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An efficient algorithm for finding the D-stability bound of discrete singularly perturbed systems with multiple time delays

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In this paper, we present an original work on the D-stabilization problem of discrete singularly perturbed systems with multiple time delays. A new robust D-stability criterion in terms of stability radius is first derived to guarantee that all poles of the discrete multiple time-delay systems remain inside the specific disk $D(\alpha, r)$ in the presence of parametric uncertainties. Then, by using the technique of time-scale separation, we derive the corresponding slow and fast subsystems of a discrete multiple time-delay singularly perturbed system. The state feedback controls for the D-stabilization of the slow and the fast subsystems are separately designed and a composite state feedback control for the original system is subsequently synthesized from these state feedback controls. Thereafter, we derive a frequency domain ε-dependent D-stability criterion for the original discrete multiple time-delay singularly perturbed system under the composite state feedback control. If any one of the conditions of this criterion is fulfilled, D-stability of the original closed-loop system is thus investigated by establishing that of its corresponding slow and fast closed-loop subsystems. Finally, an efficient algorithm is proposed to obtain a less conservative D-stability bound of the singular perturbation parameter and to reduce the computation time.

1. Introduction

In the analysis of dynamic systems, we are often faced with parametric uncertainties originating from various sources, e.g. identification errors, ageing of devices, variation of operating points, etc. Therefore, the problem of maintaining the stability of a nominally stable system subject to uncertainties has been of considerable interest to researchers and a number of significant results concerning this issue have been reported in the literature. On the other hand, the problem of pole assignment in linear systems theory has been discussed by many authors and solved in various ways. However, one cannot place the poles at a specific location, due to parametric uncertainties. Therefore, placing all poles in a specific region rather than choosing an exact assignment may be satisfactory in practical applications. One such specific region for discrete systems is a disk $D(\alpha, r)$ centred at $(\alpha, 0)$ with radius r, where $|\alpha| + r < 1$. The assignment of all poles of a system in the specific disk $D(\alpha, r)$ shown in figure 1 is referred to as a D-pole placement problem (Furuta and Kim 1987). This subject has received much attention in previous reports (Furuta and Kim 1987, Lee and Lee 1987, Kolla *et al.* 1989, Vicino 1989, Lee *et al.* 1992, Su and Shyr 1994).

The stability radius is a tool to describe the distance from instability. There are two distances from instability for a real square matrix—the complex stability radius and the real stability radius; they can differ considerably. In general, the real stability radius is more important in applications but is more difficult to determine (Hinrichsen and Pritchard 1986). Kharitonov (1991) has been concerned with the analysis of the complex stability radius of a matrix with respect to the unit disk of the complex plane.

The problem of stabilizing time-delay systems has been explored over the years because delay is commonly encountered in various engineering systems, such as chemical processes—e.g. steel smelting and refining—or in long transmission lines, in pneumatic, hydraulic or electric networks. Its existence may produce undesirable system responses. Consequently, researchers on stability analysis of time-delay systems become essential to practical applications. This question has been raised by Mori *et al.* (1982), Mori (1985), Mori and Kokame (1989), Oucheriah (1995), Wang and Wang (1995), Hsiao and Hwang (1996a), and others. Since the number of poles of the closed-loop system increases due to time delays, the introduction of a time-delay factor makes the D-pole placement problem much more complicated. The D-stability problem for discrete time-delay systems has been discussed by Lee *et al.* (1992) and Su and Shyr (1994). However, their results are too conservative. Furthermore, there exist multiple time delays in some physical systems and delays in practice are not exact integer multiples of the sampling interval. Thus, for the purpose of general application, two cases of a new robust D-stability criterion in terms of complex stability radius are proposed for discrete uncertain systems with multiple time delays which may not be exact integer

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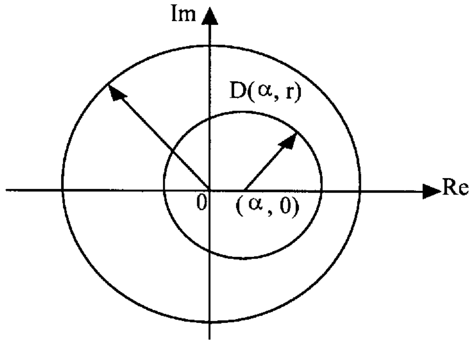

Figure 1. A disk $D(\alpha, r)$ centred at $(\alpha, 0)$ with radius r.

multiples of the sampling interval. One is a direct test (i.e., check $d_1 < d_s$) and the other is a boundary test. The robust D-stability is first checked by the direct test. If it fails, resort to the boundary test.

Singularly perturbed systems have been studied by many researchers (see, for example, Saksena *et al.* 1984, Kokotovic *et al.* 1986, Su and Hsieh 1990, Chen and Hsieh 1994, Chen *et al.* 1994, Venkatasubramanian 1994, Hsiao and Hwang 1996b). This is due not only to theoretical interest but also to the relevance of this topic to the control of engineering applications. The singular perturbation parameters often result from the presence of small parameters in various physical systems; e.g. in power system models the singular perturbation parameters can represent machine reactances or transients in voltage regulators. In industrial control systems they may represent time constants of drives and actuators, and in nuclear reactor models they are due to fast neutrons, etc. (Kokotovic *et al.* 1976). Indeed, the singular perturbation approach has been proven to be a powerful tool for system analysis and control design (Corless and Glielmo 1991). A fundamental feature of such an approach is that the feedback design problem can be broken into two design subproblems for the slow and fast subsystems. The two designs are then combined to give a design for the original systems (Khalil 1989).

A key to the analysis of singularly perturbed systems thus lies in the construction of the slow and fast subsystems. It is noted that the approximation of an original, singularly perturbed system via its corresponding slow and fast subsystems is valid only when the singular perturbation parameters of this system are sufficiently small. Therefore, it is important to find the stability bound of singular perturbation parameters such that the stability of the original system can be investigated by establishing that of its corresponding slow and fast subsystems, provided that the singular perturbation parameters are sufficiently small to be within this bound. For a continuous-time system, Klimushchev and Krasovskii (1962) found the ε-bound of singularly perturbed systems. In Feng (1988), Chen and Lin (1990) and Lin and Chen (1992), a frequency-domain approach was proposed to find the ε-bound of singularly perturbed systems. Shao and Rowland (1995) considered a linear time-invariant singularly perturbed system with single time delay in the slow states. Their work gave a delay-independent sufficient condition for the stability bound of ε. In cases of discrete time, the two-time-scale properties of weakly coupled discrete systems and control of these systems were discussed by Mahmoud (1982). The stability bound of singular perturbation parameters for the asymptotic stability analysis of singularly perturbed systems with a single parameter was discussed by Li and Li (1992).

In this paper, the research on time-scale modelling is extended to include discrete multiple time-delay singularly perturbed systems. The stability problem of discrete multiple time-delay singularly perturbed systems was first considered by Trinh and Aldeen (1995), in whose paper time delays are exact integer multiples of the sampling interval. In their work, the delay terms are treated as the perturbations of the systems and the stability bound of ε is obtained from nominal systems. As for the singular perturbation approach, they merely dealt with discrete singularly perturbed systems without delay and subject to some plant uncertainties. In our work, the control design for discrete singularly perturbed systems with multiple time delays which may not be exact integer multiples of the sampling interval is fulfilled by using the standard procedure to analyse singularly perturbed systems. Furthermore, an efficient algorithm for finding the D-stability bound of the singular perturbation parameter is proposed.

The organization of this study is as follows. In §2, the techniques of D-pole placement and complex stability radius are combined and extended such that they can solve the $D(\alpha, r)$-stability problem of discrete uncertain multiple time-delay systems. Two cases of a new robust D-stability criterion in terms of complex stability radius are proposed to guarantee that all poles of the system remain inside the specified disk $D(\alpha, r)$ in the presence of parametric uncertainties. The corresponding slow and fast subsystems of a discrete multiple time-delay singularly perturbed system are then derived in §3. In §4, the state feedback controls for the slow and fast subsystems are separately designed such that the slow and fast closed-loop subsystems are both $D(\alpha, r)$-stable. In §5, a composite state feedback control for the original system is synthesized from the state feedback controls designed in §4 and a frequency domain ε-dependent D-stability criterion is subsequently proposed to examine whether the singular perturbation parameter ε is small enough or not. If ε is so small that any one of the conditions of this criterion is satisfied, then $D(\alpha, r)$-stability of the slow and fast closed-loop subsystems can imply the stability of the original system under the composite state feedback control. In order to obtain a less conservative D-stability bound of the singular perturbation parameter and to reduce the computation time, an efficient algorithm is proposed in §6. Finally, an example is provided to illustrate the efficient algorithm.

2. D-stability criterion

In this section, we will propose a new robust D-stability criterion in terms of complex stability radius for discrete uncertain multiple time-delay systems described by the following difference equation:

$$x(k+1) = Ax(k) + \Delta Ax(k) + \sum_{i=1}^{n} A_{di} x(k-h_i) + \sum_{i=1}^{n} \Delta A_{di} x(k-h_i) \qquad (2.1)$$

in which A and A_{di} are constant matrices with appropriate dimensions and h_i, $i = 1,2,\ldots,n$, are positive numbers: ΔA and ΔA_{di} denote the parametric uncertainties with the following upper norm-bounds:

$$\|\Delta A\| \le \beta \qquad (2.2a)$$

$$\|\Delta A_{di}\| \le \eta, \quad i = 1,2,\ldots,n, \qquad (2.2b)$$

where β and η are given constants.

Before we proceed to derive the robust D-stability criterion, some useful concepts are given in the following.

Definition 1: A feedback control system is said to be $D(\alpha, r)$-stable if all poles of the system are within the specific disk $D(\alpha, r)$ centred at $(\alpha, 0)$ with radius r. In other words, the solutions of its characteristic equation satisfy

$$|(z-\alpha)/r| < 1 \qquad (2.3)$$

in which $r > 0$ and $|\alpha| + r < 1$.

Definition 2 (Hinrichsen and Pritchard 1988, Kharitonov 1991): Let all eigenvalues of the matrix A be inside the unit circle of the complex plane; then the positive value

$$\rho(A) = \{\max_{0\le\theta\le 2\pi} \{\|[e^{j\theta} I - A]^{-1}\|\}\}^{-1} \qquad (2.4)$$

is said to be a complex stability radius of the matrix A.

Remark 1 (Kharitonov 1991): The value $\rho(A)$ depends on the choice of norm. For instance, if the Euclidean norm is used, then it is easy to show that

$$\theta(A) = \min_{0\le\theta\le 2\pi} \{\underline{\sigma}[e^{j\theta} I - A]\}, \qquad (2.5)$$

in which $\underline{\sigma}(\cdot)$ is the minimal singular value of matrix $(\cdot)$.

Lemma 1 (Kharitonov 1991): *Let all eigenvalues of the matrix M be inside the unit disk of the complex plane. All the eigenvalues of all matrices $M + \Delta M$ are inside the unit disk if and only if $\|\Delta M\| < \rho(M)$.*

Lemma 2 (Vidyasagar 1985): *Let a matrix $E(z) \in \Re_\infty^{m\times n}$, with $\Re_\infty^{m\times n}$ denoting the set of $m \times n$ matrices whose elements are proper stable rational functions; then*

$$\sup_{z\in\Omega} \|E(z)\| = \sup_{|z|\ge 1} \|E(z)\| = \sup_{\theta\in[0,2\pi]} \|E(e^{j\theta})\| \qquad (2.6)$$

where $\Omega \equiv \{z = r_* e^{j\theta}, \theta \in [0, 2\pi], |r| \ge 1\}$. Since $E(z)$ is analytic for $z \in \Omega$, this norm is well defined.

After reviewing the above definitions and lemmas, we are in the position to derive the robust D-stability criterion in terms of complex stability radius for a discrete uncertain multiple time-delay system.

Theorem 1:

(I) *Suppose that all the eigenvalues of A are within the specific disk $D(\alpha, r)$. The system (2.1) is robustly $D(\alpha, r)$-stable if*

$$\frac{1}{r}\left[\beta + \sum_{i=1}^{n} (\|A_{di}\| + \eta)(r - |\alpha|)^{-h_i}\right] \equiv d_1 < \rho\left[\frac{A - \alpha I}{r}\right] \equiv d_s, \qquad (2.7a)$$

in which $|\alpha| < r$.

(II) *If $d_1 \ge d_s$ and the function*

$$h(g) \equiv \frac{1}{r}\left[\beta + \left\|\sum_{i=1}^{n} A_{di}(rg + \alpha)^{-h_i}\right\| + \sum_{i=1}^{n} \eta (r - |\alpha|)^{-h_i}\right] \qquad (2.7b)$$

does not lie inside the interval $[d_s, d_1]$, where $|\alpha| < r$ and g takes the values in the bounded region $U_1 \equiv \{g | g = r e^{j\theta}, \theta \in [0, 2\pi], 1 \le r \le d_{1r}\}$ with

$$d_{1r} = \left\|\frac{A - \alpha I}{r}\right\| + d_1$$

then the system (2.1) is robustly $D(\alpha, r)$-stable.

Proof: See Appendix A.

Remark 2: It is easy to see that the D-stability criterion in Theorem 1 will get a less conservative result than the criteria proposed by Lee *et al.* (1992) and Su and Shyr (1994). Furthermore, since system (2.1) contains multiple time delays which may not be exact integer multiples of the sampling interval, their results cannot examine the D-stability of system (2.1).

Remark 3: Case (I) of Theorem 1 provides a neat algebraic condition to test the D-stability of system (2.1) at the cost of conservativeness. It is therefore reasonable to check the D-stability with case (I) and then, if it fails, to resort to case (II). Thus, case (I) and case (II) complement each other.

However, for a practical application, it is difficult to examine case (II) of Theorem 1. The following 'boundary test' may be helpful in examining this case.

Corollary 1: *If* $d_l \geq d_s$ *and the following inequality* (2.8) *holds:*

$$h(g) = \frac{1}{r}\left[\beta + \left\| \sum_{i=1}^{n} A_{di}(rg+\alpha)^{-h_i} \right\| + \sum_{i=1}^{n} \eta(r-|\alpha|)^{-h_i}\right] < d_s \tag{2.8}$$

where $|\alpha| < r$ *and* $g = e^{j\theta}$ *for* $\theta \in [0, 2\pi]$, *then the system* (2.1) *is robustly* $D(\alpha, r)$*-stable.*

Proof: The matrix $\sum_{i=1}^{n} A_{di}(rg+\alpha)^{-h_i}$ of which all poles of the elements have the modulus

$$|g| = \frac{|\alpha|}{r} < 1 \tag{2.9}$$

belongs to $\Re_{\infty}^{m\times m}$. Consequently, on the basis of Lemma 2, the term $\| \sum_{i=1}^{n} A_{di}(rg+\alpha)^{-h_i} \|$ in (2.8) takes on its supremum in the range given by $g = e^{j\theta}$ for $\theta \in [0, 2\pi]$. Therefore, if the inequality (2.8) holds, $h(g)$ does not lie inside the interval $[d_s, d_l]$ for all $g \in U_1$ and then the system (2.1) is robustly $D(\alpha, r)$-stable (according to the case (II) of Theorem 1). This completes the proof. □

3. Problem formulation

Consider the following discrete multiple time-delay singularly perturbed system:

$$x_1(k+1) = \sum_{i=0}^{n} A_{1i} x_1(k-h_i) + \varepsilon \sum_{i=0}^{n} \tilde{A}_{1i} x_2(k-h_i) + B_1 u(k) \tag{3.1a}$$

$$x_2(k+1) = \sum_{i=0}^{n} A_{2i} x_1(k-h_i) + \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i} x_2(k-h_i) + B_2 u(k) \tag{3.1b}$$

where A_{1i}, A_{2i}, $\tilde{A}_{1i}$, $\tilde{A}_{2i}$, $i = 0, 1, 2, \ldots, n$, B_1 and B_2 are constant matrices with appropriate dimensions and h_i, $i = 1, 2, \ldots, n$, are positive numbers $(h_0 = 0)$; the pair (A_{10}, B_1) is assumed to be controllable. System (3.1) is referred to as the C-model (p. 45, Naidu and Rao 1985) and can be obtained from the slow sampling rate model as a result of discretization or sampled-data control of singularly perturbed continuous-time systems (Li and Li 1992). The small positive scalar ε is a singular perturbation parameter subject to the following constraint:

$$\varepsilon \sum_{i=0}^{n} \|\tilde{A}_{2i}\| < 1. \tag{3.2}$$

Before we discuss the main result, a lemma is first given in the following.

Lemma 3 (Chou and Chen 1990)**:** *For any matrix* $A \in R^{m\times m}$, *if* $\sigma[A] < 1$†, *then* $|\det(I \in A)| > 0$.

On the basis of Lemma 3 and the fact that

$$\sigma\left[\varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right] \leq \varepsilon \left\| \sum_{i=0}^{n} \tilde{A}_{2i} \right\| \leq \varepsilon \sum_{i=0}^{n} \|\tilde{A}_{2i}\| < 1$$

it is clear that the matrix

$$\left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)$$

is non-singular. Now, according to the time-scale analysis in Mahmoud (1982), the slow and fast subsystems of the original system (3.1) can then be derived as follows.

3.1. *The slow subsystem*

Let $x_2(k-h_i) = x_2(k) = \bar{x}_2(k)$ for $i = 1, 2, \ldots, n$; the system (3.1) thus reduces to

$$x_s(k+1) = \sum_{i=0}^{n} A_{1i} x_s(k-h_i) + \varepsilon\left(\sum_{i=0}^{n} \tilde{A}_{1i}\right)\bar{x}_2(k) + B_1 u_s(k) \tag{3.3a}$$

$$\bar{x}_2(k) = \sum_{i=0}^{n} A_{2i} x_s(k-h_i) + \varepsilon\left(\sum_{i=0}^{n} \tilde{A}_{2i}\right)\bar{x}_2(k) + B_2 u_s(k) \tag{3.3b}$$

where x_s, $\bar{x}_2$ and u_s are the slow components of x_1, x_2 and u, respectively. From equation (3.3b), we have

† The notation $\sigma[A]$ denotes the spectral radius of the matrix A.

$$\bar{x}_2(k) = \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} \sum_{i=0}^{n} A_{2i} x_s(k-h_i) + \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} B_2 u_s(k) \qquad (3.4)$$

Substituting (3.4) into (3.3a), the slow subsystem of the original system (3.1) can be expressed as

$$x_s(K+1) = \sum_{i=0}^{n} A_{si} x_s(k-h_i) + B_s u_s(k) \qquad (3.5a)$$

where

$$A_{si} = A_{1i} + \varepsilon \left(\sum_{j=0}^{n} \tilde{A}_{1j}\right)\left(I - \varepsilon \sum_{j=0}^{n} \tilde{A}_{2j}\right)^{-1} A_{2i} \quad \text{for } i = 0, 1, 2, \ldots, n \qquad (3.5b)$$

$$B_s = B_1 + \varepsilon \left(\sum_{i=0}^{n} \tilde{A}_{1i}\right)\left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} B_2. \qquad (3.5c)$$

3.2. *The fast subsystem*

Let

$$x_f(k) = x_2(k) - \bar{x}_2(k), u_f(k) = u(k) - u_s(k), u_s(k) = u_s(k-h_i)$$

and

$$x_1(k) = x_1(k-h_i) = x_s(k-h_i) = x_s(k)$$

for $i = 1, 2, \ldots, n$; we have (from equation (3.4))

$$x_f(k) = x_2(k) - \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1}\left(\sum_{i=0}^{n} A_{2i}\right) x_s(k) - \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} B_2 u_s(k) \qquad (3.6)$$

According to equation (3.1b), the fast subsystem of the original system (3.1) is derived as follows:

$$\begin{aligned}
x_f(k+1) &= \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i} x_f(k-h_i) + \varepsilon \left(\sum_{i=0}^{n} \tilde{A}_{2i}\right) \times \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1}\left(\sum_{i=0}^{n} A_{2i}\right) x_s(k) \\
&\quad + \varepsilon \left(\sum_{i=0}^{n} \tilde{A}_{2i}\right)\left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} B_2 u_s(k) \\
&\quad + \left(\sum_{i=0}^{n} A_{2i}\right) x_s(k) + B_2[u_f(k) + u_s(k)] \\
&\quad - \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1}\left(\sum_{i=0}^{n} A_{2i}\right) x_s(k) \\
&\quad - \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} B_2 u_s(k) \\
&= \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i} x_f(k-h_i) \\
&\quad + \left[\left(\sum_{i=0}^{n} A_{2i}\right) - \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right) \times \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1}\left(\sum_{i=0}^{n} A_{2i}\right)\right] x_s(k) + B_2 u_f(k) \\
&\quad + \left[I - \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)\left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1}\right] B_2 u_s(k) \\
&= \sum_{i=0}^{n} A_{fi} x_f(k-h_i) + B_f u_f(k) \qquad (3.7a)
\end{aligned}$$

where

$$A_{fi} = \varepsilon A_{2i} \quad \text{and} \quad B_f = B_2. \qquad (3.7b)$$

4. State feedback controls for the slow and fast subsystems

In this section, the state feedback controls for the slow subsystem (3.5) and for the fast subsystem (3.7) are separately designed such that both slow and fast closed-loop subsystems are $D(\alpha, r)$-stable.

4.1. *State feedback control for the slow subsystem*

Introducing the slow control

$$u_s(k) \equiv \sum_{i=0}^{n} k_{si} x_s(k-h_i) \qquad (4.1)$$

in which k_{si}, $i = 0, 1, 2, \ldots, n$, are the state feedback gains into the slow subsystem (3.5), we have

$$\begin{aligned}
x_s(k+1) &= \sum_{i=0}^{n} (A_{si} + B_s k_{si}) x_s(k-h_i) \\
&= \sum_{i=0}^{n} \{[A_{1i} + \Delta A_{1i}(\varepsilon)] + [B_1 + \Delta B_1(\varepsilon)] k_{si}\} x_s(k-h_i) \\
&= \sum_{i=0}^{n} [\bar{A}_{si} + \Delta \bar{A}_{si}(\varepsilon)] x_s(k-h_i) \qquad (4.2)
\end{aligned}$$

in which

$$\bar{A}_{si} = A_{1i} + B_1 k_{si}, \ \Delta \bar{A}_{si}(\varepsilon) = \Delta A_{1i}(\varepsilon) + \Delta B_1(\varepsilon) k_{si} \tag{4.3}$$

with

$$\Delta A_{1i}(\varepsilon) = \varepsilon\left(\sum_{i=0}^{n} \tilde{A}_{1j}\right)\left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2j}\right)^{-1} A_{2i}, \quad i = 0, 1, 2, \ldots, n$$

$$\Delta B_1(\varepsilon) = \varepsilon\left(\sum_{i=0}^{n} \tilde{A}_{1j}\right)\left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2j}\right)^{-1} B_2.$$

On the basis of the constraint (3.2), we can derive the following inequality:

$$\begin{aligned}\left\|\left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1}\right\| &= \left\| I + \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i} + \varepsilon^2 \left(\sum_{i=0}^{n} \tilde{A}_{2i}\right)^2 + \cdots \right\| \\ &\leq \|I\| + \varepsilon \left\| \sum_{i=0}^{n} \tilde{A}_{2i} \right\| + \varepsilon^2 \left\| \left(\sum_{i=0}^{n} \tilde{A}_{2i}\right)^2 \right\| + \cdots \\ &\leq 1 + \varepsilon \sum_{i=0}^{n} \|\tilde{A}_{2i}\| + \varepsilon^2 \left(\sum_{i=0}^{n} \|\tilde{A}_{2i}\|\right)^2 + \cdots \\ &= \frac{1}{1 - \varepsilon \sum_{i=0}^{n} \|\tilde{A}_{2i}\|}\end{aligned} \tag{4.4a}$$

According to the inequality (4.4 a), we have

$$\begin{aligned}\|\Delta A_{1i}(\varepsilon)\| &= \left\| \varepsilon \left(\sum_{j=0}^{n} \tilde{A}_{1j}\right)\left(I - \varepsilon \sum_{j=0}^{n} \tilde{A}_{2j}\right)^{-1} A_{2i} \right\| \\ &\leq \varepsilon \left\| \sum_{j=0}^{n} \tilde{A}_{1j} \right\| \times \left\| \left(I - \varepsilon \sum_{j=0}^{n} \tilde{A}_{2j}\right)^{-1} \right\| \|A_{2i}\| \\ &\leq \frac{\varepsilon \left(\sum_{k=0}^{n} \|\tilde{A}_{1j}\|\right) \|A_{2i}\|}{1 - \varepsilon \sum_{j=0}^{n} \|\tilde{A}_{2j}\|} \equiv \alpha_i(\varepsilon),\end{aligned}$$

$$i = 0, 1, 2, \ldots, n \tag{4.4b}$$

$$\begin{aligned}\|\Delta B_1(\varepsilon)\| &= \left\| \varepsilon \left(\sum_{j=0}^{n} \tilde{A}_{1j}\right)\left(I - \varepsilon \sum_{j=0}^{n} \tilde{A}_{2j}\right)^{-1} B_2 \right\| \\ &\leq \varepsilon \left\| \sum_{j=0}^{n} \tilde{A}_{1j} \right\| \times \left\| \left(I - \varepsilon \sum_{j=0}^{n} \tilde{A}_{2j}\right)^{-1} \right\| \|B_2\| \\ &\leq \frac{\varepsilon \left(\sum_{j=0}^{n} \|\tilde{A}_{1j}\|\right) \|B_2\|}{1 - \varepsilon \sum_{j=0}^{n} \|\tilde{A}_{2j}\|} \equiv \beta(\varepsilon),\end{aligned} \tag{4.4c}$$

and hence

$$\begin{aligned}\|\Delta \bar{A}_{si}(\varepsilon)\| &= \|\Delta A_{1i}(\varepsilon) + \Delta B_1(\varepsilon) k_{si}\| \\ &\leq \|\Delta A_{1i}(\varepsilon)\| + \|\Delta B_1(\varepsilon)\| \|k_{si}\| \\ &\leq \alpha_i(\varepsilon) + \beta(\varepsilon)\|k_{si}\| \equiv \delta_i(\varepsilon), \quad i = 0, 1, 2, \ldots, n.\end{aligned} \tag{4.5}$$

Consequently, according to case (I) of Theorem 1, if the state feedback gains $k_{si}, i = 0, 1, 2, \ldots, n$, are chosen such that

$$d_{1s} \equiv \frac{1}{r}\left[\delta_0(\varepsilon) + \sum_{i=1}^{n} (\|\bar{A}_{si}\| + \delta_i(\varepsilon))(r - |\alpha|)^{-h_i}\right] < \rho\left(\frac{\bar{A}_{s0} - \alpha I}{r}\right) = d_{ss}, \tag{4.6a}$$

then the slow closed-loop subsystem (4.2), or equivalently the slow subsystem (3.5) under the control (4.1), is $D(\alpha, r)$-stable with $r > |\alpha|$. Substituting (4.5) into (4.6 a) and according to (4.4 b) and (4.4 c), the inequality (4.6 a) is equivalent to

$$\varepsilon < \frac{n_s}{m_s + n_s\left(\sum_{i=0}^{n} \|\tilde{A}_{2i}\|\right)} \equiv \varepsilon_1^* \tag{4.6b}$$

where

$$n_s \equiv r d_{ss} - \sum_{i=1}^{n} \|\bar{A}_{si}\|(r - |\alpha|)^{-h_i} \dagger$$

and

$$m_s \equiv \left(\sum_{i=0}^{n} \|\tilde{A}_{1i}\|\right)\left[\sum_{i=0}^{n} (\|A_{2i}\| + \|B_2\|\|k_{si}\|)(r - |\alpha|)^{-h_i}\right].$$

† According to (4.6 a), it is obvious that n_s is positive.

However, if the condition (4.6 *a*), or equivalently (4.6 *b*), fails, then we resort to checking the following condition (see Corollary 1):

$$h_s(g) \equiv \frac{1}{r}\left[\delta_0(\varepsilon) + \left\|\sum_{i=1}^{n} \bar{A}_{si}(rg+\alpha)^{-h_i}\right\| \times \sum_{i=1}^{n} \delta_i(\varepsilon)(r-|\alpha|)^{-h_i}\right] < d_{ss}, \qquad (4.7)$$

where $r > |\alpha|$ and $g = e^{j\theta}$ for $\eta \in [0, 2\pi]$.

Remark 4: For the purpose of $D(\alpha, r)$-stabilization of slow closed-loop subsystem (4.2), the state feedback gains k_{si}, $i = 0, 1, 2, \ldots, n$, are adjusted such that ε_1^* is as large as possible. This can be fulfilled by choosing k_{s0} to place all the poles of $\bar{A}_{s0} = A_{10} + B_1 k_{s0}$ at $(\alpha, 0)$ (i.e. to maximize d_{ss}) and choosing k_{si} to minimize $\|\bar{A}_{si}\| = \|A_{1i} + B_1 k_{si}\|$ for $i = 1, 2, \ldots, n$. However, there are various choices of k_{si}, $i = 0, 1, 2, \ldots, n$, to make ε_1^* as large as possible, but only one of them is chosen here.

Remark 5: The inequality (4.6 *b*) provides a neat algebraic equation to find the upper bound of ε, which guarantees the $D(\alpha, r)$-stability of the slow closed-loop subsystem (4.2) at the cost of conservativeness. However, a less conservative upper bound, called $\tilde{\varepsilon}_1^*$, can be obtained by finding the upper bound of ε that fulfils the inequality (4.7) with much more computation time.

4.2. *State feedback control for the fast subsystem*

Introducing the fast control

$$u_f \equiv \varepsilon \sum_{i=0}^{n} k_{fi} x_f(k-h_i) \qquad (4.8)$$

where k_{fi}, $i = 0, 1, 2, \ldots, n$, are the state feedback gains into the fast subsystem (3.7), we have

$$x_f(k+1) = \sum_{i=0}^{n} A_{fi} x_f(k-h_i) + \varepsilon B_f \sum_{i=0}^{n} k_{fi} x_f(k-h_i)$$

$$= \sum_{i=0}^{n} (A_{fi} + \varepsilon B_f k_{fi}) x_f(k-h_i)$$

$$= \sum_{i=0}^{n} \varepsilon(\tilde{A}_{2i} + B_f k_{fi}) x_f(k-h_i)$$

$$= \sum_{i=0}^{n} \Delta\bar{A}_{fi}(\varepsilon) x_f(k-h_i) \qquad (4.9a)$$

in which

$$\Delta\bar{A}_{fi}(\varepsilon) \equiv \varepsilon(\tilde{A}_{2i} + B_f k_{fi}) \qquad (4.9b)$$

Consequently, according to case (I) of Theorem 1, if the state feedback gains k_{fi}, $i = 0, 1, 2, \ldots, n$, are chosen such that

$$d_{1f} \equiv \frac{1}{r}[\|\Delta\bar{A}_{f0}\| + \sum_{i=0}^{n} \|\Delta\bar{A}_{fi}\|(r-|\alpha|)^{-h_i}]$$

$$= \frac{1}{r}\left[\sum_{i=0}^{n} \|\Delta\bar{A}_{fi}\|(r-|\alpha|)^{-h_i}\right] < \rho\left(\frac{-\alpha I}{r}\right) \equiv d_{sf} \qquad (4.10a)$$

then the fast closed-loop subsystem (4.9), or equivalently the fast subsystem (3.7) under the control (4.8), is $D(\alpha, r)$-stable with $r > |\alpha|$. It is noted that the inequality (4.10 *a*) is equivalent to

$$\varepsilon < \frac{r d_{sf}}{\sum_{i=0}^{n} \|\tilde{A}_{2i} + B_f k_{fi}\|(r-|\alpha|)^{-h_i}} \equiv \varepsilon_2^* \qquad (4.10b)$$

However, if the condition (4.10 *a*), or equivalently (4.10 *b*), fails, then we resort to checking the following condition:

$$h_f(g) \equiv \frac{1}{r}\left\|\sum_{i=0}^{n} \Delta\bar{A}_{fi}(rg+\alpha)^{-h_i}\right\| < d_{sf} \qquad (4.11)$$

where $r > |\alpha|$ and $h = e^{j\theta}$ for $\theta \in [0, 2\pi]$.

Remark 6: In order to make ε_2^* as large as possible, according to the similar discussion in the preceding subsection, we need only to choose k_{fi} to minimize $\|\tilde{A}_{2i} + B_f k_{fi}\|$ for $i = 0, 1, 2, \ldots, n$. Moreover, ε_2^* obtained from the more conservative inequality (4.10 *b*) (i.e. (4.10 *a*)) is less than the upper bound of ε, called $\tilde{\varepsilon}_2^*$, that fulfils the inequality (4.11).

5. *D*(α, r)-stabilization of the original system

In this section, a composite state feedback control for the D-stabilization of the original discrete multiple time-delay singularly perturbed system (3.1) is subsequently synthesized from the slow control (4.1) and the fast control (4.8). Furthermore, a frequency domain ε-dependent D-stability criterion is derived such that the $D(\alpha, r)$-stability of the original system (3.1) under the composite state feedback control can be investigated by establishing that of its corresponding slow closed-loop subsystems (4.2) and fast closed-loop subsystem (4.9).

According to the slow control (4.1) and the assertion $x_s(k-h_i) = x_s(k)$ in §3.2, equation (3.6) can be rewritten as

$$x_f(k) = x_2(k) - \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} \left(\sum_{i=0}^{n} A_{2i}\right) x_s(k) - \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} B_2 \sum_{i=0}^{n} k_{si} x_s(k - h_i)$$
$$= x_2(k) - \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} \left(\sum_{i=0}^{n} A_{2i}\right) x_s(k) - \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} B_2 \sum_{i=0}^{n} k_{si} x_s(k) \quad (5.1)$$

i.e.

$$x_2(k) = x_f(k) + \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} \times \left(\sum_{i=0}^{n} A_{2i} + B_2 \sum_{i=0}^{n} k_{si}\right) x_s(k) \quad (5.2)$$

Hence, we have

$$x_2(k - h_i) = x_f(k - h_i) + \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i}\right)^{-1} \times \left(\sum_{i=0}^{n} A_{2i} + B_2 \sum_{i=0}^{n} k_{si}\right) x_s(k - h_i) \quad \text{for } i = 0, 1, 2, \ldots, n \quad (5.3)$$

Consequently, the composite state feedback control is of the following form:

$$u(k) = u_s(k) + u_f(k) = \sum_{i=0}^{n} k_{si} x_s(k - h_i) + \sum_{i=0}^{n} \varepsilon k_{fi} x_f(k - h_i)$$
$$= \sum_{i=0}^{n} \left[k_{si} - \varepsilon k_{fi} \left(I - \varepsilon \sum_{j=0}^{n} \tilde{A}_{2j}\right)^{-1} \times \left(\sum_{j=0}^{n} A_{2j} + B_2 \sum_{j=0}^{n} k_{sj}\right)\right] x_s(k - h_i)$$
$$+ \sum_{i=0}^{n} \varepsilon k_{fi} \left[\left(I - \varepsilon \sum_{j=0}^{n} \tilde{A}_{2j}\right)^{-1} \times \left(\sum_{j=0}^{n} A_{2j} + B_2 \sum_{j=0}^{n} k_{sj}\right) x_s(k - h_i) + x_f(k - h_i)\right] \quad (5.4)$$

On the basis of equation (5.3) and the assertion $x_s(k - h_i) = x_1(k - h_i)$ in §3.2 for $i = 0, 1, 2, \ldots, n$, the composite feedback control (5.4) becomes

$$u(k) = \sum_{i=0}^{n} k_{1i} x_1(k - h_i) + \sum_{i=0}^{n} k_{2i} x_2(k - h_i) \quad (5.5a)$$

where

$$k_{1i} \equiv k_{si} - \varepsilon k_{fi} \left(I - \varepsilon \sum_{i=0}^{n} \tilde{A}_{2j}\right)^{-1} \left(\sum_{i=0}^{n} A_{2j} + B_2 \sum_{i=0}^{n} k_{sj}\right) \quad \text{for } i = 0, 1, 2, \ldots, n \quad (5.5b)$$

$$k_{2i} \equiv \varepsilon k_{fi} \quad \text{for } i = 0, 1, 2, \ldots, n \quad (5.5c)$$

Applying the composite state feedback control (5.5) to the original system (3.1), we have

$$x_1(k + 1) = \sum_{i=0}^{n} M_{1i} x_1(k - h_i) + \sum_{i=0}^{n} M_{2i} x_2(k - h_i) \quad (5.6a)$$

$$x_2(k + 1) = \sum_{i=0}^{n} M_{3i} x_1(k - h_i) + \sum_{i=0}^{n} M_{4i} x_2(k - h_i) \quad (5.6b)$$

where

$$\left.\begin{aligned} M_{1i} &= A_{1i} + B_1 k_{1i}, & M_{2i} &= \varepsilon \tilde{A}_{1i} + B_1 k_{2i} \\ M_{3i} &= A_{2i} + B_2 k_{1i}, & M_{4i} &= \varepsilon \tilde{A}_{2i} + B_2 k_{2i} \end{aligned}\right\} \quad (5.6c)$$

Prior to discussing $D(\alpha, r)$-stabilization problem of the closed-loop system (5.6), we first introduce a useful lemma as follows.

Lemma 4 (maximum modulus theorem) (John 1967): *If $f(z)$ is analytic in a bounded domain D and continuous in $\bar{D}$ (i.e. the closure of D), then $|f(z)|$ takes its maximum on the boundary of D.*

We now proceed to derive a frequency domain ε-dependence D-stability criterion for the closed-loop system (5.6).

Theorem 2: *If the state feedback gains k_{si} and k_{fi} for $i = 0, 1, 2, \ldots, n$ are chosen such that the slow closed-loop subsystem (4.2) and the fast closed-loop subsystem (4.9) are both $D(\alpha, r)$-stable with $r > |\alpha|$, the original system (3.1) under the composite state feedback control (5.5), or equivalently system (5.6), is $D(\alpha, r)$-stable with $r > |\alpha|$ if the singular perturbation parameter ε satisfies*

(I) $\rho[\phi_{\tilde{g}}(\varepsilon, e^{j\theta})] < 1 \quad \text{for } \theta \in [0, 2\pi]$ (5.7a)

or

(II) $\rho[\tilde{\tilde{\phi}}_{\tilde{g}}(\varepsilon, e^{j\theta})] < 1 \quad \text{for } \theta \in [0, 2\pi]$ (5.7b) *where*

$$\tilde{\phi}_{\tilde{g}}(\varepsilon, e^{j\theta}) = \tilde{\Lambda}_{\tilde{g}}^{-1}(\varepsilon, e^{j\theta})\left\{\tilde{R}_{\tilde{g}}(\varepsilon, e^{j\theta}) - \left[\sum_{i=0}^{n} M_{2i}(\varepsilon)(r e^{-j\theta} + \alpha)^{-h_i}\right]\right.$$
$$\times\left[(r e^{j\theta} + \alpha)I - \sum_{i=0}^{n} M_{4i}(\varepsilon)(r e^{-j\theta} + \alpha)^{-h_i}\right]^{-1}$$
$$\left.\times\left[\sum_{i=0}^{n} M_{3i}(\varepsilon)(r e^{-j\theta} + \alpha)^{-h_i}\right]\right\}$$

and

$$\tilde{\tilde{\phi}}_{\tilde{g}}(\varepsilon, e^{j\theta}) = \left\{\tilde{R}_{\tilde{g}}(\varepsilon, e^{j\theta}) - \left[\sum_{i=0}^{n} M_{2i}(\varepsilon)(r e^{-j\theta} + \alpha)^{-h_i}\right]\right.$$
$$\times\left[(r e^{-j\theta} + \alpha)I - \sum_{i=0}^{n} M_{4i}(\varepsilon)(r e^{-j\theta} + \alpha)^{-h_i}\right]^{-1}$$
$$\left.\times\left[\sum_{i=0}^{n} M_{3i}(\varepsilon)(r e^{-j\theta} + \alpha)^{-h_i}\right]\right\}\tilde{\Lambda}_{\tilde{g}}^{-1}(\varepsilon, e^{j\theta})$$

with

$$\tilde{\Lambda}_{\tilde{g}}(\varepsilon, e^{j\theta}) = (r e^{-j\theta} + \alpha)I - \sum_{i=0}^{n}(A_{si}(\varepsilon) + B_s(\varepsilon)k_{si})(r e^{-j\theta} + \alpha)^{-h_i}$$

and

$$\tilde{R}_{\tilde{g}}(\varepsilon, e^{j\theta}) = \sum_{i=0}^{n}\left[\varepsilon\left(\sum_{i=0}^{n}\tilde{A}_{1j}\right)\left(I - \varepsilon\sum_{i=0}^{n}\tilde{A}_{2j}\right)^{-1}\right.$$
$$\times (A_{2i} + B_2 k_{si}) + \varepsilon B_1 k_{fi}\left(I - \varepsilon\sum_{i=0}^{n}\tilde{A}_{2j}\right)^{-1}$$
$$\left.\times\sum_{i=0}^{n}(A_{2j} + B_2 k_{sj})\right](r e^{-j\theta} + \alpha)^{-h_i}$$

Proof: See Appendix B.

Remark 7: The dependence of the matrices M_{ji} for $j = 2,\ 3,\ 4$ and $i = 0, 1, 2, \ldots, n$ upon ε, although omitted elsewhere, is indicated here in Theorem 2 for the purpose of clarification.

Remark 8: In principle, both the two conditions in (5.7) can be used to test the D-stability of the closed-loop system (5.6). It is therefore reasonable to check the D-stability with any one of the inequalities and then, if it fails, to resort to another.

Remark 9: Let ε_3^* and $\tilde{\varepsilon}_3^*$ be the upper bounds of ε that fulfil the D-stability conditions (5.7 *a*) and (5.7 *b*), respectively. Since there is no explicit information to indicate the conservativeness of ε_3^* and $\tilde{\varepsilon}_3^*$, the less conservative one should be used to find the D-stability bound of ε for each case in hand.

6. Algorithm for finding the D-stability bound ε^*

According to Theorem 2, the D-stability bound of ε, called ε^*, can be obtained by finding the upper ε-bound such that not only the slow closed-loop subsystem (4.2) and the fast closed-loop subsystem (4.9) are both $D(\alpha, r)$-stable but also the condition (5.7 *a*) or (5.7 *b*) is satisfied for all $\varepsilon \in (0, \varepsilon^*)$. In order to obtain a less conservative result and to reduce the computation time, on the basis of Remarks 4–6 and 9, we propose an efficient algorithm to find the D-stability bound ε^* such that $D(\alpha, r)$-stability of the slow closed-loop subsystem (4.2) and the fast closed-loop subsystem (4.9) can imply that of the original closed-loop system (5.6), provided that the singular perturbation parameter is sufficiently small to be within this bound.

Algorithm:

Step 1. Choose k_{s0} to place all the eigenvalues of $\bar{A}_{s0} = A_{10} + \underline{B}_1 k_{s0}$ at $(\alpha, 0)$ and choose k_{si} to minimize $\|\bar{A}_{si}\| = \|A_{1i} + B_1 k_{si}\|$ for $i = 1, 2, \ldots, n$, and then we can obtain ε_1^* from (4.6 *b*).

Step 2. Choose k_{fi} to minimize $\|\tilde{A}_{2i} + B_f k_{fi}\|$ for $i = 0, 1, 2, \ldots, n$, and then we can obtain ε_2^* from (4.10 *b*).

Step 3. Find the upper bound of ε_3^* such that the inequality (5.7 *a*) holds for all $\varepsilon \in (0, \varepsilon_3^*)$.

Step 4. Choose $\varepsilon^* = \min(\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*)$.

Step 5. If $\varepsilon^* = \varepsilon_j^*, j = 1, 2$ and 3, then go to (1), (2) and (3), respectively.

(1) Find the upper bound $\tilde{\varepsilon}_1^*$ such that (4.7) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_1^*)$ and $\varepsilon^* = \min(\tilde{\varepsilon}_1^*, \varepsilon_2^*, \varepsilon_3^*)$. If $\varepsilon^* = \tilde{\varepsilon}_1^*$, then stop; else go to Step 6.

(2) Find the upper bound $\tilde{\varepsilon}_2^*$ such that (4.11) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_2^*)$ and $\varepsilon^* = \min(\varepsilon_1^*, \tilde{\varepsilon}_2^*, \varepsilon_3^*)$. If $\varepsilon^* = \tilde{\varepsilon}_2^*$, then stop; else go to Step 7.

(3) Find the upper bound $\tilde{\varepsilon}_3^*$ such that (5.7 *b*) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_3^*)$. If $\tilde{\varepsilon}_3^* > \varepsilon_3^*$, then $\varepsilon^* = \min(\varepsilon_1^*, \varepsilon_2^*, \tilde{\varepsilon}_3^*)$. Under this condition, if $\varepsilon^* = \tilde{\varepsilon}_3^*$, then stop; else go to Step 8. However, if $\tilde{\varepsilon}_3^* \le \varepsilon_3^*$, then stop.

Step 6. If $\varepsilon^* = \varepsilon_j^*, j = 2$ and 3, then go to (1) and (2), respectively.

(1) Find the upper bound $\tilde{\varepsilon}_2^*$ such that (4.11) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_2^*)$ and $\varepsilon^* = \min(\tilde{\varepsilon}_1^*, \tilde{\varepsilon}_2^*, \varepsilon_3^*)$. If $\varepsilon^* = \tilde{\varepsilon}_2^*$, then stop; else go to Step 11.

(2) Find the upper bound $\tilde{\varepsilon}_3^*$ such that (5.7*b*) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_3^*)$. If $\tilde{\varepsilon}_3^* > \varepsilon_3^*$, then $\varepsilon^* = \min(\tilde{\varepsilon}_1^*, \varepsilon_2^*, \tilde{\varepsilon}_3^*)$. Under this condition, if $\varepsilon^* = \tilde{\varepsilon}_3^*$, then stop; else go to Step 10. However, if $\tilde{\varepsilon}_3^* \leq \varepsilon_3^*$, then stop.

Step 7. If $\varepsilon^* = \varepsilon_j^*$, $j = 1$ and 3, then go to (1) and (2), respectively.

(1) Find the upper bound $\tilde{\varepsilon}_1^*$ such that (4.7) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_1^*)$ and $\varepsilon^* = \min(\tilde{\varepsilon}_1^*, \tilde{\varepsilon}_2^*, \varepsilon_3^*)$. If $\varepsilon^* = \tilde{\varepsilon}_1^*$, then stop; else go to Step 11.

(2) Find the upper bound $\tilde{\varepsilon}_3^*$ such that (5.7*b*) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_3^*)$. If $\tilde{\varepsilon}_3^* > \varepsilon_3^*$, then $\varepsilon^* = \min(\varepsilon_1^*, \tilde{\varepsilon}_2^*, \tilde{\varepsilon}_3^*)$. Under this condition, if $\varepsilon^* = \tilde{\varepsilon}_3^*$, then stop; else go to Step 9. However, if $\tilde{\varepsilon}_3^* \leq \varepsilon_3^*$, then stop.

Step 8. If $\varepsilon^* = \varepsilon_j^*$, $j = 1$ and 2, then go to (1) and (2), respectively.

(1) Find the upper bound $\tilde{\varepsilon}_1^*$ such that (4.7) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_1^*)$ and $\varepsilon^* = \min(\tilde{\varepsilon}_1^*, \varepsilon_2^*, \tilde{\varepsilon}_3^*)$. If $\varepsilon^* = \tilde{\varepsilon}_1^*$, then stop; else go to Step 10.

(2) Find the upper bound $\tilde{\varepsilon}_2^*$ such that (4.11) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_2^*)$ and $\varepsilon^* = \min(\varepsilon_1^*, \tilde{\varepsilon}_2^*, \tilde{\varepsilon}_3^*)$. If $\varepsilon^* = \tilde{\varepsilon}_2^*$, then stop; else go to Step 9.

Step 9. Find the upper bound $\tilde{\varepsilon}_1^*$ such that (4.7) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_1^*)$ and $\varepsilon^* = \min(\tilde{\varepsilon}_1^*, \tilde{\varepsilon}_2^*, \tilde{\varepsilon}_3^*)$.

Step 10. Find the upper bound $\tilde{\varepsilon}_2^*$ such that (4.11) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_2^*)$ and $\varepsilon^* = \min(\tilde{\varepsilon}_1^*, \tilde{\varepsilon}_2^*, \tilde{\varepsilon}_3^*)$.

Step 11. Find the upper bound $\tilde{\varepsilon}_3^*$ such that (5.7b) holds for all $\varepsilon \in (0, \tilde{\varepsilon}_3^*)$. If $\tilde{\varepsilon}_3^* > \varepsilon_3^*$, then $\varepsilon^* = \min(\tilde{\varepsilon}_1^*, \tilde{\varepsilon}_2^*, \tilde{\varepsilon}_3^*)$; else $\varepsilon^* = \min(\tilde{\varepsilon}_1^*, \tilde{\varepsilon}_2^*, \varepsilon_3^*)$.

Remark 10: For some cases, the preceding algorithm may avoid the examination of conditions (4.7), (4.11) and (5.7*a*), (or (5.7*b*)) simultaneously to get a less conservative D-stability bound ε^*. For example, if the design algorithm stops at Step 5 (1) then it is not necessary to examine the conditions (4.11) and (5.7*b*). This obviously reduces the computation time.

Remark 11: Consider the following discrete multiple time-delay singularly perturbed system which is referred to as the R-model (Naidu and Rao 1985, p. 47):

$$x_{1r}(k+1) = \sum_{i=0}^{n} A_{1i} x_{1r}(k-h_i) + \sum_{i=0}^{n} \tilde{A}_{1i} x_{2r}(k-h_i) + B_1 u(k) \quad (6.1\,a)$$

$$x_{2r}(k+1) = \varepsilon \sum_{i=0}^{n} A_{2i} x_{1r}(k-h_i) + \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i} x_{2r}(k-h_i) + \varepsilon B_2 u(k) \quad (6.1\,b)$$

Introducing the following state-variable transformation (6.2) into system (6.1):

$$\begin{bmatrix} x_{1r}(k) \\ x_{2r}(k) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} x_{1c}(k) \\ x_{2c}(k) \end{bmatrix} \quad (6.2)$$

the system can then be converted into the following C-model:

$$x_{1c}(k+1) = \sum_{i=0}^{n} A_{1i} x_{1c}(k-h_i) + \varepsilon \sum_{i=0}^{n} \tilde{A}_{1i} x_{2c}(k-h_i) + B_1 u(k) \quad (6.3\,a)$$

$$x_{2c}(k+1) = \sum_{i=0}^{n} A_{2i} x_{1c}(k-h_i) + \varepsilon \sum_{i=0}^{n} \tilde{A}_{2i} x_{2c}(k-h_i) + B_2 u(k) \quad (6.3\,b)$$

Hence, the design algorithm proposed in this study can also solve the D-stabilization problem of the R-model system (6.1) by a state-variable transformation.

7. Example

Consider a discrete time-delay singularly perturbed system described by the following equations:

$$x_1(k+1) = \sum_{i=0}^{2} A_{1i} x_1(k-h_i) + \varepsilon \sum_{i=0}^{2} \tilde{A}_{1i} x_2(k-h_i) + B_1 u(k) \quad (7.1\,a)$$

$$x_2(k+1) = \sum_{i=0}^{2} A_{2i} x_1(k-h_i) + \varepsilon \sum_{i=0}^{2} \tilde{A}_{2i} x_2(k-h_i) + B_2 u(k) \quad (7.1\,b)$$

in which

$$h_0 = 0, h_1 = 0.2, h_2 = 1.1;$$

$$A_{10} = \begin{bmatrix} 2.3 & 2 \\ 4 & 4.3 \end{bmatrix}, A_{11} \begin{bmatrix} 1 & 1.5 \\ 0 & 3 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 5 & 10 \\ 0 & 20 \end{bmatrix}, \tilde{A}_{10} = \begin{bmatrix} 0.4 & 0.23 \\ 0 & 0.17 \end{bmatrix},$$

$$\tilde{A}_{11} = \begin{bmatrix} 0.5 & 0 \\ 0.65 & 0.31 \end{bmatrix}, \tilde{A}_{12} = \begin{bmatrix} 0.27 & 0.71 \\ 0.18 & 0.28 \end{bmatrix},$$

$$A_{20} = \begin{bmatrix} 2 & 2 \\ 2 & 2.001 \end{bmatrix}, A_{21} = \begin{bmatrix} 1 & 1.5 \\ 1 & 1.501 \end{bmatrix},$$

$$A_{22}=\begin{bmatrix}5 & 10\\ 5 & 10\end{bmatrix}, \tilde{A}_{20}=\begin{bmatrix}5 & 1\\ 1.5 & 1\end{bmatrix},$$

$$\tilde{A}_{21}=\begin{bmatrix}6 & 1.2\\ 6 & 1\end{bmatrix}, \tilde{A}_{22}=\begin{bmatrix}2 & 2\\ 2.2 & 2\end{bmatrix},$$

$$B_1=\begin{bmatrix}1\\ 2\end{bmatrix}, B_2=\begin{bmatrix}1\\ 1\end{bmatrix}$$

It is desired to find a composite state feedback control $u(k)$ such that the time-domain performance of system (7.1) satisfies the following specifications:

(a) overshoot $\le 15\%$, or equivalently, damping ratio $\zeta \ge 0.5$; (7.2 *a*)

(b) rise time ≥ 8 s, or equivalently, natural frequency $\omega_n \ge 0.3125$; (7.2 *b*)

(c) settling time ≤ 20 s, or equivalently, all poles less than 0.8 (the sampling interval $T = 1$ s) (7.2 *c*)

These constraints (a)–(c) may be interpreted as pole locations inside the specific disk $D(0.3, 0.46)$ (Lee and Lee 1987). Subsequently, the design algorithm proposed in §6 is used to find the D-stability bound ε^* such that $D(0.3, 0.46)$-stability of the slow and fast closed-loop subsystems can imply that of the original system (7.1) under the composite state feedback control (5.5) for all $\varepsilon \in (0, \varepsilon^*)$.

Step 1. Choose $k_{s0} = [-2 \ \ -2]$ to place all the eigenvalues of $\bar{A}_{s0} = A_{10} + B_1 k_{s0}$ at (0.3, 0) and choose $k_{s1} = [-1 \ \ -1.5]$ and $k_{s2} = [-5 \ \ -10]$ to minimize $\|\bar{A}_{si}\| = \|A_{1i} + B_1 k_{si}\|$† for $i = 1, 2$, respectively; we then get $\varepsilon_1^* = 0.0448$ from equation (4.6 *b*).

Step 2. Choose $k_{f0} = [-5 \ \ -1]$, $k_{f1} = [-6 \ \ -1]$ and $k_{f2} = [-2 \ \ -2]$ to minimize $\|\tilde{A}_{2i} + B_f k_{fi}\|$ for $i = 0, 1, 2$, respectively; we then get $\varepsilon_2^* = 0.0699$ from equation (4.10 *b*).

Step 3. In figure 2, the supremum of the function $\rho[\tilde{\phi}_{\tilde{g}}(\varepsilon, e^{j\theta})]$ in the range $\theta \in [0, 2\pi]$ is depicted with respect to ε. According to this figure and the inequality (5.7 *a*), we have $\varepsilon_3^* = 0.0128$.

Step 4. Choose $\varepsilon^* = \min(\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*) = 0.0128$.

Step 5. Since $\varepsilon^* = \varepsilon_3^* = 0.0128$, we resort to case (3). In figure 3, the supremum of the function $\rho[\tilde{\tilde{\phi}}_{\tilde{g}}(\varepsilon, e^{j\theta})]$ in the range $\theta \in [0, 2\pi]$ is depicted with respect to ε. According to this figure and the inequality (5.7 *b*), $\tilde{\varepsilon}_3^* = 0.007$ is obtained. Since $\tilde{\varepsilon}_3^* > \varepsilon_3^*$, we stop.

† In this example, the Euclidean norm is considered.

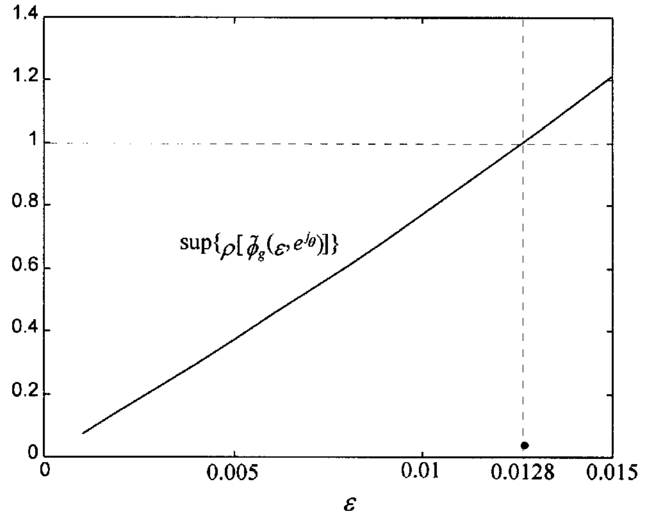

Figure 2. The supremum of the function $\rho[\tilde{\phi}_{\tilde{g}}(\varepsilon, e^{j\theta})]$ in the range $\theta \in [0, 2\pi]$.

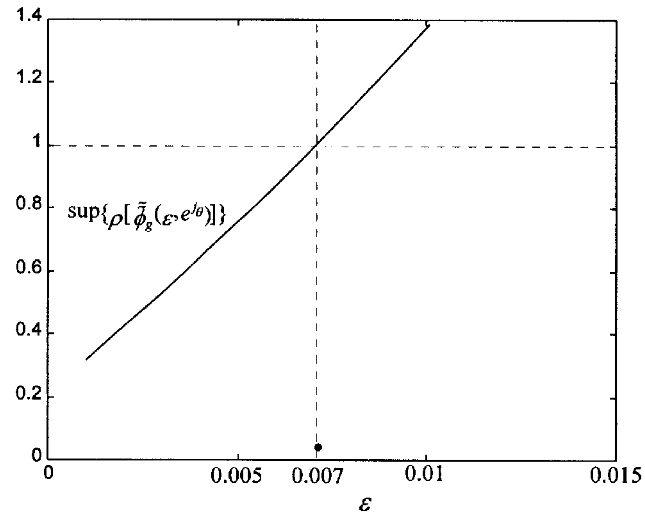

Figure 3. The supremum of the function $\rho[\tilde{\tilde{\phi}}_{\tilde{g}}(\varepsilon, e^{j\theta})]$ in the range $\theta \in [0, 2\pi]$.

According to the above discussion, we conclude that the discrete time-delay singularly perturbed system (7.1) under the composite state feedback control (5.5) with the state feedback gains obtained in Step 1 and Step 2 is D(0.3, 0.46)-stable for all $\varepsilon < \varepsilon^* = 0.0128$.

8. Conclusion

In this paper, we investigate the D-stabilization problem of a discrete multiple time-delay singularly perturbed system. Two cases of a new robust D-stability criterion in terms of complex stability radius are first proposed for discrete uncertain multiple time-delay systems. One is a direct test (i.e. $d_1 < d_s$) and the other is a boundary test. Then, the corresponding slow and fast subsystems of a discrete multiple time-delay singularly perturbed system are derived by using the technique of time-scale separation. the state feedback controls for the D-stabilization of the slow and fast subsystems are separately designed, and a composite state feedback control for the original system is subsequently synthesized from these state feedback controls. Thereafter, a frequency domain ε-dependent D-stability criterion is proposed for the original discrete multiple time-delay singularly perturbed system under the composite state feedback control. If any one of the conditions of this criterion is fulfilled, the D-stability of the original closed-loop system is thus investigated by establishing that of its corresponding slow and fast closed-loop subsystems. Finally, an efficient algorithm is proposed to obtain a less conservative D-stability bound of the singular perturbation parameter and to reduce the computation time.

Appendix A. Proof of Theorem 1:

(I): The necessary and sufficient condition to guarantee that all poles of the system (2.1) lie inside the specific disk $D(\alpha, r)$ is that all solutions of the characteristic equation

$$\det\left\{zI - \left[A + \Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})z^{-h_i}\right]\right\} = 0 \tag{A 1}$$

satisfy $|z - \alpha| < r$. Let $(z - \alpha)/r$ be replaced by a variable g (i.e. $z = rg + \alpha$); then (A 1) becomes

$$\det\left\{(rg+\alpha)I - \left[A + \Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(rg+\alpha)^{-h_i}\right]\right\} = 0$$

or, equivalently,

$$\det\left\{gI - \left[\frac{A-\alpha I}{r} + \frac{1}{r}\left(\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(rg+\alpha)^{-h_i}\right)\right]\right\} = 0 \tag{A 2}$$

If $|g| \geq 1$, we have $r - |\alpha| \leq |rg + \alpha|$. From equation (2.7 a), the following inequality (A 3) can be achieved:

$$\frac{1}{r}\left[\|\Delta A\| + \sum_{i=1}^{n}(\|A_{di}\| + \|\Delta A_{di}\|)|rg+\alpha|^{-h_i}\right] < \rho\left(\frac{A-\alpha I}{r}\right) \quad \text{for } |g| \geq 1 \tag{A 3}$$

$$\Rightarrow \left\|\frac{1}{r}\left[\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(rg+\alpha)^{-h_i}\right]\right\| < \rho\left(\frac{A-\alpha I}{r}\right) \quad \text{for } |g| \geq 1 \tag{A 4}$$

Since all the eigenvalues of A are within the disk $D(\alpha, r)$, all the eigenvalues of $(A - \alpha I)/r$ lie inside the unit circle. Hence, from (A 4) and Lemma 1, all eigenvalues of the matrix

$$\frac{A-\alpha I}{r} + \frac{1}{r}\left[\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(rg+\alpha)^{-h_i}\right] \tag{A 5}$$

remain inside the unit circle for $|g| \geq 1$. That is,

$$\left|\lambda\left\{\frac{A-\alpha I}{r} + \frac{1}{r}\left[\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(rg+\alpha)^{-h_i}\right]\right\}\right| < 1 \quad \text{for } |g| \geq 1 \tag{A 6}$$

This implies that

$$|g| \neq \left|\lambda\left\{\frac{A-\alpha I}{r} + \frac{1}{r}\left[\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(rg+\alpha)^{-h_i}\right]\right\}\right| \quad \text{for } |g| \geq 1 \tag{A 7}$$

In view of (A 7), all solutions of the characteristic equation (A 2) satisfy $|g| < 1$ (i.e. $|(z-\alpha)/r| < 1$). This completes the proof of case (I).

(II): If the system (2.1) is not $D(\alpha, r)$-stable, then there exists a solution $\hat{g}$ of the characteristic equation (A 2) satisfying

$$|\hat{g}| = \left|\lambda\left\{\frac{A-\alpha I}{r} + \frac{1}{r}\left[\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(r\hat{g}+\alpha)^{-h_i}\right]\right\}\right| \geq 1 \tag{A 8}$$

On the basis of Lemma 1 and equation (A 8), the following inequality is obtained:

$$\left\|\frac{1}{r}\left[\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(r\hat{g}+\alpha)^{-h_i}\right]\right\| \geq \rho\left(\frac{A-\alpha I}{r}\right) \tag{A 9}$$

Since

$$\begin{aligned}\left\|\frac{1}{r}\left[\Delta A + \sum_{i=1}^{n}(A_{di} + \Delta A_{di})(r\hat{g}+\alpha)^{-h_i}\right]\right\| &\leq \frac{1}{r}\left[\beta + \left\|\sum_{i=1}^{n} A_{di}(r\hat{g}+\alpha)^{-h_i}\right\| + \sum_{i=1}^{n}\eta(r-|\alpha|)^{-h_i}\right] \\ &\leq \frac{1}{r}\left[\beta + \sum_{i=1}^{n}(\|A_{di}\| + \eta)(r-|\alpha|)^{-h_i}\right]\end{aligned} \tag{A 10}$$

we have

$$d_s=\rho\left(\frac{A-\alpha I}{r}\right)\le\left\|\frac{1}{r}\left[\Delta A+\sum_{i=1}^{n}(A_{di}+\Delta A_{di})(rg+\alpha)^{-h_i}\right]\right\|$$
$$\le\frac{1}{r}\left[\beta+\left\|\sum_{i=1}^{n}A_{di}(rg+\alpha)^{-h_i}\right\|+\sum_{i=1}^{n}\eta(r-|\alpha|)^{-h_i}\right]=h(g)$$
$$\le\frac{1}{r}\left[\beta+\sum_{i=1}^{n}(\|A_{di}\|+\eta)(r-|\alpha|)^{-h_i}\right]=d_1\quad\text{for}\,|g|\ge 1 \tag{A 11}$$

Moreover, according to (A 8), we can obtain the following inequality:

$$1\le|\hat g|=\left|\lambda\left\{\frac{A-\alpha I}{r}+\frac{1}{r}\left[\Delta A+\sum_{i=1}^{n}(A_{di}+\Delta A_{di})(r\hat g+\alpha)^{-h_i}\right]\right\}\right|$$
$$\le\left\|\frac{A-\alpha I}{r}+\frac{1}{r}\left[\Delta A+\sum_{i=1}^{n}(A_{di}+\Delta A_{di})(r\hat g+\alpha)^{-h_i}\right]\right\|$$
$$\le\left\|\frac{A-\alpha I}{r}\right\|+\frac{1}{r}\left[\beta+\sum_{i=1}^{n}(\|A_{di}\|+\eta)(r-|\alpha|)^{-h_i}\right]=d_{1r} \tag{A 12}$$

This implies that, if the system (2.1) is not $D(\alpha,r)$-stable, then all the unstable poles of this system must be within the bounded region U_1. Hence, if the inequality (A 11) is not true (i.e. $h(g)$ does not lie inside the interval $[d_s,d_1]$) for all $g\in U_1$, then the system (2.1) is robustly $D(\alpha,r)$-stable. This completes the proof of case (II).

Appendix B. Proof of Theorem 2:

(I): Applying a z-transform to the closed-loop system (5.6) yields

$$zX_1(z)=\sum_{i=0}^{n}M_{1i}z^{-h_i}X_1(z)+\sum_{i=0}^{n}M_{2i}z^{-h_i}X_2(z)+xz_1(0) \tag{B 1a}$$

$$zX_2(z)=\sum_{i=0}^{n}M_{3i}z^{-h_i}X_1(z)+\sum_{i=0}^{n}M_{4i}z^{-h_i}X_2(z)+xz_2(0) \tag{B 1b}$$

in which $x_1(0)$ and $x_2(0)$ are the bounded initial conditions of the states $x_1(k)$ and $x_2(k)$, respectively. According to (B 1b), we have

$$X_2(z)=\left(zI-\sum_{i=0}^{n}M_{4i}z^{-h_i}\right)^{-1}\left(\sum_{i=0}^{n}M_{3i}z^{-h_i}\right)X_1(z)+z\left(zI-\sum_{i=0}^{n}M_{4i}z^{-h_i}\right)^{-1}x_2(0) \tag{B 2}$$

Substituting (B 2) into (B 1a), $X_1(z)$ is obtained as

$$X_1(z)=z\Phi^{-1}(z)\left(\sum_{i=0}^{n}M_{2i}z^{-h_i}\right)\left(zI-\sum_{i=0}^{n}M_{4i}z^{-h_i}\right)^{-1}x_2(0)+z\Phi^{-1}(z)x_1(0) \tag{B 3}$$

where

$$\Phi(z)\equiv zI-\sum_{i=0}^{n}M_{1z}z^{-h_i}-\left(\sum_{i=0}^{n}M_{2i}z^{-h_i}\right)\times\left(zI-\sum_{i=0}^{n}M_{4i}z^{-h_i}\right)^{-1}\left(\sum_{i=0}^{n}M_{3i}z^{-h_i}\right) \tag{B 4}$$

Since the fast closed-loop subsystem (4.9) is $D(\alpha,r)$-stable, and the basis of the fact that $M_{4i}=\Delta\bar{_{fi}}$†, all poles of the term

$$\left(zI-\sum_{i=0}^{n}M_{4i}z^{-h_i}\right)^{-1}$$

in (B 3) lie inside the disk $D(\alpha,r)$. Moreover, all poles of the term $\sum_{i=0}^{n}M_{2i}z^{-h_i}$ in (B 3) are $z=0$ which is also inside the disk $D(\alpha,r)$ ($\because r>|\alpha|$). Therefore, to let all poles of $X_1(z)$ be within the disk $D(\alpha,r)$ (and likewise those of $X_2(z)$), we need only to find the condition which guarantees that all the poles of $\Phi^{-1}(z)$ are within the disk $D(\alpha,r)$. Substituting (5.5 b) and (5.6 c) into (B4), we have

$$\Phi(z)=zI-\sum_{i=0}^{n}(A_{1i}+B_1k_{1i})z^{-h_i}-\left(\sum_{i=0}^{n}M_{2i}z^{-h_i}\right)\times\left(zI-\sum_{i=0}^{n}M_{4i}z^{-h_i}\right)^{-1}\left(\sum_{i=0}^{n}M_{3i}z^{-h_i}\right)$$
$$=zI-\sum_{i=0}^{n}\left\{A_{1i}+B_1\left[k_{si}-\varepsilon k_{fi}\left(I-\varepsilon\sum_{j=0}^{n}\tilde A_{2j}\right)^{-1}\times\left(\sum_{j=0}^{n}A_{2j}+B_2\sum_{j=0}^{n}k_{sj}\right)\right]\right\}z^{-h_i}-\left(\sum_{i=0}^{n}m_{2i}z^{-h_i}\right)$$

† The fact that $M_{4i}=\Delta\bar A_{fi}$ can be observed by comparing (4.9 b) with (5.6 c) and using the matrices defined in (3.7 b) and (5.5 c).

$$\times\left(zI-\sum_{i=0}^{n}M_{4i}z^{-h_i}\right)^{-1}\left(\sum_{i=0}^{n}M_{3i}z^{-h_i}\right)$$

$$=zI-\sum_{i=0}^{n}\left\{\left[A_{1i}+\varepsilon\left(\sum_{j=0}^{n}\tilde{A}_{1j}\right)\left(I-\varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1}A_{2i}\right]\right.$$

$$\left.+\left[B_1+\varepsilon\left(\sum_{j=0}^{n}\tilde{A}_{1j}\right)\left(I-\varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1}B_2\right]k_{si}\right\}z^{-h_i}$$

$$+\sum_{i=0}^{n}\left[\varepsilon\left(\sum_{j=0}^{n}\tilde{A}_{1j}\right)\left(I-\varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1}(A_{2i}+B_2k_{si})\right]z^{-h_i}$$

$$+\sum_{i=0}^{n}\left[\varepsilon B_1k_{fi}\left(I-\varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1}\right.$$

$$\left.\times\left(\sum_{j=0}^{n}A_{2j}+B_2\sum_{j=0}^{n}k_{sj}\right)\right]z^{-h_i}-\left(\sum_{i=0}^{n}M_{2i}z^{-h_i}\right)$$

$$\times\left(zI-\sum_{i=0}^{n}M_{4i}z^{-h_i}\right)^{-1}\left(\sum_{i=0}^{n}M_{3i}z^{-h_i}\right)$$

$$=zI-\sum_{i=0}^{n}(A_{si}+B_sk_{si})z^{-h_i}+R(z)$$

$$-\left(\sum_{i=0}^{n}M_{2i}z^{-h_i}\right)\left(zI-\sum_{i=0}^{n}M_{4i}z^{-h_i}\right)^{-1}\left(\sum_{i=0}^{n}M_{3i}z^{-h_i}\right)$$

see (3.5 *b*) and (3.5 *c*)

$$=\Lambda(z)\left\{I+\Lambda^{-1}(z)\left[R(z)-\left(\sum_{i=0}^{n}M_{2i}z^{-h_i}\right)\right.\right.$$

$$\left.\left.\times\left(zI-\sum_{i=0}^{n}M_{4i}z^{-h_i}\right)^{-1}\left(\sum_{i=0}^{n}M_{3i}z^{-h_i}\right)\right]\right\}$$

where

$$\Lambda(z)=zI-\sum_{i=0}^{n}(A_{si}+B_sk_{si})z^{-h_i}$$

and

$$R(z)=\sum_{i=0}^{n}\left[\varepsilon\left(\sum_{j=0}^{n}\tilde{A}_{1j}\right)\left(I-\varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1}\right.$$

$$\times(A_{2i}+B_2k_{si})+\varepsilon B_1k_{fi}\left(I-\varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1}$$

$$\left.\times\left(\sum_{j=0}^{n}A_{2j}+B_2k_{sj}\right)\right]z^{-h_i}$$

Hence, we have

$$\Phi^{-1}(z)=[I+\phi(z)]^{-1}\Lambda^{-1}(z)=\psi^{-1}(z)\Lambda^{-1}(z) \quad (\text{B}\,5)$$

where

$$\psi(z)\equiv I+\phi(z)$$

with

$$\phi(z)\equiv\Lambda^{-1}(z)\left[R(z)-\left(\sum_{i=0}^{n}M_{2i}z^{-h_i}\right)\right.$$

$$\left.\times\left(zI-\sum_{i=0}^{n}M_{4i}z^{-h_i}\right)^{-1}\left(\sum_{i=0}^{n}M_{3i}z^{-h_i}\right)\right]$$

Since the slow closed-loop system (4.2) is $D(\alpha,r)$-stable, the term $\Lambda^{-1}(z)$ in (B 5) has all poles lying inside the disk $D(\alpha,r)$. Consequently, if all poles of the term $\psi^{-1}(z)=[I+\phi(z)]^{-1}$ in (B 5) lie inside the disk $D(\alpha,r)$, we can guarantee that $\Phi^{-1}(z)$ has all poles lying inside the disk $D(\alpha,r)$.

Let $(z-\alpha)/r$ be replaced by a variable g (i.e. $z=rg+\alpha$); then the term $\psi^{-1}(z)$ becomes

$$\psi^{-1}(z)=[I+\phi(z)]^{-1}=[I+\phi(rg+\alpha)]^{-1}$$
$$=[I+\phi_g(g)]^{-1}\equiv\psi_g^{-1}(g) \quad (\text{B}\,6a)$$

where

$$\phi_g(g)=\Lambda_g^{-1}(g)\left[R_g(g)=\left(\sum_{i=0}^{n}M_{2i}(rg+\alpha)^{-h_i}\right)\right.$$

$$\times\left((rg+\alpha)I-\sum_{i=0}^{n}M_{4i}(rg+\alpha)^{-h_i}\right)^{-1}$$

$$\left.\times\left(\sum_{i=0}^{n}M_{3i}(rg+\alpha)^{-h_i}\right)\right] \quad (\text{B}\,6b)$$

with

$$\Lambda_g(g)=(rg+\alpha)I-\sum_{i=0}^{n}(A_{si}+B_sk_{si})(rg+\alpha)^{-h_i}$$

and

$$R_g(g)=\sum_{i=0}^{n}\left[\varepsilon\left(\sum_{j=0}^{n}\tilde{A}_{1j}\right)\left(I-\varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1}\right.$$

$$\times(A_{2i}+B_2k_{si})+\varepsilon B_1k_{fi}\left(I-\varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1}$$

$$\left.\times\left(\sum_{j=0}^{n}A_{2j}+B_2k_{sj}\right)\right](rg+\alpha)^{-h_i}$$

If the following inequality holds:

$$|\det\psi_g(g)|=|\det[I+\phi_g(g)]|>0 \qquad \forall|g|\geq 1 \quad (\text{B}\,7)$$

(i.e. all the poles of $\psi_{\tilde{g}}^{-1}(g)$ are within the unit disk), then all the poles of $\psi^{-1}(z)$ lie inside the disk $D(\alpha, r)$. Let $g = \tilde{g}^{-1}$; then $\psi_s(g)$ becomes

$$\psi_g(g) = \psi_g(\tilde{g}^{-1}) = I + \phi_g(\tilde{g}^{-1}) = I + \tilde{\phi}_{\tilde{g}}(\tilde{g}) \equiv \tilde{\psi}_{\tilde{g}}(\tilde{g}) \quad \text{(B 8}a\text{)}$$

where

$$\tilde{\phi}_{\tilde{g}}(\tilde{g}) = \tilde{\Lambda}_{\tilde{g}}^{-1}(\tilde{g})\left[\tilde{R}_{\tilde{g}}(\tilde{g}) - \left(\sum_{i=0}^{n} M_{2i}(r\tilde{g}^{-1}+\alpha)^{-h_i}\right) \times \left((r\tilde{g}^{-1}+\alpha)I - \sum_{i=0}^{n} M_{4i}(r\tilde{g}^{-1}+\alpha)^{-h_i}\right)^{-1} \times \left(\sum_{i=0}^{n} M_{3i}(r\tilde{g}^{-1}+\alpha)^{-h_i}\right)\right] \quad \text{(B 8}b\text{)}$$

with

$$\tilde{\Lambda}_{\tilde{g}}(\tilde{g}) = (r\tilde{g}^{-1}+\alpha)I - \sum_{i=0}^{n}(A_{si} + B_s k_{si})(r\tilde{g}^{-1}+\alpha)^{-h_i}$$

and

$$\tilde{R}_{\tilde{g}}(\tilde{g}) = \sum_{i=0}^{n}\left[\varepsilon\left(\sum_{j=0}^{n}\tilde{A}_{1j}\right)\left(I - \varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1} \times (A_{2i} + B_2 k_{si}) + \varepsilon B_1 k_{fi}\left(I - \varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1} \times \left(\sum_{j=0}^{n} A_{2j} + B_2 k_{sj}\right)\right](r\tilde{g}^{-1}+\alpha)^{-h_i}$$

Therefore, the examination of (B 7) is equivalent to investigating the following inequality:

$$|\det\tilde{\psi}_{\tilde{g}}(\tilde{g})| = |\det[I + \tilde{\phi}_{\tilde{g}}(\tilde{g})]| > 0 \qquad \forall|\tilde{g}| \le 1 \quad \text{(B 9)}$$

Introducing the singular perturbation parameter ε into (B 8b), $\tilde{\phi}_{\tilde{g}}(\tilde{g})$ can then be rewritten as

$$\tilde{\phi}_{\tilde{g}}(\varepsilon, \tilde{g}) \equiv \tilde{\Lambda}_{\tilde{g}}^{-1}(\varepsilon, \tilde{g})\left[\tilde{R}_{\tilde{g}}(\varepsilon, \tilde{g}) - \left(\sum_{i=0}^{n} M_{2i}(\varepsilon)(r\tilde{g}^{-1}+\alpha)^{-h_i}\right) \times \left((r\tilde{g}^{-1}+\alpha)I - \sum_{i=0}^{n} M_{4i}(\varepsilon)(r\tilde{g}^{-1}+\alpha)^{-h_i}\right)^{-1} \times \left(\sum_{i=0}^{n} M_{3i}(\varepsilon)(r\tilde{g}^{-1}+\alpha)^{-h_i}\right)\right] \quad \text{(B 10)}$$

with

$$\tilde{\Lambda}_{\tilde{g}}(\varepsilon, \tilde{g}) = (r\tilde{g}^{-1}+\alpha)I - \sum_{i=0}^{n}(A_{si}(\varepsilon) + B_s(\varepsilon)k_{si})(r\tilde{g}^{-1}\alpha)^{-h_i}$$

and

$$\tilde{R}_{\tilde{g}}(\varepsilon, \tilde{g}) = \sum_{i=0}^{n}\left[\varepsilon\left(\sum_{j=0}^{n}\tilde{A}_{1j}\right)\left(I - \varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1}(A_{2i} + B_2 k_{si}) + \varepsilon B_1 k_{fi}\left(I - \varepsilon\sum_{j=0}^{n}\tilde{A}_{2j}\right)^{-1}\left(\sum_{j=0}^{n} A_{2j} + B_2 k_{sj}\right)\right] \times (r\tilde{g}^{-1}+\alpha)^{-h_i}$$

Since all poles of the term $(zI - \sum_{i=0}^{n} M_{4i} z^{-h_i})^{-1}$ in (B 3) and the term $\Lambda^{-1}(z)$ in (B 5) lie inside the disk $D(\alpha, r)$, the terms $[(rg+\alpha)I - \sum_{i=0}^{n} M_{4i}(rg+\alpha)^{-h_i}]^{-1}$ and $\Lambda_g^{-1}(g)$ in (B 6b) do not have any pole lying inside the region $|g| \ge 1$. Moreover, the term $R_g(g)$ in (B 6b) does not have any pole lying inside the region $|g| \ge 1$ ($\because$ the multiple poles of $R_g(g)$ are at $g = -\alpha/r$ and $r > |\alpha|$) either. We can then conclude that $\phi_g(g)$ does not have any pole lying inside the region $|g| \ge 1$. Consequently, $\tilde{\phi}_{\tilde{g}}(\varepsilon, \tilde{g}) = \tilde{\phi}_{\tilde{g}}(\tilde{g}) = \phi_g(\tilde{g}^{-1})$ has no poles lying inside the region $|\tilde{g}| \le 1$ and the function $\lambda_i[\tilde{\phi}_{\tilde{g}}(\varepsilon, \tilde{g})]$† is hence analytic and continuous in the bounded domain $|\tilde{g}| \le 1$. Therefore, if (5.7 a) holds,

$$\text{i.e.} \quad \sigma[\tilde{\phi}_{\tilde{g}}(\varepsilon, \tilde{g})] = \max_i |\lambda_i[\tilde{\phi}_{\tilde{g}}(\varepsilon, \tilde{g})]| < 1 \qquad \forall|\tilde{g}| = 1 \quad \text{(B 11)}$$

then we have (according to Lemma 4)

$$\sigma[\tilde{\phi}_{\tilde{g}}(\varepsilon, \tilde{g})] < 1 \qquad \forall|\tilde{g}| \le 1 \quad \text{(B 12)}$$

On the basis of (B 12) and Lemma 3, the following inequality is obtained:

$$|\det\tilde{\psi}_{\tilde{g}}(\tilde{g})| = |\det[I + \tilde{\phi}_{\tilde{g}}(\tilde{g})]| = |\det[I + \tilde{\phi}_{\tilde{g}}(\varepsilon, \tilde{g})]| > 0 \qquad \forall|\tilde{g}| \le 1 \quad \text{(B 13)}$$

and then the inequality (B 9), or equivalently (B 7), is fulfilled. This implies that the closed-loop system (5.6) is stable, thus completing the proof of case (I).

(II): Using the matrix inversion formula

$$(I + \Sigma\Pi)^{-1} = I - \Sigma(I + \Pi\Sigma)^{-1}\Pi \quad \text{(B 14)}$$

the function $\psi^{-1}(z)$ in (B 5) can be rewritten as

† The notation $\lambda_i\ (A)$ denotes the eigenvalue of the matrix A.

$$\psi^{-1}(z) = \left\{ I + \Lambda^{-1}(z) \left[R(z) - \left(\sum_{i=0}^{n} M_{2i} z^{-h_i} \right) \right.\right.$$

$$\left.\left. \times \left(zI = \sum_{i=0}^{n} M_{4i} z^{-h_i} \right)^{-1} \left(\sum_{i=0}^{n} M_{3i} z^{-h_i} \right) \right] \right\}^{-1}$$

$$= I - \Lambda^{-1}(z)[I + \bar{\phi}(z)]^{-1}$$

$$\times \left[R(z) - \left(\sum_{i=0}^{n} M_{2i} z^{-h_i} \right) \left(zI - \sum_{i=0}^{n} M_{4i} z^{-h_i} \right)^{-1} \right.$$

$$\left. \times \left(\sum_{i=0}^{n} M_{3i} z^{-h_i} \right) \right] \quad \text{(B 15)}$$

where

$$\bar{\phi}(z) \equiv \left[R(z) - \left(\sum_{i=0}^{n} M_{2i} z^{-h_i} \right) \left(zI - \sum_{i=0}^{n} M_{4i} z^{-h_i} \right)^{-1} \right.$$

$$\left. \times \left(\sum_{i=0}^{n} m_{3i} z^{-h_i} \right) \right] \Lambda^{-1}(z) \quad \text{(B 16)}$$

Since all poles of the matrices

$$\left[R(z) - \left(\sum_{i=0}^{n} M_{2i} z^{-h_i} \right) \left(zI - \sum_{i=0}^{n} M_{4i} z^{-h_i} \right)^{-1} \left(\sum_{i=0}^{n} M_{3i} z^{-h_i} \right) \right]$$

and $\Lambda^{-1}(z)$ lie inside the disk $D(\alpha, r)$, therefore, to ensure the $D(\alpha, r)$-stability of the system (5.6), we need only to find a condition which guarantes that all the poles of $[I + \bar{\phi}(z)]^{-1}$ are within the disk $D(\alpha, r)$. Following the same procedure as that in case (I), the proof of case (II) is thereby completed.

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