7a}$$

or

$$\hat{C}'_{pk} = \frac{\bar{X} - \text{LSL}}{3\hat{\sigma}} \tag{7b}$$

according to whether $\mu$ is greater than or less than $\frac{1}{2}(\text{USL} + \text{LSL}) = \mu_0$.

As Bissell notes, in either case the distribution of $3\hat{C}'_{pk}\sqrt{n}$ is non-central $t$ with $f$ degrees of freedom and non-centrality parameter $(d - |\mu - \mu_0|)\sqrt{n}$. We shall consider, without loss of generality, $\mu > \mu_0$. It is to be expected that $\hat{C}'_{pk}$ will have a greater variance than $\hat{C}_{pk}$ because, when $\bar{X} \geq \mu_0$, $\hat{C}'_{pk} = \hat{C}_{pk}$, however, when $\bar{X} < \mu_0$ the numerator of $\hat{C}'_{pk}$ is greater (by $2(\mu_0 - \bar{X})$) than that of $\hat{C}_{pk}$. For the same reason, we expect $E(\hat{C}'_{pk})$ to exceed $E(\hat{C}_{pk})$, leading to greater positive bias when $\mu \neq \mu_0$. However, these effects will not be large (except when $\mu$ differs little from $\mu_0$) because the probability

$$\text{prob}(\bar{X} < \mu_0) = \Phi\left(-\frac{|\mu - \mu_0|\sqrt{n}}{\sigma}\right)$$

is indeed quite small, except for small values of $n$.

The most noticeable effect will be the reduction in the variance when $\mu = \mu_0$ (which serves as a justification of the title of this paper).

The expected value of $\hat{C}'^{r}_{pk}$ is

$$E(\hat{C}'^{r}_{pk}) = \frac{f^{r/2}}{3^r} E\{(d - \bar{X} + \mu_0)^r\} E(\chi_f^{-f})\sigma^{-r}.$$

In particular

$$E(\hat{C}'_{pk}) = \frac{1}{3}\left(\frac{1}{2}f\right)^{1/2}\left(\frac{d}{\sigma} - \frac{\mu - \mu_0}{\sigma}\right)\Gamma\left(\frac{f-1}{2}\right)\Big/\Gamma\left(\frac{f}{2}\right) \tag{8a}$$

(compare equation (5a)) and

$$\text{var}(\hat{C}'_{pk}) = \frac{1}{9}\,\frac{f}{f-2}\left\{\left(\frac{d}{\sigma} - \frac{\mu - \mu_0}{\sigma}\right)^2 + \frac{1}{n}\right\} - E(\hat{C}'_{pk})^2. \tag{8b}$$

For $\mu = \mu_0$ we find that

$$E(\hat{C}'_{pk}) = E(\hat{C}_{pk})\left\{1 - \left(\frac{\sigma}{d}\right)\sqrt{\left(\frac{2}{\pi n}\right)}\right\}^{-1}$$

and

$$\text{var}(\hat{C}_{pk}) = \text{var}(\hat{C}'_{pk}) - \frac{f}{9}\,\frac{d}{\sigma}\left[\frac{2}{f-2} - \left\{\Gamma\left(\frac{f-1}{2}\right)\Big/\Gamma\left(\frac{f}{2}\right)\right\}^2\right]$$
$$- \frac{f}{9\pi n}\Gamma\left(\frac{f-1}{2}\right)\Big/\Gamma\left(\frac{f}{2}\right) < \text{var}(\hat{C}'_{pk}),$$

since

$$\left\{\Gamma\left(\frac{f-1}{2}\right)\Big/\Gamma\left(\frac{f}{2}\right)\right\}^2 < \frac{2}{f-2}.$$

(Note that $\Gamma^2(z) < \Gamma(z - \frac{1}{2})\,\Gamma(z + \frac{1}{2})$.)

Bissell (1990) obtains an approximate formula for $\text{var}(\hat{C}'_{pk})$ by using the method of statistical differentials. It is (in our notation)

$$\text{var}(\hat{C}'_{pk}) \doteqdot C^2_{pk}\left\{\frac{\text{var}(d - \bar{X} + \mu_0)}{(d - \mu + \mu_0)^2} + \frac{\text{var}(\hat{\sigma})}{\sigma^2}\right\} \doteqdot \frac{1}{9n} + \frac{C^2_{pk}}{2f}. \tag{8c}$$

(This formula does not allow for bias in $\hat{C}'_{pk}$ as an estimator of $C_{pk}$; however, the effect of this will be of a higher order in $n^{-1}$ or $f^{-1}$.)

When $f = n - 1$,

TABLE 4
*Moments of* $\hat{C}'_{pk}$†

| $d/\sigma$ | 0.0 EV | 0.0 Var | 0.5 EV | 0.5 Var | 1.0 EV | 1.0 Var | 1.5 EV | 1.5 Var | 2.0 EV | 2.0 Var |
|---|---|---|---|---|---|---|---|---|---|---|
| | *Results for the following values of* $(\mu-\mu_0)/\sigma$: | | | | | | | | | |
| *n = 10* | | | | | | | | | | |
| 3 | 1.094 | 0.103 | 0.912 | 0.076 | 0.729 | 0.054 | 0.547 | 0.036 | 0.365 | 0.024 |
| 4 | 1.459 | 0.171 | 1.277 | 0.135 | 1.094 | 0.103 | 0.912 | 0.076 | 0.729 | 0.054 |
| 5 | 1.824 | 0.260 | 1.641 | 0.213 | 1.459 | 0.171 | 1.277 | 0.135 | 1.094 | 0.103 |
| 6 | 2.188 | 0.368 | 2.006 | 0.311 | 1.824 | 0.260 | 1.641 | 0.213 | 1.459 | 0.171 |
| *n = 20* | | | | | | | | | | |
| 3 | 1.041 | 0.048 | 0.868 | 0.029 | 0.694 | 0.021 | 0.521 | 0.015 | 0.347 | 0.010 |
| 4 | 1.388 | 0.065 | 1.215 | 0.052 | 1.041 | 0.048 | 0.868 | 0.029 | 0.694 | 0.021 |
| 5 | 1.736 | 0.099 | 1.562 | 0.081 | 1.388 | 0.065 | 1.215 | 0.052 | 1.041 | 0.048 |
| 6 | 2.083 | 0.139 | 1.909 | 0.118 | 1.736 | 0.099 | 1.562 | 0.099 | 1.388 | 0.065 |
| *n = 30* | | | | | | | | | | |
| 3 | 1.027 | 0.024 | 0.855 | 0.019 | 0.685 | 0.013 | 0.513 | 0.009 | 0.342 | 0.006 |
| 4 | 1.369 | 0.039 | 1.198 | 0.031 | 1.027 | 0.024 | 0.855 | 0.018 | 0.685 | 0.013 |
| 5 | 1.711 | 0.059 | 1.540 | 0.048 | 1.369 | 0.039 | 1.198 | 0.031 | 1.027 | 0.024 |
| 6 | 2.054 | 0.083 | 1.883 | 0.070 | 1.711 | 0.059 | 1.540 | 0.048 | 1.369 | 0.039 |
| *n = 40* | | | | | | | | | | |
| 3 | 1.020 | 0.017 | 0.850 | 0.013 | 0.680 | 0.009 | 0.510 | 0.006 | 0.340 | 0.005 |
| 4 | 1.360 | 0.028 | 1.190 | 0.022 | 1.020 | 0.017 | 0.850 | 0.013 | 0.680 | 0.009 |
| 5 | 1.700 | 0.042 | 1.530 | 0.032 | 1.360 | 0.028 | 1.190 | 0.022 | 1.020 | 0.017 |
| 6 | 2.040 | 0.060 | 1.870 | 0.051 | 1.700 | 0.042 | 1.530 | 0.032 | 1.360 | 0.028 |
| *n = 50* | | | | | | | | | | |
| 3 | 1.016 | 0.013 | 0.846 | 0.010 | 0.677 | 0.007 | 0.508 | 0.005 | 0.339 | 0.004 |
| 4 | 1.354 | 0.022 | 1.185 | 0.017 | 1.016 | 0.013 | 0.846 | 0.010 | 0.677 | 0.007 |
| 5 | 1.693 | 0.033 | 1.523 | 0.027 | 1.354 | 0.022 | 1.185 | 0.017 | 1.016 | 0.013 |
| 6 | 2.031 | 0.046 | 1.862 | 0.039 | 1.693 | 0.033 | 1.523 | 0.027 | 1.354 | 0.022 |

†$f=n-1$; EV, expected value; Var, variance. It can be shown that the two moments depend only on $d/\sigma - (\mu-\mu_0)/\sigma$ and not on $d/\sigma$ and $(\mu-\mu_0)/\sigma$ separately. We could therefore just have a one-way table with the argument $d/\sigma - (\mu-\mu_0)/\sigma$ (=0, 0.5, 1.0, . . ., 6.0), but this would not be easily comparable with Table 1.

$$\mathrm{var}(\hat{C}'_{pk}) \doteqdot \frac{1}{9n} + \frac{C_{pk}^2}{2(n-1)}.$$

Table 4 gives values of $E(\hat{C}'_{pk})$ and $\mathrm{var}(\hat{C}'_{pk})$, calculated from formulae (8a) and (8b). The discrepancies between these values and the corresponding values in Table 1 are noticeable when $\mu=\mu_0$, but decrease rapidly as $(\mu-\mu_0)/\sigma$ increases. For Bissell's (1990) example (p. 337) the differences are negligible, on the basis of the assumption that the true process mean is $\bar{X}$ and the standard deviation is $S$. Approximation (8c) for $\mathrm{var}(\hat{C}'_{pk})$ gives values rather less than the exact values in Table 4.

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