

$n \in N$ $W_2(t) = 0$ for all $t \in [t_0, t_n]$, and at time t_n queue 1 forms a homogeneous layer of size w_n and composition $a_1 = 1$, $a_{J+3} = x_n$. Moreover, $x_n \in \mathcal{I}$, hence $w_n \leq \theta^n w_0$ for some $\theta < 1$, and t_n tends to a finite limit, which completes the proof.

APPENDIX B

PROOF OF PROPOSITION 2.3

Set $b = \sum_{s=3}^{J+2} \frac{1}{c^{s-1}}$. Assume that for some j , $0 \leq j < J$, the following property (\mathcal{P}_j) is satisfied: $W_1(t_j) = 0$, $W_2(t_j) > 0$, $Q_{j+2}(t_j) = Q_2(t_0)$, and for all $t \in [t_j, t_j + W_2(t_j)]$ (t regular)

$$\begin{aligned} \frac{\dot{D}_2(t)}{c^{s-2}(1+b\delta)^j} &\leq \dot{D}_s(t) \leq \frac{\dot{D}_2(t)}{c^{s-2}}, & 2 \leq s \leq j+2 \\ \dot{D}_s(t) &\leq c^{J-(s-2)} \dot{D}_2(t), & j+3 \leq s \leq J+2. \end{aligned}$$

For $t \in [t_j, t_j + W_2(t_j)]$ (t regular), $2 \leq s \leq J+2$

$$\dot{A}_{s+1}(t) = \dot{D}_s(t) = \frac{\dot{D}_s(t)}{c\dot{D}_2(t) + \delta \sum_{s'=3}^{J+2} \dot{D}_{s'}(t)} \quad \text{[by (8)].}$$

Our assumption yields on one hand, for $2 \leq s \leq j+2$

$$\begin{aligned} \dot{A}_{s+1}(t) &\geq \frac{1}{c^{s-2}(1+b\delta)^j} \\ &\quad \times \frac{1}{c + \delta \sum_{s'=3}^{j+2} 1/c^{s'-2} + \delta \sum_{s'=j+3}^{J+2} c^{J-(s'-2)}} \\ &\geq \frac{1}{c^{s-2}(1+b\delta)^j} \frac{1}{c + \delta \sum_{s'=3}^{J+2} 1/c^{s'-2}} \\ &= \frac{1}{c^{(s+1)-2}(1+b\delta)^{j+1}}. \end{aligned} \tag{15}$$

On the other hand, we get

$$\dot{A}_{s+1}(t) \leq \begin{cases} \frac{1}{c^{s-2}} \frac{1}{c} = \frac{1}{c^{(s+1)-2}}, & 2 \leq s \leq j+2, \\ \frac{1}{c^{J-(s-2)}} \frac{1}{c} = c^{J-[(s+1)-2]}, & j+2 < s \leq J+2. \end{cases} \tag{16}$$

A finer estimation for $s = J+2$ yields $\dot{A}_{J+3}(t) \leq 1/(c+\delta)$, hence $\delta \dot{A}_1(t) + c \dot{A}_{J+3}(t) \leq \delta + c/(c+\delta) < 1$ (the latter inequality being equivalent to $c+\delta < 1$). So by Lemma 1.1, for $t \in [t_j, t_j + W_2(t_j)]$: $W_1(t) = 0$, hence

$$\dot{A}_2(t) = \dot{A}_1(t) = 1 \tag{17}$$

and $W_1(t_{j+1}) = 0$. Finally, since $t \mapsto t + W_2(t)$ maps $[t_j, t_j + W_2(t_j)]$ onto $[t_{j+1}, t_{j+1} + W_2(t_{j+1})]$, and for $t \in [t_j, t_j + W_2(t_j)]$ (t regular)

$$\frac{\dot{D}_s(t + W_2(t))}{\dot{D}_2(t + W_2(t))} = \frac{\dot{A}_s(t)}{\dot{A}_2(t)}, \quad 2 \leq s \leq J+2 \quad \text{[by (6)]}$$

formulas (15)–(17) show that property (\mathcal{P}_{j+1}) is induced by (\mathcal{P}_j) . [that $Q_{j+3}(t_{j+1}) = Q_{j+2}(t_j)$ follows directly from $t_{j+1} = t_j + W_2(t_j)$]. Since (\mathcal{P}_0) is valid by assumption, a straightforward induction completes the proof of Proposition 2.3.

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Optimal Solution of the Two-Stage Kalman Estimator

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Abstract—The two-stage Kalman estimator was originally proposed to reduce the computational complexity of the augmented state Kalman filter. Recently, it was also applied to the tracking of maneuvering targets by treating the target acceleration as a bias term. Except in certain restrictive conditions, the conventional two-stage estimators are suboptimal in the sense that they are not equivalent to the augmented state filter. In this paper, the authors propose a new two-stage Kalman estimator, i.e., new structure, which is an extension of Friedland’s estimator and is optimal in general conditions. In addition, we provide some analytic results to demonstrate the computational advantages of two-stage estimators over augmented ones.

Index Terms—Augmented state Kalman filter, bias-free filter, dynamical bias, optimal filter, two-stage Kalman estimator.

I. INTRODUCTION

Consider the problem of estimating the state of a dynamic system in the presence of a dynamical bias. It is common to treat the bias as part of the system state and then estimate the bias as well as the system state. This leads to an *augmented state Kalman filter* (ASKF) whose implementation can be computationally intensive. To reduce the computational cost, Friedland [1] proposed to employ the two-stage Kalman estimator to decouple the augmented filter into two parallel reduced-order filters. In recent years, the computational efficiency of the two-stage estimator is also appreciated when it is used to address the maneuvering target tracking problem, in which the target acceleration is treated as a random bias [14]. While Friedland’s decomposition is optimal for the case of a constant bias, it is suboptimal for a random/dynamical bias unless an algebraic constraint on the statistics of the bias process is satisfied [10], [12]. Since this

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algebraic constraint is seldom satisfied for practical systems, the two-stage Kalman estimator cannot exactly implement the ASKF. The motivation for our work is generalization of the two-stage structure to recover the optimal performance when the bias is a random process.

Here we review some previous works. After [1], many researchers have also contributed in this area, e.g., Tacker *et al.* [2], Tanaka [4], Mendel *et al.* [6], and Ignagni [7]. Recently, Ignagni [8] considered the case of a bias driven by a white noise which is uncorrelated with the system noise. However, the result he obtained is suboptimal. In [12], Alouani *et al.* considered a random bias in which the bias noise is correlated with the system noise. It was proved that under an algebraic constraint on the correlation between the system noise and the bias noise, the proposed two-stage Kalman estimator is optimal. Since almost all practical systems will not satisfy this algebraic constraint, they also concluded that all two-stage Kalman estimators are suboptimal. In [10], Alouani *et al.* extended the result of [12] to color noises. The two-stage Kalman estimator is also applied to the maneuvering target tracking problems (e.g., [9], [11], and [14]) and the nonlinear estimation problems (e.g., [3], [5], and [13]).

The objectives of this paper are to propose an *optimal two-stage Kalman estimator* (OTSKE) to evaluate its performance and to describe its applications. As shown in [12], the conventional two-stage Kalman estimator (CTSKE) is suboptimal unless an algebraic constraint is satisfied. Using the matrix transformation technique, we generalize the CTSKE to obtain the OTSKE, in which the algebraic constraint [12] is removed and the optimal performance is guaranteed. The OTSKE is optimal in the minimum mean square error (MMSE) sense, and this is verified in a theorem by proving that it is equivalent to the ASKF. This paper is organized as follows. In Section II, we state the problem of interest. In Section III, the OTSKE is derived for state estimation in the presence of a dynamical bias without any constraint. Performance and applications of the proposed OTSKE filter are given in Sections IV and V, respectively. Section VI is the conclusion. A detailed proof is provided in the Appendix.

II. STATEMENT OF THE PROBLEM

The problem of interest is described by the discretized equation set

$$X_{k+1} = A_k X_k + B_k \gamma_k + W_k^x \quad (1)$$

$$\gamma_{k+1} = C_k \gamma_k + W_k^\gamma \quad (2)$$

$$Y_k = H_k X_k + D_k \gamma_k + \eta_k \quad (3)$$

where $X_k \in R^n$ is the system state, $\gamma_k \in R^p$ is the dynamical bias, and $Y_k \in R^m$ is the measurement vector. Matrices A_k , B_k , C_k , D_k , and H_k are of appropriate dimensions with the assumption that C_k is nonsingular. The process noises W_k^x, W_k^γ and the measurement noise η_k are zero-mean white Gaussian sequences with the following covariances: $E[W_k^x (W_k^x)'] = Q_k^x \delta_{kl}$, $E[W_k^\gamma (W_k^\gamma)'] = Q_k^\gamma \delta_{kl}$, $E[W_k^x (W_k^\gamma)'] = Q_k^{x\gamma} \delta_{kl}$, $E[\eta_k (\eta_k)'] = R_k \delta_{kl}$, $E[W_k^x (\eta_k)'] = 0$, and $E[W_k^\gamma (\eta_k)'] = 0$, where $'$ denotes transpose. The initial states X_0 and γ_0 are assumed to be uncorrelated with the white noise sequences W_k^x, W_k^γ , and η_k . The initial conditions are assumed to be Gaussian random variables with $E[X_0] = \bar{X}_0$, $\text{Cov}\{X_0\} = P_0^x$, $E[\gamma_0] = \bar{\gamma}_0$, $\text{Cov}\{\gamma_0\} = P_0^\gamma > 0$, and $\text{Cov}\{X_0 \gamma_0'\} = P_0^{x\gamma}$.

Treating X_k and γ_k as the augmented system state, the ASKF is described by

$$\bar{X}_{k|k-1} = \bar{A}_{k-1} \bar{X}_{k-1|k-1} \quad (4)$$

$$X_{k|k}^a = X_{k|k-1}^a + K_k (Y_k - \bar{H}_k X_{k|k-1}^a) \quad (5)$$

$$P_{k|k-1} = \bar{A}_{k-1} P_{k-1|k-1} \bar{A}_{k-1}' + Q_{k-1} \quad (6)$$

$$K_k = P_{k|k-1} \bar{H}_k' \{ \bar{H}_k P_{k|k-1} \bar{H}_k' + R_k \}^{-1} \quad (7)$$

$$P_{k|k} = (I - K_k \bar{H}_k) P_{k|k-1} \quad (8)$$

where

$$X_{(\cdot)}^a = \begin{bmatrix} X_{(\cdot)} \\ \gamma_{(\cdot)} \end{bmatrix}, \quad K_k = \begin{bmatrix} K_k^x \\ K_k^\gamma \end{bmatrix}$$

$$P_{(\cdot)} = \text{Cov}\{X_{(\cdot)}^a\} = \begin{bmatrix} P_{(\cdot)}^x & P_{(\cdot)}^{x\gamma} \\ (P_{(\cdot)}^{x\gamma})' & P_{(\cdot)}^\gamma \end{bmatrix}$$

$$\bar{A}_k = \begin{bmatrix} A_k & B_k \\ 0 & C_k \end{bmatrix}, \quad \bar{H}_k = [H_k \quad D_k], \quad Q_k = \begin{bmatrix} Q_k^x & Q_k^{x\gamma} \\ (Q_k^{x\gamma})' & Q_k^\gamma \end{bmatrix}.$$

The computational cost of the ASKF increases with the augmented state dimension. Hence, the filter model (4)–(8) may be impractical to implement. The reason for this computational complexity is the extra computation of $P_{(\cdot)}^{x\gamma}$ terms. Therefore, if these $P_{(\cdot)}^{x\gamma}$ terms can be eliminated, we can reduce the complexity from implementational point of view. In the next section, we propose an optimal two-stage implementation of the above filter without explicitly calculating these $P_{(\cdot)}^{x\gamma}$ terms.

III. DERIVATION OF THE OPTIMAL TWO-STAGE KALMAN ESTIMATOR

The design of a new two-stage estimator is described as follows. First, form a *modified bias-free filter* by ignoring the bias term and by adding an external bias-compensating input. Second, take the bias into account and derive a *bias filter* to compensate the *modified bias-free filter* in order to reconstruct the original filter. These two filters are used to build a new algorithm which is equivalent to the ASKF. This new algorithm is named the OTSKE.

If the bias term is ignored ($\gamma = 0$), the bias-free filter is just a Kalman filter based on the model (1) and (3). Hence, the bias-free filter is given by

$$\bar{X}_{k|k-1} = A_{k-1} \bar{X}_{k-1|k-1} \quad (9)$$

$$\bar{X}_{k|k} = \bar{X}_{k|k-1} + \bar{K}_k^x (Y_k - H_k \bar{X}_{k|k-1}) \quad (10)$$

$$\bar{P}_{k|k-1} = A_{k-1} \bar{P}_{k-1|k-1} A_{k-1}' + Q_{k-1}^x \quad (11)$$

$$\bar{K}_k^x = \bar{P}_{k|k-1} H_k' \{ H_k \bar{P}_{k|k-1} H_k' + R_k \}^{-1} \quad (12)$$

$$\bar{P}_{k|k} = (I - \bar{K}_k^x H_k) \bar{P}_{k|k-1} \quad (13)$$

where $\bar{X}_{k|k}$ represents the estimate of the state process when the bias is ignored and $\bar{P}_{k|k}$ is the error covariance of $\bar{X}_{k|k}$. Accounting for the bias noise effect, we modify the bias-free filter by changing the predicted state and covariance equations, i.e., (9) and (11), into

$$\bar{X}_{k|k-1} = A_{k-1} \bar{X}_{k-1|k-1} + u_{k-1} \quad (14)$$

$$\bar{P}_{k|k-1} = A_{k-1} \bar{P}_{k-1|k-1} A_{k-1}' + \bar{Q}_{k-1}^x \quad (15)$$

where u_k , a new external input, and \bar{Q}_k^x , a new statistic for W_k^x , are yet to be determined. To distinguish this modified filter from Friedland's bias-free filter, the new filter [(14), (10), (15), (12), and (13)] is called the *modified bias-free filter*.

The *modified bias-free filter* \bar{X} can be corrected by adding a *bias filter*, denoted by $\{\bar{\gamma}, \bar{K}^\gamma, \bar{P}^\gamma\}$, to reconstruct the original filter. This creates the OTSKE filter which will later be presented as a linear combination of the estimates of the *modified bias-free filter* and the *bias filter*. The *bias filter* is derived in the following. First, we propose the following two-stage U - V transformation:

$$X_{k|k-1}^a = T(U_k) \bar{X}_{k|k-1}^a \quad (16)$$

$$X_{k|k}^a = T(V_k) \bar{X}_{k|k}^a \quad (17)$$

$$P_{k|k-1} = T(U_k) \bar{P}_{k|k-1} T'(U_k) \quad (18)$$

$$K_k = T(V_k) \bar{K}_k \quad (19)$$

$$P_{k|k} = T(V_k) \bar{P}_{k|k} T'(V_k) \quad (20)$$

where

$$\begin{aligned}\bar{X}_{(\cdot)}^a &= \begin{bmatrix} \bar{X}_{(\cdot)}^x \\ \bar{\gamma}_{(\cdot)} \end{bmatrix}, & \bar{K}_k &= \begin{bmatrix} \bar{K}_k^x \\ \bar{K}_k^\gamma \end{bmatrix} \\ \bar{P}_{(\cdot)} &= \begin{bmatrix} \bar{P}_{(\cdot)}^x & 0 \\ 0 & \bar{P}_{(\cdot)}^\gamma \end{bmatrix}, & T(M) &= \begin{bmatrix} I & M \\ 0 & I \end{bmatrix}\end{aligned}$$

and U_k and V_k are blending matrices defined by $U_k \equiv P_{k|k-1}^{x\gamma} (P_{k|k-1}^\gamma)^{-1}$ and $V_k \equiv P_{k|k}^{x\gamma} (P_{k|k}^\gamma)^{-1}$, respectively. The main advantage of using the T transformation is that the inverse transformation $T^{-1}(M) = T(-M)$ involves only a change of sign. Using this inverse transformation, (16)–(20) become

$$\bar{X}_{k|k-1}^a = T(-U_k)X_{k|k-1}^a \quad (21)$$

$$\bar{X}_{k|k}^a = T(-V_k)X_{k|k}^a \quad (22)$$

$$\bar{P}_{k|k-1} = T(-U_k)P_{k|k-1}T'(-U_k) \quad (23)$$

$$\bar{K}_k = T(-V_k)K_k \quad (24)$$

$$\bar{P}_{k|k} = T(-V_k)P_{k|k}T'(-V_k). \quad (25)$$

Next, based on the above [(16)–(20) and (21)–(25)], the *bias filter* can be obtained via the following *two-steps iterative substitution* method.

Step 1: Substituting (4)–(8) into the right-hand side of (21)–(25), we have

$$\bar{X}_{k|k-1}^a = T(-U_k)\bar{A}_{k-1}X_{k-1|k-1}^a \quad (26)$$

$$\bar{X}_{k|k}^a = T(-V_k)(X_{k|k-1}^a + K_k(Y_k - \bar{H}_k X_{k|k-1}^a)) \quad (27)$$

$$\begin{aligned}\bar{P}_{k|k-1} &= T(-U_k)(\bar{A}_{k-1}P_{k-1|k-1}\bar{A}_{k-1}' + Q_{k-1})T'(-U_k) \\ &\quad (28)\end{aligned}$$

$$\bar{K}_k = T(-V_k)P_{k|k-1}\bar{H}_k' \{ \bar{H}_k P_{k|k-1}\bar{H}_k' + R_k \}^{-1} \quad (29)$$

$$\bar{P}_{k|k} = (T(-V_k) - \bar{K}_k\bar{H}_k)P_{k|k-1}T'(-V_k). \quad (30)$$

Step 2: Substituting (16)–(20) into the right-hand side of (26)–(30), we have

$$\bar{X}_{k|k-1}^a = T(-U_k)\bar{A}_{k-1}T(V_{k-1})\bar{X}_{k-1|k-1}^a \quad (31)$$

$$\begin{aligned}\bar{X}_{k|k}^a &= T(U_k - V_k)\bar{X}_{k|k-1}^a + \bar{K}_k(Y_k - \bar{H}_k T(U_k)\bar{X}_{k|k-1}^a) \\ &\quad (32)\end{aligned}$$

$$\begin{aligned}\bar{P}_{k|k-1} &= T(-U_k)(\bar{A}_{k-1}T(V_{k-1})\bar{P}_{k-1|k-1} \\ &\quad \times T'(V_{k-1})\bar{A}_{k-1}' + Q_{k-1})T'(-U_k) \\ &\quad (33)\end{aligned}$$

$$\begin{aligned}\bar{K}_k &= T(U_k - V_k)\bar{P}_{k|k-1}T'(U_k)\bar{H}_k' \\ &\quad \times \{ \bar{H}_k T(U_k)\bar{P}_{k|k-1}T'(U_k)\bar{H}_k' + R_k \}^{-1} \\ &\quad (34)\end{aligned}$$

$$\bar{P}_{k|k} = (T(U_k - V_k) - \bar{K}_k\bar{H}_k T(U_k))\bar{P}_{k|k-1}T'(U_k - V_k). \quad (35)$$

Using (33), (35), and the decoupled structure of $\bar{P}_{(\cdot)}$, we obtain the following constraints of U_k and V_k :

$$\begin{aligned}0 &= \bar{U}_k C_{k-1} \bar{P}_{k-1|k-1}^\gamma C_{k-1}' + Q_{k-1}^\gamma \\ &\quad - U_k (C_{k-1} \bar{P}_{k-1|k-1}^\gamma C_{k-1}' + Q_{k-1}^\gamma) \\ &\quad (36)\end{aligned}$$

$$0 = \bar{U}_k - V_k - \bar{K}_k^x S_k \quad (37)$$

where \bar{U}_k and S_k are defined as

$$\bar{U}_k = (A_{k-1}V_{k-1} + B_{k-1})C_{k-1}^{-1} \quad (38)$$

$$S_k = H_k U_k + D_k. \quad (39)$$

Lastly, we obtain the *bias filter*, by expanding (31)–(35) and using (36)–(39) as

$$\bar{\gamma}_{k|k-1} = C_{k-1}\bar{\gamma}_{k-1|k-1} \quad (40)$$

$$\bar{\gamma}_{k|k} = \bar{\gamma}_{k|k-1} + \bar{K}_k^\gamma (Y_k - H_k \bar{X}_{k|k-1} - S_k \bar{\gamma}_{k|k-1}) \quad (41)$$

$$\bar{P}_{k|k-1}^\gamma = C_{k-1}\bar{P}_{k-1|k-1}^\gamma C_{k-1}' + Q_{k-1}^\gamma \quad (42)$$

$$\bar{K}_k^\gamma = \bar{P}_{k|k-1}^\gamma S_k' \{ H_k \bar{P}_{k|k-1}^x H_k' + R_k + S_k \bar{P}_{k|k-1}^\gamma S_k' \}^{-1} \quad (43)$$

$$\bar{P}_{k|k}^\gamma = (I - \bar{K}_k^\gamma S_k) \bar{P}_{k|k-1}^\gamma \quad (44)$$

where $\bar{X}_{k|k-1}$ and $\bar{P}_{k|k-1}^x$ are given by (14) and (15), respectively. The U_k and V_k are obtained by solving (36) and (37) and using (42) as

$$U_k = \bar{U}_k + (Q_{k-1}^\gamma - \bar{U}_k Q_{k-1}^\gamma) (\bar{P}_{k|k-1}^\gamma)^{-1} \quad (45)$$

$$V_k = U_k - \bar{K}_k^x S_k. \quad (46)$$

The external input u_k and the error covariance matrix \bar{Q}_k^x of the *modified bias-free filter* are obtained from (31), (33), (36), and (38) as

$$u_k = (\bar{U}_{k+1} - U_{k+1})C_k \bar{\gamma}_{k|k} \quad (47)$$

$$\bar{Q}_k^x = Q_k^x - Q_k^{x\gamma} \bar{U}_{k+1}' - U_{k+1} (Q_k^{x\gamma} - \bar{U}_{k+1} Q_k^\gamma)'. \quad (48)$$

From (47) and (48), it is clear that the difference between the *modified bias-free filter* and the bias-free filter is that the former is coupled with the *bias filter*, while the latter is decoupled from the *bias filter*. These coupled terms exist in the calculations of the blending matrix U_k , the external input u_k , and the error covariance matrix \bar{Q}_k^x [see (45), (47), and (48)]. Although these coupled terms would increase the computational load, in Section IV we will verify that actually the computation of the *modified bias-free filter* is only mildly increased over the bias-free filter.

Now, we are in the position to define the OTSKE based on the outputs of the *modified bias-free filter* and the *bias filter*

$$\hat{X}_{k|k-1} = \bar{X}_{k|k-1} + U_k \bar{\gamma}_{k|k-1} \quad (49)$$

$$\hat{X}_{k|k} = \bar{X}_{k|k} + V_k \bar{\gamma}_{k|k} \quad (50)$$

$$\begin{aligned}P_{k|k-1}^{11} &= \bar{P}_{k|k-1}^x + U_k \bar{P}_{k|k-1}^\gamma U_k' \\ &= E[(X_k - \hat{X}_{k|k-1})(X_k - \hat{X}_{k|k-1})'] \\ &\quad (51)\end{aligned}$$

$$\begin{aligned}P_{k|k}^{11} &= \bar{P}_{k|k}^x + V_k \bar{P}_{k|k}^\gamma V_k' \\ &= E[(X_k - \hat{X}_{k|k})(X_k - \hat{X}_{k|k})'] \\ &\quad (52)\end{aligned}$$

$$\begin{aligned}P_{k|k-1}^{12} &= U_k \bar{P}_{k|k-1}^\gamma \\ &= E[(X_k - \hat{X}_{k|k-1})(\gamma_k - \bar{\gamma}_{k|k-1})'] \\ &\quad (53)\end{aligned}$$

$$P_{k|k}^{12} = V_k \bar{P}_{k|k}^\gamma = E[(X_k - \hat{X}_{k|k})(\gamma_k - \bar{\gamma}_{k|k})'] \quad (54)$$

$$\begin{aligned}P_{k|k-1}^{22} &= \bar{P}_{k|k-1}^\gamma \\ &= E[(\gamma_k - \bar{\gamma}_{k|k-1})(\gamma_k - \bar{\gamma}_{k|k-1})'] \\ &\quad (55)\end{aligned}$$

$$P_{k|k}^{22} = \bar{P}_{k|k}^\gamma = E[(\gamma_k - \bar{\gamma}_{k|k})(\gamma_k - \bar{\gamma}_{k|k})'] \quad (56)$$

with the following initial conditions:

$$\begin{aligned}V_0 &= P_0^{x\gamma} (P_0^\gamma)^{-1}, & \bar{X}_{0|0} &= \bar{X}_0 - V_0 \bar{\gamma}_0, & \bar{\gamma}_{0|0} &= \bar{\gamma}_0 \\ \bar{P}_{0|0}^x &= P_0^x - V_0 P_0^\gamma V_0', & \bar{P}_{0|0}^\gamma &= P_0^\gamma.\end{aligned} \quad (57)$$

The structure of the proposed OTSKE filter is shown in Fig. 1.

The OTSKE is optimal in the MMSE sense as stated in the following theorem. The proof of the theorem, which is given in the Appendix, shows that the OTSKE is equivalent to the ASKF.

TABLE I
 KALMAN ESTIMATOR ARITHMETIC OPERATION REQUIREMENTS

Variable	Number of Multiplications($M(n, m)$)	Number of Additions($A(n, m)$)
$x_{k k-1}$	n^2	$n^2 - n$
$P_{k k-1}^x$	$2n^3$	$2n^3 - n^2$
K_k^x	$n^2m + 2nm^2 + m^3$	$n^2m + 2nm^2 + m^3 - 2nm$
$x_{k k}$	$2nm$	$2nm$
$P_{k k}^x$	$n^3 + n^2m$	$n^3 + n^2m - n^2$
Totals	$3n^3 + 2n^2m + 2nm^2 + m^3 + n^2 + 2nm$	$3n^3 + 2n^2m + 2nm^2 + m^3 - n^2 - n$

 TABLE II
 AUXILIARY MATRICES ARITHMETIC OPERATION REQUIREMENTS FOR THE OTSKE

Variable	Number of Multiplications($M^o(n, p, m)$)	Number of Additions($A^o(n, p, m)$)
U_k	$n^2p + np^2 + p^3$	$n^2p + np^2 + p^3 - np$
U_k	$2np^2 + p^3$	$2np^2 + p^3$
S_k	npm	npm
V_k	npm	npm
\bar{Q}_{k-1}^x	$2n^2p$	$2n^2p$
u_{k-1}	$np + p^2$	$2np + p^2 - p$
Totals	$3n^2p + 3np^2 + 2npm + 2p^3 + np + p^2$	$3n^2p + 3np^2 + 2npm + 2p^3 + np + p^2 - p$

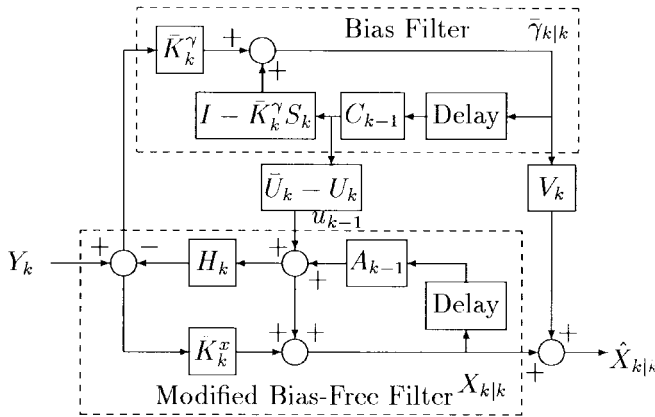


Fig. 1. Block diagram of the OTSKE.

Theorem: If the error covariance \bar{Q}_k^x of the process noise of the modified bias-free filter given by (48) is positive semidefinite, the OTSKE, which is given by (49)–(56), gives the MMSE estimate of the system state.

Note that the algebraic constraint of [12], i.e., $Q_k^{x\gamma} - \bar{U}_{k+1}Q_k^\gamma = 0$, is not required to guarantee the optimality of the proposed OTSKE filter. However, if this algebraic constraint is satisfied, the external input u_k will vanish and the error covariance matrix \bar{Q}_k^x becomes $Q_k^x - Q_k^{x\gamma}\bar{U}_{k+1}'$. Then, the *modified bias-free filter* will be identical to the bias-free filter of [12], and hence the OTSKE will be equivalent to the CTSKE [12]. Furthermore, if the bias is a constant one, the error covariance matrix \bar{Q}_k^x becomes Q_k^x . Then, the *modified bias-free filter* will be identical to the bias-free filter of [1], and hence the OTSKE filter will become Friedland's filter [1].

IV. PERFORMANCE EVALUATIONS

To demonstrate the computational advantage of the two-stage estimators over the ASKF, we use the number of arithmetic operations, i.e., multiplications and additions, as a measure of computational

complexity. To facilitate the discussion, we first list in Table I the arithmetic operation of a standard Kalman estimator which has state dimension n and measurement dimension m . The arithmetic operations of the auxiliary matrices specifically needed by the OTSKE and the CTSKE are shown in Tables II and III, respectively. It is clear from Table I that the arithmetic operations required for the ASKF are $M(n + p, m)$ for multiplications and $A(n + p, m)$ for additions, those for the bias-free filter are $M(n, m)$ and $A(n, m)$, and those for the bias filter are $M(p, m)$ and $A(p, m)$. The arithmetic operations required for the auxiliary matrices, from Tables II and III, are $M^o(n, p, m)$ and $A^o(n, p, m)$ for the OTSKE, and $M^c(n, p, m)$ and $A^c(n, p, m)$ for the CTSKE. Therefore, the operational savings of the OTSKE are

$$\begin{aligned} P_M^o &= M(n + p, m) - M(n, m) - M(p, m) - M^o(n, p, m) \\ &= 6n^2p + 6np^2 + (2m + 1)np - 2p^3 - p^2 \end{aligned} \quad (58)$$

$$\begin{aligned} P_A^o &= A(n + p, m) - A(n, m) - A(p, m) - A^o(n, p, m) \\ &= 6n^2p + 6np^2 + (2m - 3)np - 2p^3 - m^3 - p^2 + p \end{aligned} \quad (59)$$

and the operational savings of the CTSKE are

$$\begin{aligned} P_M^c &= M(n + p, m) - M(n, m) - M(p, m) - M^c(n, p, m) \\ &= 7n^2p + 8np^2 + 2npm + 2np - p^3 \end{aligned} \quad (60)$$

$$\begin{aligned} P_A^c &= A(n + p, m) - A(n, m) - A(p, m) - A^c(n, p, m) \\ &= 7n^2p + 8np^2 + 2npm - p^3 - m^3 - np. \end{aligned} \quad (61)$$

It is clear from (58)–(61) that the savings of the arithmetic operation of the proposed OTSKE and the CTSKE as opposed to the ASKF are approximately $12(n^2p + np^2)$ and $14(n^2p + np^2)$, respectively. Roughly speaking, the computational savings of the two-stage structure is due to system-order reduction from $n + p$ to n and p . The operational savings suggested here will be tested in Example 2 of Section V. Note that if parallel structure is employed, further reduction in the computation time can be achieved. However, this is not the issue of this paper.

TABLE III
AUXILIARY MATRICES ARITHMETIC OPERATION REQUIREMENTS FOR THE CTSKE

Variable	Number of Multiplications($M^c(n, p, m)$)	Number of Additions($A^c(n, p, m)$)
U_k	$n^2p + np^2 + p^3$	$n^2p + np^2 + p^3 - np$
S_k	npm	npm
V_k	npm	npm
\bar{Q}_{k-1}^x	n^2p	n^2p
Totals	$2n^2p + np^2 + 2npm + p^3$	$2n^2p + np^2 + 2npm + p^3 - np$

V. EXAMPLES

In this section, we demonstrate the applications of the proposed OTSKE to solve two problems which appeared in [8] and [4].

Example 1: A problem that appeared in [8] is described as follows. If the bias undergoes some random variation with time and the process noise is uncorrelated with the bias noise, then $Q_k^\gamma \neq 0$ and $Q_k^{x\gamma} = 0$. In this case, it can be proved that the CTSKE is not an optimal solution. However, if U_k , V_k , and \bar{Q}_k^x are calculated by (45), (46), and (48), respectively, then the OTSKE [(49)–(56)] will provide the optimal solution.

Example 2: A problem discussed in [4] is to try to extend Friedland's filter to a dynamical bias. However, the result obtained in [4], named as the parallel filtering algorithm (PFA), and the CTSKE both are not optimal solutions because these estimators are not equivalent to the ASKF. Instead, the proposed OTSKE will provide the optimal solution. To illustrate the performance degradation of the PFA and the CTSKE, we conduct the following target tracking simulation. Consider that a target maneuvers a slow 90° turn with acceleration $\ddot{x} = \ddot{y} = 0.075 \text{ m/s}^2$. The initial position and velocity of the target are given by $x(0) = 100 \text{ m}$, $\dot{x}(0) = 0 \text{ m/s}$, $y(0) = 100 \text{ m}$, and $\dot{y}(0) = -15 \text{ m/s}$. The sampling interval is $T = 10 \text{ s}$. The simulation time is 500 s . The Cartesian target position is measured. The process and measurement noise covariance matrices are

$$Q_k^x = \begin{bmatrix} 500 & 100 & 0 & 0 \\ 100 & 20 & 0 & 0 \\ 0 & 0 & 500 & 100 \\ 0 & 0 & 100 & 20 \end{bmatrix}, \quad Q_k^{x\gamma} = \begin{bmatrix} 10 & 0 \\ 2 & 0 \\ 0 & 10 \\ 0 & 2 \end{bmatrix}$$

$$Q_k^\gamma = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad R_k = \begin{bmatrix} 10^4 & 500 \\ 500 & 10^4 \end{bmatrix}.$$

All filters are initialized by taking the following values: $\bar{X}_0 = 0$, $\bar{\gamma}_0 = 0$, $P_0^x = Q_0^x$, $P_0^{x\gamma} = 0$, and $P_0^\gamma = Q_0^\gamma$. A Monte Carlo simulation of 50 runs (using Matlab) was performed. The simulation results in Table IV show the average root mean square tracking errors of the estimators in the X -axis and the Y -axis. The number of flops (using Matlab) counted for the filters is also included. It can be seen from Table IV that the performances of the ASKF and the OTSKE are the same, but the flops counted for the OTSKE are fewer than that of the ASKF. Although the flops counted for the CTSKE are fewer than that of the OTSKE, the estimates of the CTSKE are degraded. This performance degradation is due to the inherent suboptimality of the CTSKE filter. Note that if we substitute $n = 4$, $p = 2$, and $m = 2$ into (58)–(61) and use the fact that $\text{flops} = P_M + P_A$, the flops savings of the OTSKE and the CTSKE are 578 and 752, respectively. These results are very close to the simulation results which are 543 and 731, respectively.

VI. CONCLUSION

In this paper, the OTSKE is derived, and the complexity and performance of various estimators are analyzed and compared to show the advantage of the OTSKE. The OTSKE is mathematically

TABLE IV
PERFORMANCES OF THE ASKF, OTSKE, PFA, AND CTSKE FILTERS

Performance	ASKF	OTSKE	PFA	CTSKE
RMS X_Position Error	88.2110	88.2110	89.3026	99.2182
RMS X_Velocity Error	8.9598	8.9598	9.1833	16.2566
RMS Y_Position Error	92.4970	92.4970	94.2884	101.9359
RMS Y_Velocity Error	9.6452	9.6452	10.1775	17.8116
flops(one iteration)	2081	1538	3451	1350

equivalent to the ASKF without requiring the system constraint imposed on the CTSKE [12]. Another advantage of the OTSKE is that it is less computationally intensive than the ASKF. Furthermore, the OTSKE can be equivalent to the CTSKE if a system constraint is satisfied. Although the proposed OTSKE is slightly more complex than the CTSKE, it prevents the performance degradation inherent in the CTSKE. Therefore, the proposed OTSKE is the best balance between the performance of the ASKF and the efficiency of the CTSKE.

In order for the derived OTSKE filter to be stable, one necessary condition is that the *modified bias-free filter* covariance matrix \bar{Q}_k^x remain positive semidefinite for all time. This stability requirement is under investigation.

APPENDIX

Before proving the theorem, the following relationships are needed.

1) From (42) and (45)

$$\bar{U}_{k+1} C_k \bar{P}_{k|k}^\gamma C_k' = U_{k+1} \bar{P}_{k+1|k}^\gamma - Q_k^{x\gamma}. \quad (62)$$

2) From (12) and (43)

$$\bar{K}_k^x M_k = \bar{P}_{k|k-1}^x H_k' + \bar{K}_k^x S_k \bar{P}_{k|k-1}^\gamma S_k' \quad (63)$$

$$\bar{K}_k^\gamma M_k = \bar{P}_{k|k-1}^\gamma S_k' \quad (64)$$

where $M_k = H_k \bar{P}_{k|k-1}^x H_k' + S_k \bar{P}_{k|k-1}^\gamma S_k' + R_k$.

3) From (44), (63), and (64)

$$\begin{aligned} \bar{P}_{k|k}^\gamma S_k' (\bar{K}_k^x)' &= (\bar{P}_{k|k-1}^\gamma S_k' - \bar{K}_k^\gamma M_k) (\bar{K}_k^x)' + \bar{K}_k^\gamma H_k \bar{P}_{k|k-1}^x \\ &= \bar{K}_k^\gamma H_k \bar{P}_{k|k-1}^x. \end{aligned} \quad (65)$$

By inductive reasoning, assume that at time k

$$\begin{aligned} \bar{X}_{k|k} &= \hat{X}_{k|k}, & \gamma_{k|k} &= \bar{\gamma}_{k|k} \\ P_{k|k}^x &= P_{k|k}^{11}, & P_{k|k}^{x\gamma} &= P_{k|k}^{12}, & P_{k|k}^\gamma &= P_{k|k}^{22}. \end{aligned} \quad (66)$$

Using (4), (66), (50), (14), (47), (38), (40), and (49), we obtain

$$\begin{aligned} \bar{X}_{k+1|k} &= A_k (\bar{X}_{k|k} + V_k \bar{\gamma}_{k|k}) + B_k \bar{\gamma}_{k|k} \\ &= \bar{X}_{k+1|k} - (\bar{U}_{k+1} - U_{k+1}) C_k \bar{\gamma}_{k|k} + \bar{U}_{k+1} C_k \bar{\gamma}_{k|k} \\ &= \bar{X}_{k+1|k} + U_{k+1} \bar{\gamma}_{k+1|k} = \hat{X}_{k+1|k}. \end{aligned} \quad (67)$$

Using (4), (66), and (40), we obtain

$$\gamma_{k+1|k} = C_k \gamma_{k|k} = C_k \bar{\gamma}_{k|k} = \bar{\gamma}_{k+1|k}. \quad (68)$$

Using (6), (66), (52), (54), (56), (38), (15), (48), (62), (45), and (51), we obtain

$$\begin{aligned} P_{k+1|k}^x &= A_k(\bar{P}_{k|k}^x + V_k \bar{P}_{k|k}^\gamma V_k') A_k' + B_k \bar{P}_{k|k}^\gamma (A_k V_k + B_k)' \\ &\quad + A_k V_k \bar{P}_{k|k}^\gamma B_k' + Q_k^x \\ &= A_k \bar{P}_{k|k}^x A_k' + Q_k^x + \bar{U}_{k+1} C_k \bar{P}_{k|k}^\gamma C_k' \bar{U}_{k+1}' \\ &= \bar{P}_{k+1|k}^x + U_{k+1} (Q_k^{x\gamma} - \bar{U}_{k+1} Q_k^\gamma + \bar{U}_{k+1} \bar{P}_{k+1|k}^\gamma)' \\ &= \bar{P}_{k+1|k}^x + U_{k+1} \bar{P}_{k+1|k}^\gamma U_{k+1}' = P_{k+1|k}^{11}. \end{aligned} \quad (69)$$

Using (6), (66), (54), (56), (38), (62), and (53), we obtain

$$P_{k+1|k}^{x\gamma} = \bar{U}_{k+1} C_k \bar{P}_{k|k}^\gamma C_k' + Q_k^{x\gamma} = U_{k+1} \bar{P}_{k+1|k}^\gamma = P_{k+1|k}^{12}. \quad (70)$$

Using (6), (66), (42), and (55), we obtain

$$P_{k+1|k}^\gamma = C_k \bar{P}_{k|k}^\gamma C_k' + Q_k^\gamma = \bar{P}_{k+1|k}^\gamma = P_{k+1|k}^{22}. \quad (71)$$

Using (7), (69)–(71), (39), (46), (63), and (64), we obtain

$$\begin{aligned} K_{k+1}^x &= (\bar{P}_{k+1|k}^x H_{k+1}' + U_{k+1} \bar{P}_{k+1|k}^\gamma S_{k+1}') \{\bullet\}^{-1} \\ &= (\bar{P}_{k+1|k}^x H_{k+1}' + \bar{K}_{k+1}^x S_{k+1}' \bar{P}_{k+1|k}^\gamma S_{k+1}') M_{k+1}^{-1} \\ &\quad + V_{k+1} \bar{P}_{k+1|k}^\gamma S_{k+1}' M_{k+1}^{-1} \\ &= \bar{K}_{k+1}^x + V_{k+1} \bar{K}_{k+1}^\gamma \end{aligned} \quad (72)$$

where

$$\{\bullet\} = H_{k+1} (\bar{P}_{k+1|k}^x H_{k+1}' + U_{k+1} \bar{P}_{k+1|k}^\gamma S_{k+1}') + D_{k+1} \bar{P}_{k+1|k}^\gamma S_{k+1}' + R_{k+1}.$$

Using (7), (70)–(71), (39), and (64), we obtain

$$\bar{K}_{k+1}^\gamma = \bar{P}_{k+1|k}^\gamma S_{k+1}' M_{k+1}^{-1} = \bar{K}_{k+1|k}^\gamma. \quad (73)$$

Next, show that (66) holds at time $k+1$. Using (67), (68), and (39), we obtain

$$\begin{aligned} r_{k+1} &\equiv Y_{k+1} - H_{k+1} X_{k+1|k} - D_{k+1} \gamma_{k+1|k} \\ &= Y_{k+1} - H_{k+1} \bar{X}_{k+1|k} - S_{k+1} \bar{\gamma}_{k+1|k}. \end{aligned} \quad (74)$$

Using (5), (67), (72), (74), (10), (46), (41), and (50), we obtain

$$\begin{aligned} X_{k+1|k+1} &= \bar{X}_{k+1|k} + U_{k+1} \bar{\gamma}_{k+1|k} + (\bar{K}_{k+1}^x + V_{k+1} \bar{K}_{k+1}^\gamma) r_{k+1} \\ &= \bar{X}_{k+1|k} + \bar{K}_{k+1}^x (Y_{k+1} - H_{k+1} \bar{X}_{k+1|k}) \\ &\quad + (U_{k+1} - \bar{K}_{k+1}^x S_{k+1}') \bar{\gamma}_{k+1|k} + V_{k+1} \bar{K}_{k+1}^\gamma r_{k+1} \\ &= \bar{X}_{k+1|k+1} + V_{k+1} (\bar{\gamma}_{k+1|k} + \bar{K}_{k+1}^\gamma r_{k+1}) \\ &= \bar{X}_{k+1|k+1} + V_{k+1} \bar{\gamma}_{k+1|k+1} = \hat{X}_{k+1|k+1}. \end{aligned} \quad (75)$$

Using (5), (74), (68), (73), and (41), we obtain

$$\begin{aligned} \gamma_{k+1|k+1} &= \gamma_{k+1|k} + \bar{K}_{k+1}^\gamma r_{k+1} \\ &= \bar{\gamma}_{k+1|k} + \bar{K}_{k+1}^\gamma r_{k+1} = \bar{\gamma}_{k+1|k+1}. \end{aligned} \quad (76)$$

Using (8), (69), (70), (72), (39), (13), (46), (44), (65), and (52), we

obtain

$$\begin{aligned} P_{k+1|k+1}^x &= \bar{P}_{k+1|k}^x + U_{k+1} \bar{P}_{k+1|k}^\gamma U_{k+1}' - (\bar{K}_{k+1}^x + V_{k+1} \bar{K}_{k+1}^\gamma) \\ &\quad \times (H_{k+1} \bar{P}_{k+1|k}^x + S_{k+1} \bar{P}_{k+1|k}^\gamma U_{k+1}') \\ &= \bar{P}_{k+1|k+1}^x + V_{k+1} (I - \bar{K}_{k+1}^\gamma S_{k+1}') \bar{P}_{k+1|k}^\gamma U_{k+1}' \\ &\quad - V_{k+1} \bar{K}_{k+1}^\gamma H_{k+1} \bar{P}_{k+1|k}^x \\ &= \bar{P}_{k+1|k+1}^x + V_{k+1} \bar{P}_{k+1|k+1}^\gamma V_{k+1}' + V_{k+1} \\ &\quad \times (\bar{P}_{k+1|k+1}^\gamma S_{k+1}' (\bar{K}_{k+1}^x)' - \bar{K}_{k+1}^\gamma H_{k+1} \bar{P}_{k+1|k}^x) \\ &= \bar{P}_{k+1|k+1}^x + V_{k+1} \bar{P}_{k+1|k+1}^\gamma V_{k+1}' = P_{k+1|k+1}^{11}. \end{aligned} \quad (77)$$

Using (8), (70)–(72), (39), (46), (44), and (54), we obtain

$$\begin{aligned} P_{k+1|k+1}^{x\gamma} &= (U_{k+1} - \bar{K}_{k+1}^x S_{k+1}' - V_{k+1} \bar{K}_{k+1}^\gamma S_{k+1}') \bar{P}_{k+1|k}^\gamma \\ &= V_{k+1} (I - \bar{K}_{k+1}^\gamma S_{k+1}') \bar{P}_{k+1|k}^\gamma \\ &= V_{k+1} \bar{P}_{k+1|k+1}^\gamma = P_{k+1|k+1}^{12}. \end{aligned} \quad (78)$$

Using (8), (70), (71), (73), (39), (44), and (56), we obtain

$$P_{k+1|k+1}^\gamma = (I - \bar{K}_{k+1}^\gamma S_{k+1}') \bar{P}_{k+1|k}^\gamma = P_{k+1|k+1}^{22}. \quad (79)$$

Finally, show that (66) holds at time $k=0$. This can be verified by the initial parameters in (57).

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