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Generalized Diameters and Rabin Numbers of Networks

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Abstract. Reliability and efficiency are important criteria in the design of interconnection networks. Recently, the w-wide diameter $d_w(G)$, the $(w-1)$-fault diameter $D_w(G)$, and the w-Rabin number $r_w(G)$ have been used to measure network reliability and efficiency. In this paper, we study $d_w(G)$, $D_w(G)$ and $r_w(G)$ using the strong w-Rabin number $r^*_w(G)$ for $1 \leq w \leq k(G)$ and G is a circulant network $G(d^n; \{1, d, \ldots, d^{n-1}\})$, a d-ary cube network $C(d, n)$, a generalized hypercube $GH(m_{n-1}, \ldots, m_0)$, a folded hypercube $FH(n)$ or a WK-recursive network $WK(d, t)$.

Keywords: diameter, connectivity, Rabin number, circulant network, cube, hypercube, WK-recursive network

1. Introduction

Reliability and efficiency are important criteria in the design of interconnection networks. Connectivity is widely used to measure network fault-tolerance capacity, while diameter determines routing efficiency along individual paths. In practice, we are interested in high-connectivity, small-diameter networks.

By a network we mean a graph or a digraph. The *distance* $d_G(x, y)$ from a vertex x to another vertex y in a network G is the minimum number of edges of a (di)path from x to y. The *diameter* $d(G)$ of a network G is the maximum distance from one vertex to another. The *connectivity* $k(G)$ of a network G is the minimum number of vertices whose removal results in a disconnected or trivial network. According to the Menger's theorem, there exist k (internally) vertex-disjoint paths from a vertex x to another vertex y in a network of connectivity k. Throughout this paper, "vertex-disjoint" always means "internally vertex-disjoint."

For a network G with connectivity $k(G)$ and $w \leq k(G)$, the three parameters $d_w(G)$, $D_w(G)$ and $r_w(G)$ (defined below) arise from the study of parallel routing, fault-tolerant systems, and randomized routing, respectively (see (Hsu, 1994; Krishnamoorthy and Krishnamurthy, 1987; Liaw and Chang, 1998)). Due to widespread use of (and demand for) reliable, efficient and fault-tolerant networks, these three parameters have been the subjects of extensive study over the past decade (see (Hsu, 1994)).

The *w-wide diameter* $d_w(G)$ of a network G is the minimum l such that for any two distinct vertices x and y there exist w vertex-disjoint (di)paths of length at most l from

x to y. The notion of w-wide diameter was introduced by Hsu (1994) to unify the concepts of diameter and connectivity.

The $(w-1)$-*fault diameter* of G is $D_w(G) = \max\{d(G-S) : |S| \le w-1\}$. This notion was defined by Hsu (1994), and the special case in which $w = k(G)$ was first defined by Krishnamoorthy and Krishnamurthy (1987) who studied the fault-tolerant properties of graphs and networks.

The w-*Rabin number* $r_w(G)$ of a network G is the minimum l such that for any $w+1$ distinct vertices $x, y_1, \ldots, y_w$ there exist w vertex-disjoint (di)paths of length at most l from x to $y_1, y_2, \ldots, y_w$. This concept was first defined by Hsu (1994) and the special case in which $w = k(G)$ was studied by Rabin (1989) in conjunction with a randomized routing algorithm.

It is clear that when $w = 1$, $d_1(G) = D_1(G) = r_1(G) = d(G)$ for any network G. On the other hand, these parameters can be very large, as in the case in which $w = k(G)$. For example, Hsu and Luczak (1994) showed that $d_k(G) = \frac{n}{2}$ for some regular graph G having connectivity and degree k and n vertices. The following are basic properties and relationships among $d_w(G)$, $D_w(G)$, and $r_w(G)$.

Proposition 1 (Liaw et al., 1998). *The following statements hold for any network G of connectivity k.*
(1) $D_1(G) \le D_2(G) \le \cdots \le D_k(G)$.
(2) $d_1(G) \le d_2(G) \le \cdots \le d_k(G)$.
(3) $r_1(G) \le r_2(G) \le \cdots \le r_k(G)$.
(4) $D_w(G) \le d_w(G)$ *and* $D_w(G) \le r_w(G)$ *for* $1 \le w \le k$.

In this paper, we study w-wide diameters, w-fault diameters and w-Rabin numbers for a class of circulant networks, d-ary cube networks, generalized hypercube networks, folded hypercube networks and WK-recursive networks. The first two networks are digraphs and the other three are graphs. Partial results for these networks were obtained in (Duh and Chen, 1999). For technical reasons we need a more general concept we call a strong w-Rabin number. The *strong w-Rabin number* $r^*_w(G)$ of a network G is the minimum l such that for any $w+1$ (not necessarily distinct) vertices $x, y_1, \ldots, y_w$ there exist w vertex-disjoint (di)paths of length at most l from x to $y_1, y_2, \ldots, y_w$. Clearly, we have

Proposition 2. *The following statements hold for any network G of connectivity k.*
(1) $r^*_1(G) \le r^*_2(G) \le \cdots \le r^*_k(G)$.
(2) $d_w(G) \le r^*_w(G)$ *and* $r_w(G) \le r^*_w(G)$ *for* $1 \le w \le k$.

The main purpose of this paper is to determine $d_w(G)$, $D_w(G)$, $r_w(G)$ and $r^*_w(G)$ for the above-mentioned networks G and $1 \le w \le k(G)$.

2. Circulant networks

For a positive integer N, let Z_N be the additive group of residue classes modulo N. The *circulant digraph* $G(N; A)$ associated with N and a subset $A \subseteq Z_N - \{0\}$ is a digraph with a vertex set Z_N and an edge set $\{xy : x, y \in Z_N \text{ and } y - x \in A\}$. It is clear that $G(N; A)$ is a vertex-transitive digraph in which every vertex has an indegree and an outdegree equal to $|A|$.

This section studies $G(d^n; A)$ for $A = \{1, d, \ldots, d^{n-1}\}$ with $d \geq 2$ and $n \geq 1$. Hamidoune (1984) showed that $k(G(d^n; \{1, d, \ldots, d^{n-1}\})) = n$. It is easy to show that $d(G(d^n; \{1, d, \ldots, d^{n-1}\})) = n(d-1)$; in fact, $d(0, d^n - 1) = n(d-1)$ in this digraph. Hsu and Lyuu (1994) and Duh and Chen (1997) have previously shown:

Theorem 3 (Hsu and Lyuu, 1994; Duh and Chen, 1997). *If* $A = \{1, d, d^2, \ldots, d^{n-1}\}$, *then* $d_n(G(d^n; A)) = r_n(G(d^n; A)) = n(d-1) + 1$.

Moreover, Liaw et al. (1998) proved:

Theorem 4 (Liaw et al., 1998). *If* $A = \{1, d, d^2, \ldots, d^{n-1}\}$, *then*

$$D_w(G(d^n; A)) = d_w(G(d^n; A)) = \begin{cases} n(d-1), & \text{for } 1 \leq w \leq n-1, \\ n(d-1)+1, & \text{for } w = n. \end{cases}$$

For any vertex x in $G(d^n; A)$, we can write $x = x_{n-1}d^{n-1} + \cdots + x_1 d + x_0$, where $0 \leq x_i \leq d-1$ for $0 \leq i \leq n-1$. x is often denoted by $(x_{n-1}, x_{n-2}, \ldots, x_0)$, where $c_i(x) = x_i$ is called the *ith coordinate* of x. Denote $p_i(x) = (x_{n-1}, x_{n-2}, \ldots, x_1, i)$ for $x = (x_{n-1}, x_{n-2}, \ldots, x_1, x_0)$. G^n is short-form notation for $G(d^n; A)$. For $0 \leq i \leq n-1$, the *ith unit vector* in G^n is the vector e_i^n with $c_i(e_i^n) = 1$ and $c_j(e_i^n) = 0$ for $0 \leq j \leq n-1$, with $j \neq i$. Similar notions will be used in following sections for different networks.

The vertex set $V(G^n)$ of G^n can be decomposed into $V_0^{n-1} \cup V_1^{n-1} \cup \cdots \cup V_{d-1}^{n-1}$, where $V_i^{n-1} = \{x \in V(G^n) : c_0(x) = i\}$. Note that V_i^{n-1} induces a subdigraph G_i^{n-1} of G^n that is isomorphic to G^{n-1} according to the natural mapping:

$$(x_{n-1}, x_{n-2}, \ldots, x_1, i) \rightarrow (x_{n-1}, x_{n-2}, \ldots, x_1).$$

For instance, $p_i(0)$ in G_i^{n-1} corresponds to 0 in G^{n-1}, and $p_i(e_k^n)$ in G_i^{n-1} corresponds to e_{k-1}^{n-1} in G^{n-1} for $1 \leq k \leq n-1$.

Theorem 5. *If* $A = \{1, d, d^2, \ldots, d^{n-1}\}$, *then*

$$r_w(G(d^n; A)) = r_w^*(G(d^n; A)) = \begin{cases} n(d-1), & \text{for } 1 \leq w \leq n-1, \\ n(d-1)+1, & \text{for } w = n. \end{cases}$$

Proof: From Theorem 4 and Propositions 1 (4) and 1 (2), it suffices to prove by induction on n that $r_w^*(G^n) \leq n(d-1) + \delta$, where $\delta = 1$ for $w = n$ and $\delta = 0$ for $1 \leq w \leq n-1$. Since the network is vertex-transitive, we only need to prove that for $x = 0, y^1, y^2, \ldots, y^w$, there exist w vertex-disjoint dipaths from x to $y^1, y^2, \ldots, y^w$ that are of length at most $n(d-1) + \delta$. The claim is trivial for $n = 1$. Suppose it holds for $n-1$ and we consider the case of n. The desired dipaths are constructed according to the following three cases. Without loss of generality, we may assume $y_0^1 \le y_0^2 \le \cdots \le y_0^w$.

Case 1. $y_0^1 < y_0^w$. Let $a_i = y_0^i$ for $1 \le i \le w$ and $j = a_1$. Let s be the largest index such that $y_0^1 = y_0^s$. By the induction hypothesis, there exist $w-1$ vertex-disjoint dipaths $q_2', q_3', \ldots, q_w'$ from 0 to $p_0(y^2), p_0(y^3), \ldots, p_0(y^w)$, respectively, of length at most $(n-1)(d-1)+\delta$, where

$$q_i' : 0 \to e^n_{k_i'} \to u^i \to \cdots \to p_0(y^i) \quad \text{for } 2 \le i \le w.$$

Without loss of generality, we may assume that q_a' has a length less than or equal to the length of q_b' when $p_0(y^a) = p_0(y^b)$ and $2 \le a < b \le w$. Note that $k_2', k_3', \ldots, k_w'$ are distinct. So there exists at most one $e^n_{k_i'} = p_0(y^1)$. For the case in which $e^n_{k_i'} = p_0(y^1)$ for some $i > s$, exchange the roles of y^i and y^w in the following argument. According to the induction hypothesis, there exist $w-1$ vertex-disjoint dipaths $q_1'', q_2'', \ldots, q_{w-1}''$ from $p_j(0)$ to $p_j(y^1)$, $p_j(y^2), \ldots, p_j(y^s)$, $p_j(e^n_{k_{s+1}'}), \ldots, p_j(e^n_{k_{w-1}'})$, respectively, of length at most $(n-1)(d-1)+\delta$, where

$$q_i'' : p_j(0) \to p_j\left(e^n_{k_i''}\right) \to v^i \to \cdots \to p_j(y^i) \quad \text{for } 1 \le i \le s.$$

Note that $k_1'', k_2'', \ldots, k_s'', k_{s+1}', k_{s+2}', \ldots, k_{w-1}'$ are distinct. Then the following w dipaths are vertex-disjoint and of length at most $(n-1)(d-1)+\delta+a_i \le n(d-1)+\delta$.

$$\begin{aligned}
q_i &: 0 \to e^n_{k_i''} \to p_1\left(e^n_{k_i''}\right) \to \cdots \to p_j\left(e^n_{k_i''}\right) \to v^i \to \cdots \to p_j(y^i) = y^i \\
&\quad \text{for } 1 \le i \le s; \\
q_i &: 0 \to e^n_{k_i'} \to p_1\left(e^n_{k_i'}\right) \to \cdots \to p_{a_i}\left(e^n_{k_i'}\right) \to p_{a_i}(u^i) \to \cdots \to p_{a_i}(y^i) = y^i \\
&\quad \text{for } s < i < w; \\
q_w &: 0 \to p_1(0) \to p_2(0) \to \cdots \to p_{a_w}(0) \to p_{a_w}\left(e^n_{k_w'}\right) \to p_{a_w}(u^w) \to \cdots \\
&\quad \to p_{a_w}(y^w) = y^w.
\end{aligned}$$

Case 2. $y_0^1 = y_0^2 = \cdots = y_0^w = j > 0$. Use the argument in Case 1 to obtain w vertex-disjoint dipaths from 0 to $y^1-1, y^2, \ldots, y^w$ of length at most $n(d-1)+\delta$. In fact, the dipath from 0 to y^1-1 is of length at most $n(d-1)+\delta-1$. Replace this dipath by adding vertex y^1 at the end to obtain the desired dipaths.

Case 3. $y_0^1 = y_0^2 = \cdots = y_0^w = j = 0$. According to the induction hypothesis, there exist $w-\delta$ vertex-disjoint dipaths

$$q_i : 0 \to e^n_{k_i} \to u^i \to \cdots \to y^i \quad (1 \le i \le w-\delta)$$

of length at most $(n-1)(d-1)+1 \le n(d-1)+\delta$ from 0 to $y^1, y^2, \ldots, y^{w-\delta}$, respectively.

So the case of $w \leq n-1$ is done. For the case in which $\delta = 1$ (i.e., $w = n$), consider the 0-y^n dipath

$$q_n : 0 \to p_1(0) \to p_2(0) \to \cdots \to p_{d-1}(0) \to \cdots \to y^n - 1 \to y,$$

where $p_{d-1}(0) \to \cdots \to y^n - 1$ is a shortest dipath in G_{d-1}^{n-1} with length at most $(n-1)(d-1)$. Note that q_n is of length at most $(d-1) + (n-1)(d-1) + 1 = n(d-1) + 1$.

□

3. *d*-Ary cube networks

A *d-ary cube network* $C(d, n)$ (see (Hsu and Lyuu, 1994)) is a digraph of d^n vertices, in which any vertex x has the form $(x_{n-1}, x_{n-2}, \ldots, x_0)$ where $0 \leq x_i \leq d-1$ for $0 \leq i \leq n-1$, and x is adjacent to $(x_{n-1}, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_0)$ for $0 \leq j \leq n-1$ where additions are taken modulo d. We can define $c_i(x)$, $p_i(x)$ and e_i^n as in Section 2. The vertex set of $C(d, n)$ can be viewed as a module over Z_d. Then x can also be written as $x = x_{n-1}e_{n-1}^n + \cdots + x_0 e_0^n$.

It is straightforward to show that the diameter $d(C(d, n)) = n(d-1)$. Hsu and Lyuu (1994) showed that $d_n(C(d, n)) = n(d-1) + 1$ for $d \geq 2$ and $r_n(C(d, n)) = n(d-1) + 1$ for $d \geq 3$. Note that $C(2, n)$ is the n-dimensional binary hypercube Q_n, and Rabin (1989) proved that $r_n(Q_n) = n + 1$. The above results, and the fact that each vertex has degree n in $C(d, n)$, imply that $k(C(d, n)) = n$. In this section we consider $D_w(C(d, n))$, $d_w(C(d, n))$, $r_w(C(d, n))$ and $r_w^*(C(d, n))$ for $d \geq 2$ and $1 \leq w \leq n$.

Theorem 6. *If $d \geq 2$ and $1 \leq w \leq n$, then*

$$D_w(C(d, n)) = d_w(C(d, n)) = \begin{cases} n(d-1), & \text{for } 1 \leq w \leq n-1, \\ n(d-1) + 1, & \text{for } w = n. \end{cases}$$

Proof: We claim that between any two vertices in $C(d, n)$, there exist n vertex-disjoint dipaths of length at most $n(d-1)$, except that at most, one of them has a length of $n(d-1) + 1$. Since the network is vertex-transitive, it suffices to prove the claim for vertices $0 = (0, 0, \ldots, 0)$ and $x = (x_{n-1}, x_{n-2}, \ldots, x_0)$. Suppose x has exactly k nonzero coordinates, say $x_i > 0$ for $0 \leq i \leq k-1$ and $x_i = 0$ for $k \leq i \leq n-1$. Construct n vertex-disjoint 0-x dipaths $q_0, q_1, \ldots, q_{n-1}$ as follows. Suppose $a_1, a_2, \ldots, a_r$ are positive integers and $i_1, i_2, \ldots, i_r$ are (not necessarily distinct) nonnegative integers. Denote by $\langle\langle a_1 e_{i_1}^n, a_2 e_{i_2}^n, \ldots, a_r e_{i_r}^n \rangle\rangle$ the following dipath from 0 to $\sum_{j=1}^{r} a_j e_{i_j}^n$:

$$\begin{aligned} 0 &\to e_{i_1}^n \to 2e_{i_1}^n \to \cdots \to a_1 e_{i_1}^n \\ &\to a_1 e_{i_1}^n + e_{i_2}^n \to a_1 e_{i_1}^n + 2e_{i_2}^n \to \cdots \to a_1 e_{i_1}^n + a_2 e_{i_2}^n \\ &\to \cdots \to \sum_{j=1}^{r-1} a_j e_{i_j}^n + e_{i_r}^n \to \sum_{j=1}^{r-1} a_j e_{i_j}^n + 2e_{i_r}^n \to \cdots \to \sum_{j=1}^{r-1} a_j e_{i_j}^n + a_r e_{i_r}^n. \end{aligned}$$

The desired n dipaths are:

$$q_s : \langle\langle x_s e^n_s, x_{s+1}e^n_{s+1}, \ldots, x_{k-1}e^n_{k-1}, x_0 e^n_0, x_1 e^n_1, \ldots, x_{s-1}e^n_{s-1}\rangle\rangle \quad \text{for } 0 \le s \le k-1;$$
$$q_s : \langle\langle (d-1)e^n_s, x_0 e^n_0, x_1 e^n_1, \ldots, x_{k-1}e^n_{k-1}, 1e^n_s\rangle\rangle \quad \text{for } k \le s \le n-1.$$

Note that each dipath q_s has a length of at most $n(d-1)$, except for the case in which $x = (d-1)e^n_0 + (d-1)e^n_1 + \cdots + (d-1)e^n_{n-2}$ the dipath q_{n-1} has a length $n(d-1)+1$. Therefore, $d_{n-1}(C(d,n)) \le n(d-1)$ and $d_n(C(d,n)) \le n(d-1)+1$.

$d_{n-1}(C(d,n)) \le n(d-1)$, along with $d(C(d,n)) = n(d-1)$ and Proposition 1, lead to $D_w(C(d,n)) = d_w(C(d,n)) = n(d-1)$ for $1 \le w \le n-1$.

Let $x' = \sum_{i=0}^{n-2}(d-1)e^n_i$ and $S = \{e^n_0, e^n_1, \ldots, e^n_{n-2}\}$. Since $d_{C(d,n)}(e^n_{n-1}, x') = n(d-1)$, we have $d_{C(d,n)-S}(0, x') = n(d-1)+1$, and so, $D_n(C(d,n)) \ge n(d-1)+1$. This, together with Proposition 1, implies $D_n(C(d,n)) = d_n(C(d,n)) = n(d-1)+1$. □

Theorem 7. *If $d \ge 2$ and $1 \le w \le n$, then*

$$r_w(C(d,n)) = r^*_w(C(d,n)) = \begin{cases} n(d-1), & \text{for } 1 \le w \le n-1, \\ n(d-1)+1, & \text{for } w = n. \end{cases}$$

Proof: The proof is the same as that for Theorem 5, except $y^1 - 1$ is replaced by $p_{y^1_0 - 1}(y^1)$ in Case 2 and $y^n - 1$ by $p_{d-1}(y^n)$ in Case 3. □

4. Generalized hypercube networks

Generalized hypercubes (see (Bhuyan and Agrawal, 1984)) are natural generalizations of (binary) hypercubes. Suppose $m_0, m_1, \ldots, m_{n-1}$ are positive integers greater than or equal to 2. The *generalized hypercube* $GH(m_{n-1}, \ldots, m_0)$ is the graph whose vertices are those $x = (x_{n-1}, x_{n-2}, \ldots, x_0)$ with $0 \le x_i \le m_i - 1$ for $0 \le i \le n-1$, and two vertices are adjacent if and only if they differ by exactly one coordinate. As in Section 2, we use the notation $c_i(x)$, $p_i(x)$ and e^n_i. We also use G^n for the network $GH(m_{n-1}, \ldots, m_0)$, and decompose the vertex set $V(G^n)$ of G^n into $V^{n-1}_0 \cup V^{n-1}_1 \cup \cdots \cup V^{n-1}_{d-1}$, where $V^{n-1}_i = \{x \in V(G^n) : c_0(x) = i\}$. Each V^{n-1}_i then induces a subgraph G^{n-1}_i of G^n that is isomorphic to $G^{n-1} = GH(m_{n-1}, \ldots, m_1)$ according to the natural mapping: $(x_{n-1}, x_{n-2}, \ldots, x_1, i) \to (x_{n-1}, x_{n-2}, \ldots, x_1)$.

It is straightforward to show that the diameter $d(GH(m_{n-1}, \ldots, m_0)) = n$. Duh et al. (1996) showed that the connectivity $k(GH(m_{n-1}, \ldots, m_0)) = \sum_{i=0}^{n-1}(m_i - 1) \equiv k$ and the k-wide diameter $d_k(GH(m_{n-1}, \ldots, m_0)) = n+1$. In this section we completely determine $D_w(GH(m_{n-1}, \ldots, m_0))$, $d_w(GH(m_{n-1}, \ldots, m_0))$, $r_w(GH(m_{n-1}, \ldots, m_0))$ and $r^*_w(GH(m_{n-1}, \ldots, m_0))$ for $1 \le w \le \sum_{i=0}^{n-1}(m_i - 1)$.

Theorem 8. *If $m_i \ge 2$ for $0 \le i \le n-1$, then*

$$D_w(GH(m_{n-1},\ldots,m_0)) = d_w(GH(m_{n-1},\ldots,m_0))$$
$$= \begin{cases} n, & \text{for } 1 \le w \le n-1; \\ n, & \text{for } w = n \text{ and the existence of at least two } m_i \ge 3; \\ n+1, & \text{for } w = n \text{ and the existence of at most one } m_i \ge 3; \\ n+1, & \text{for } n+1 \le w \le \sum_{i=0}^{n-1}(m_i - 1). \end{cases}$$

Proof: We first claim that between any two vertices in $GH(m_{n-1},\ldots,m_0)$, there exist $n-1$ (or n if there exist at least two $m_i \ge 3$) vertex-disjoint paths of length at most n. Since the graph is vertex-transitive, it suffices to prove the claim for vertices $0 = (0, 0, \ldots, 0)$ and $x = (x_{n-1}, x_{n-2}, \ldots, x_0)$. Suppose x has exactly h nonzero coordinates, say, $x_i > 0$ for $0 \le i \le h-1$ and $x_i = 0$ for $h \le i \le n-1$. Construct $n-1$ (or n) vertex-disjoint 0-x paths $q_0, q_1, \ldots, q_{n-2}$ (or $q_0, q_1, \ldots, q_{n-1}$) as follows.

Let ae_i^n denote the vertex for which $c_i(ae_i^n) = a$ and $c_j(ae_i^n) = 0$ for $0 \le j \le n-1$, with $j \ne i$. For any vertex $y = (y_{n-1}, y_{n-2}, \ldots, y_0)$ and positive integer a, let $y + ae_i^n$ denote the vertex $(y_{n-1}, y_{n-2}, \ldots, y_{i+1}, a, y_{i-1}, y_{i-2}, \ldots, y_0)$. Suppose $a_1, a_2, \ldots, a_r$ are positive integers and $i_1, i_2, \ldots, i_r$ are (not necessarily distinct) nonnegative integers. Denote by $\langle\langle a_1e_{i_1}^n, a_2e_{i_2}^n, \ldots, a_re_{i_r}^n\rangle\rangle$ the following path from 0 to $\sum_{j=1}^r a_je_{i_j}^n$:

$$0 \to a_1e_{i_1}^n \to a_1e_{i_1}^n + a_2e_{i_2}^n \to \cdots \to \sum_{j=1}^r a_je_{i_j}^n.$$

The desired $n-1$ (or n) paths are:

$$q_s : \left\langle\!\left\langle x_se_s^n, x_{s+1}e_{s+1}^n, \ldots, x_{h-1}e_{h-1}^n, x_0e_0^n, x_1e_1^n, \ldots, x_{s-1}e_{s-1}^n \right\rangle\!\right\rangle \quad \text{for } 0 \le s \le h-1;$$
$$q_s : \left\langle\!\left\langle x_se_s^n, x_0e_0^n, x_1e_1^n, \ldots, x_{h-1}e_{h-1}^n, 0e_s^n \right\rangle\!\right\rangle \quad \text{for } h \le s \le n-1.$$

Note that each q_s has a length of at most n, except that q_h has a length of $n+1$ for the case in which $h = n-1$. If $h = n-1$ and there exist at least two $m_i \ge 3$, implying that $m_i \ge 3$ for some $0 \le i \le h-1$, say, $m_0 \ge 3$, then there exists $0 \le x_0' \le m_0 - 1$, with $x_0' \ne x_0$. We use the 0-x path $\langle\langle x_0'e_0^n, x_1e_1^n, x_2e_2^n, \ldots, x_{h-1}e_{h-1}^n, x_0e_0^n\rangle\rangle$ as q_{n-1}, which has a length of n. Therefore, $d_n(GH(m_{n-1},\ldots,m_0)) \le n$ if there exist at least two $m_i \ge 3$ and $d_{n-1}(GH(m_{n-1},\ldots,m_0)) \le n$.

On the other hand, let $x' = (1, 1, \ldots, 1)$ and $S = \{e_0^n, e_1^n, \ldots, e_{n-1}^n\}$. Then $d_{GH(m_{n-1},\ldots,m_0)-S}(0, x') = n+1$, and so, $D_{n+1}(GH(m_{n-1},\ldots,m_0)) \ge n+1$. For the case in which there exists at most one $m_i \ge 3$, say $m_j = 2$ for all $j \ge 1$, let $x' = (1, \ldots, 1, 0)$ and $S = \{e_1^n, e_2^n, \ldots, e_{n-1}^n\}$. Then $d_{GH(m_{n-1},\ldots,m_0)-S}(0, x') = n+1$, and so, $D_n(GH(m_{n-1},\ldots,m_0)) \ge n+1$.

The above results, along with $d(GH(m_{n-1},\ldots,m_0)) = n < n+1 = d_k(GH(m_{n-1},\ldots, m_0))$, and Proposition 1, lead to

$$D_w(GH(m_{n-1},\dots,m_0)) = d_w(GH(m_{n-1},\dots,m_0))$$
$$= \begin{cases} n, & \text{for } 1 \le w \le n-1; \\ n, & \text{for } w = n \text{ and the existence of at least two } m_i \ge 3; \\ n+1, & \text{for } w = n \text{ and the existence of at most one } m_i \ge 3; \\ n+1, & \text{for } n+1 \le w \le \sum_{i=0}^{n-1}(m_i - 1). \end{cases}$$

□

Lemma 9. *If $m_i \ge 2$ for $0 \le i \le n-1$, then $r_k^*(GH(m_{n-1},\dots,m_0)) \le n+1$, where $k = \sum_{i=0}^{n-1}(m_i - 1)$.*

Proof: Since the network is vertex-transitive, it suffices to prove by induction on n that for vertices $x = 0 = (0, 0, \dots, 0)$, $y^1, y^2, \dots, y^k$, there exist k vertex-disjoint paths from x to $y^1, y^2, \dots, y^k$ that are of length at most $n+1$. The claim is trivial for $n = 1$. Suppose it holds for $n-1$ and we consider the case of n. Without loss of generality, we may assume that $y_0^1 \le y_0^2 \le \dots \le y_0^k$. Let $y_0^0 = -1$ and s_i be the maximum index such that $y_0^{s_i} \le i$ for $-1 \le i \le n-1$. Note that $s_i = s_{i-1}$ is equivalent to there being no w such that $y_0^w = i$.

Case 1. $0 = s_{-1} < s_0 < s_1 < \dots < s_{m_0-1} = k$. Note that $s_0 \le k - (m_0 - 1) = \sum_{i=1}^{n-1}(m_i - 1)$. According to the induction hypothesis, there exist s_0 vertex-disjoint paths $q_1', q_2', \dots, q_{s_0}'$ in G_0^{n-1} from 0 to $y^1, y^2, \dots, y^{s_0}$, respectively, of length at most n, where

$$q_j' : p_i(0) \to t^j = \big(0, \dots, 0, t_r^j, 0, \dots, 0, i\big) \to u^j \to \dots \to y^j, \tag{1}$$

with $i = 0$, $t_r^j > 0$ and $1 \le r \le n-1$ for $1 \le j \le s_0$. Note that all $t^1, t^2, \dots, t^{s_0}$ are distinct. Once q_j''s are constructed for $1 \le j \le s_{i-1}$, we construct q_j''s for $s_{i-1} < j \le s_i$ as follows. Consider the vertices y^j for $s_{i-1} < j \le s_i$ and $p_i(t^j)$ for $1 \le j \le s_{i-1}$ with $j \ne$ any s_a. The total number of such vertices is less than or equal to $k - (m_0 - 1 - i) + i = \sum_{i=1}^{n-1}(m_i - 1)$. According to the induction hypothesis, there exist vertex-disjoint paths in G_i^{n-1} from $p_i(0)$ to these vertices. For the case in which some $y^j =$ some $p_i(t^j)$, we may assume the path from $p_i(0)$ to y^j is at least as long as the path from $p_i(0)$ to $p_i(t^j)$. Now identify the path from $p_i(0)$ to y^j as q_j' $(s_{i-1} < j \le s_i)$, which also has the same form in (1). Note that there exists at most one $p_0(t^j) = t^{s_0}$. For the case $p_0(t^j) = t^{s_0}$ for some $j \ne s_i$, exchange the roles of y^j and y^{s_i} in the following argument. Continue this process until $i = m_0 - 1$. Then we have a 0-y^j path q_j' of length at most n for $1 \le j \le k$ such that all t^j (with $j \ne$ any s_a) are distinct. The desired paths of length at most $n+1$ are:

$$q_j : 0 \to p_i(0) \to t^j \to u^j \to \dots \to y^j \quad \text{for } j = s_i;$$
$$q_j : 0 \to p_0(t^j) \to p_i(t^j) = t^j \to u^j \to \dots \to y^j \quad \text{for } s_{i-1} < j \le s_i.$$

Case 2. $s_{i-1} = s_i$ for some i. For each such i, choose y^j such that $y_0^j = y_0^{j+1}$ and replace y^j with $p_i(y^j)$. This results in the new sequence $\bar{y}^1, \bar{y}^2, \dots, \bar{y}^k$ that satisfies the conditions in Case 1. Construct q'_j as in Case 1, except for those y^j replaced by $p_i(y^j)$, use $0 \to p_i(0) \to \cdots \to p_i(y) \to y$ for q_j, where the deletion of 0 and y from q_j is just a shortest $p_0(0)$-$p_i(y)$ path in G_i^{n-1}. □

Lemma 10. *If $m_i \geq 2$ for $0 \leq i \leq n-1$, then $r_n^*(GH(m_{n-1}, \ldots, m_0)) \leq n$ when there exist at least two $m_i \geq 3$ and $r_{n-1}^*(GH(m_{n-1}, \ldots, m_0)) \leq n$.*

Proof: Since the network is vertex-transitive, it suffices to prove by induction on n that for vertices $x = 0 = (0, 0, \ldots, 0)$, $y^1, y^2, \ldots, y^w$, there exists w vertex-disjoint paths fom x to $y^1, y^2, \ldots, y^w$ that are of length at most n, where $w = n$, when there exists at least two $m_i \geq 3$ and $w = n-1$, otherwise. The claim is trivial for $n = 1$. Suppose it is true for $n-1$, and consider the case of n. Without loss of generality, we may assume that $m_0 \leq m_1 \leq \cdots \leq m_{n-1}$, which implies that G_j^{n-1} has also at least two $m_i \geq 3$ when G^n does. The desired paths are constructed according to the following three cases. Without loss of generality, we may assume that $y_0^1 \leq y_0^2 \leq \cdots \leq y_0^w$.

Case 1. $y_0^1 < y_0^w$. Let $a_i = y_0^i$ for $1 \leq i \leq w$ and $j = a_1$. Let s be the largest index such that $y_0^1 = y_0^s$. According to the induction hypothesis, there exist $w-1$ vertex-disjoint paths $q'_2, q'_3, \ldots, q'_w$ from 0 to $p_0(y^2), p_0(y^3), \ldots, p_0(y^w)$, respectively, of length at most $n-1$, where

$$q'_i : 0 \to u_i e^n_{k'_i} \to \cdots \to p_0(y^i) \quad \text{for } 2 \leq i \leq w.$$

Without loss of generality, we may assume that q'_a has a length less than or equal to the length of q'_b when $p_0(y^a) = p_0(y^b)$ and $2 \leq a < b \leq w$. Note that $u_2 e^n_{k'_2}, u_3 e^n_{k'_3}, \ldots, u_w e^n_{k'_w}$ are distinct. So there exists at most one $u_i e^n_{k'_i} = p_0(y^1)$. For the case in which $u_i e^n_{k'_i} = p_0(y^1)$ for some $i > s$, exchange the roles of y^i and y^w in the following argument. According to the induction hypothesis, there exist $w-1$ vertex-disjoint paths $q''_1, q''_2, \ldots, q''_{w-1}$ from $p_j(0)$ to $p_j(y^1), p_j(y^2), \ldots, p_j(y^s), p_j(u_{s+1} e^n_{k'_{s+1}}), \ldots, p_j(u_{w-1} e^n_{k'_{w-1}})$, respectively, of length at most $n-1$, where

$$q''_i : p_j(0) \to p_j\left(v_i e^n_{k''_i}\right) \to \cdots \to p_j(y^i) \quad \text{for } 1 \leq i \leq s.$$

Note that $v_1 e^n_{k''_1}, v_2 e^n_{k''_2}, \ldots, v_s e^n_{k''_s}, u_{s+1} e^n_{k'_{s+1}}, u_{s+2} e^n_{k'_{s+2}}, \ldots, u_{w-1} e^n_{k'_{w-1}}$ are distinct. Then the following w paths are vertex-disjoint and of length at most n.

$$q_i : 0 \to v_i e^n_{k''_i} \to p_j\left(v_i e^n_{k''_i}\right) \to \cdots \to p_j(y^i) = y^i \quad \text{for } 1 \leq i \leq s;$$

$$q_i : 0 \to u_i e^n_{k'_i} \to p_{a_i}\left(u_i e^n_{k'_i}\right) \to \cdots \to p_{a_i}(y^i) = y^i \quad \text{for } s+1 \leq i \leq w-1;$$

$$q_w : 0 \to p_{a_w}(0) \to p_{a_w}\left(u_w e^n_{k'_w}\right) \to \cdots \to p_{a_w}(y^w) = y^w.$$

Case 2. $y_0^1 = y_0^2 = \cdots = y_0^w = j > 0$. Use the argument in Case 1 to obtain the w vertex-disjoint paths from 0 to $p_0(y^1), y^2, \ldots, y^w$ of length at most n. In fact, the path from 0 to

$p_0(y^1)$ is of length at most $n-1$. Replace this path by adding vertex y^1 at the end to get the desired paths.

Case 3. $y_0^1 = y_0^2 = \cdots = y_0^w = j = 0$. Since G_0^{n-1} is isomorphic to G^{n-1}, the claim follows from Lemma 9 and Proposition 1. □

Lemmas 9 and 10 and Propositions 1 and 2 lead to the following theorem.

Theorem 11. *If* $m_i \geq 2$ *for* $0 \leq i \leq n-1$, *then*

$$r_w(GH(m_{n-1}, \ldots, m_0)) = r_w^*(GH(m_{n-1}, \ldots, m_0))$$
$$= \begin{cases} n, & \textit{for } 1 \leq w \leq n-1; \\ n, & \textit{for } w = n \textit{ and the existence of at least two } m_i \geq 3; \\ n+1, & \textit{for } w = n \textit{ and the existence of at most one } m_i \geq 3; \\ n+1, & \textit{for } n+1 \leq w \leq \sum_{i=0}^{n-1}(m_i - 1). \end{cases}$$

5. Folded hypercube networks

A *folded hypercube network* $FH(n)$ (see (El-Amawy and Latifi, 1991)) is a graph whose vertices are binary sequences $x = (x_{n-1}, x_{n-2}, \ldots, x_0)$ with $x_i = 0$ or 1 for $0 \leq i \leq n-1$, and two vertices are adjacent if and only if they differ by exactly one coordinate or by all coordinates. Folded hypercubes are enhancements of hypercubes. They are basically binary hypercubes augmented by extra *complement edges* whose two end vertices differ by all coordinates. El-Amawy and Latifi (1991) showed that the diameter $d(FH(n)) = \lceil \frac{n}{2} \rceil$; Duh et al. (1995) that $d_{n+1}(FH(n)) = \lceil \frac{n}{2} \rceil + 1$. The above result and the fact that each vertex has degree $n+1$ in $FH(n)$ imply $k(FH(n)) = n+1$. In this section we consider $D_w(FH(n))$, $d_w(FH(n))$, $r_w(FH(n))$ and $r_w^*(FH(n))$ for $n \geq 2$ and $1 \leq w \leq n+1$.

Theorem 12. *If* $n \geq 2$, *then*

$$D_w(FH(n)) = d_w(FH(n)) = \begin{cases} \lceil \frac{n}{2} \rceil, & \textit{for } 1 \leq w \leq \lceil \frac{n}{2} \rceil - 1, \\ \lceil \frac{n}{2} \rceil + 1, & \textit{for } \lceil \frac{n}{2} \rceil \leq w \leq n+1. \end{cases}$$

Proof: We first claim that for any two vertices in $FH(n)$, there exist $\lceil \frac{n}{2} \rceil - 1$ vertex-disjoint paths of length at most $\lceil \frac{n}{2} \rceil$. Since the graph is vertex-transitive, it suffices to prove the claim for vertices $0 = (0, 0, \ldots, 0)$ and $x = (x_{n-1}, x_{n-2}, \ldots, x_0)$. Suppose x has exactly k nonzero coordinates, say, $x_i = 1$ for $0 \leq i \leq k-1$ and $x_i = 0$ for $k \leq i \leq n-1$. Construct $\lceil \frac{n}{2} \rceil - 1$ vertex-disjoint 0-x paths $q_0, q_1, \ldots, q_{\lceil \frac{n}{2} \rceil - 2}$ as follows. Let $e_n^n = (1, 1, \ldots, 1)$. Suppose $i_1, i_2, \ldots, i_r$ are (not necessarily distinct) nonnegative integers. Denote by $\langle\langle e_{i_1}^n, e_{i_2}^n, \ldots, e_{i_r}^n \rangle\rangle$ the path from 0 to $\sum_{j=1}^{r} e_{i_j}^n$, where additions are performed modulo 2,

$$0 \to e_{i_1}^n \to e_{i_1}^n + e_{i_2}^n \to \cdots \to \sum_{j=1}^{r} e_{i_j}^n.$$

The desired $\lceil \frac{n}{2} \rceil - 1$ paths are:

$$q_s : \langle\langle e^n_s, e^n_{s+1}, \ldots, e^n_{k-1}, e^n_0, e^n_1, \ldots, e^n_{s-1}\rangle\rangle \quad \text{for } 0 \le s \le k-1 \text{ and } k \le \lceil \tfrac{n}{2} \rceil;$$
$$q_s : \langle\langle e^n_s, e^n_0, e^n_1, \ldots, e^n_{k-1}, e^n_s\rangle\rangle \quad \text{for } k \le s \le \lceil \tfrac{n}{2} \rceil - 2;$$
$$q_s : \langle\langle e^n_{s+k}, e^n_{s+k+1}, \ldots, e^n_n, e^n_k, e^n_{k+1}, \ldots, e^n_{s+k-1}\rangle\rangle \quad \text{for } 0 \le s \le n-k \text{ and } k > \lceil \tfrac{n}{2} \rceil;$$
$$q_s : \langle\langle e^n_s, e^n_n, e^n_{n-1}, \ldots, e^n_{k+1}, e^n_s\rangle\rangle \quad \text{for } n-k+1 \le s \le \lceil \tfrac{n}{2} \rceil - 2 \text{ and } k > \lceil \tfrac{n}{2} \rceil.$$

Note that each q_s has a length of at most $\lceil \frac{n}{2} \rceil$. Therefore, $d_{\lceil \frac{n}{2} \rceil - 1}(FH(n)) \le \lceil \frac{n}{2} \rceil$.

$d_{\lceil \frac{n}{2} \rceil - 1}(FH(n)) \le \lceil \frac{n}{2} \rceil$, along with $d(FH(n)) = \lceil \frac{n}{2} \rceil$ (see (El-Amawy and Latifi, 1991)) and Proposition 1, leads to $D_w(FH(n)) = d_w(FH(n)) = \lceil \frac{n}{2} \rceil$ for $1 \le w \le \lceil \frac{n}{2} \rceil - 1$.

Let $x' = \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 2} e^n_i$ and $S = \{e^n_i : 0 \le i \le \lceil \frac{n}{2} \rceil - 2\}$. Then $d_{FH(n)-S}(0, x') = \lceil \frac{n}{2} \rceil + 1$, and so, $D_{\lceil \frac{n}{2} \rceil}(FH(n)) \ge \lceil \frac{n}{2} \rceil + 1$. This, along with $d_{n+1}(FH(n)) \le \lceil \frac{n}{2} \rceil + 1$ (see (Duh et al., 1995)) and Proposition 1, leads to $D_w(FH(n)) = d_w(FH(n)) = \lceil \frac{n}{2} \rceil + 1$ for $\lceil \frac{n}{2} \rceil \le w \le n+1$. □

Corollary 13. *If* $n \ge 2$, *then*

$$r^*_w(FH(n)) \ge r_w(FH(n)) \ge \begin{cases} \lceil \frac{n}{2} \rceil, & \textit{for } 1 \le w \le \lceil \frac{n}{2} \rceil - 1, \\ \lceil \frac{n}{2} \rceil + 1, & \textit{for } \lceil \frac{n}{2} \rceil \le w \le n+1. \end{cases}$$

Determining the exact values of $r^*_w(FH(n))$ and $r_w(FH(n))$ remains an open question.

6. WK-Recursive networks

WK-Recursive networks $WK(d, t)$ were proposed by Vecchia and Sanges (1988) as a general class of recursively scalable network topologies for message-passing architectures. The vertex set of $WK(d, t)$ is $\{(x_t, \ldots, x_1) : 0 \le x_i \le d-1 \text{ for } 1 \le i \le t\}$, and a vertex $x = (x_t, \ldots, x_1)$ is adjacent to the following two types of vertices:

(1) $(x_t, \ldots, x_2, x'_1)$, where $x'_1 \ne x_1$ and $0 \le x'_1 \le d-1$,

(2) $(x_t, \ldots, x_{i+1}, x_{i-1}, (x_i)^{i-1})$ when $x_i \ne x_{i-1} = x_{i-2} = \cdots = x_1$, where $(x_i)^{i-1}$ represents $i-1$ consecutive x_i's.

An *open edge* whose one end vertex is unspecified is incident to $(x_t, \ldots, x_1)$ if $x_t = x_{t-1} = \cdots = x_1$. For each $1 \le i \le t$, a vertex $x = (x_t, \ldots, x_1)$ is called an i-*frontier* if $x = (x_t, \ldots, x_{i+1}, (x_i)^i)$. So, $WK(d, 1)$ is a d-complete graph augumented by d open edges, each at a vertex; and $WK(d, t)$ with $t \ge 2$ is composed of d $WK(d, t-1)$'s that are connected by edges of type (2).

Chen and Duh (1994) showed that $d(WK(d, t)) = 2^t - 1$, $k(WK(d, t)) = d-1$, $d_{d-1}(WK(d, t)) = 3 \cdot 2^{t-1} - 1$, and the distance between any two of t-frontiers is $2^t - 1$. In this section we consider $D_w(WK(d, t))$, $d_w(WK(d, t))$, $r_w(WK(d, t))$ and $r^*_w(WK(d, t))$ for $t \ge 1$ and $1 \le w \le d-1$.

For convenience, let $G^t = WK(d, t)$ and G_i^{t-1} be the subgraph of G^t induced by $V_i^{t-1} = \{x \in V(G^t) : x_t = i\}$ for $0 \le i \le d-1$. Note that each G_i^{t-1} is isomorphic to $WK(d, t-1)$ according to the natural mapping: $(i, x_{t-1}, \ldots, x_1) \to (x_{t-1}, \ldots, x_1)$.

Lemma 14. *Suppose $1 \le w \le d-1$. If $x^1, x^2, \ldots, x^w$ are w (not necessarily distinct) vertices and $y^1, y^2, \ldots, y^w$ are w distinct t-frontiers, then there exist w vertex-disjoint paths q_i from x^i to y^i (for $1 \le i \le w$) each of length at most $2^t - 1$.*

Proof: We prove the lemma by induction on t. The proof is trivial for $t = 1$. Assume the lemma holds for $t-1$ and we consider the case of t. Let $x^i = (x_t^i, \ldots, x_1^i)$ and $y^i = ((y_t^i)^t)$ for $1 \le i \le w$. Without loss of generality, we may assume that there exists $0 \le \overline{w} \le w$ such that $y_t^i = x_t^i$ for $1 \le i \le \overline{w}$ and $y_t^i \ne x_t^j$ for $\overline{w}+1 \le i \le w$ and $1 \le j \le w$. Define $\bar{y}^i = (x_t^i, (y_t^i)^{t-1})$ for $1 \le i \le w$. According to the induction hypothesis, for any fixed $0 \le j \le d-1$ and those x^i and $\bar{y}^i$ with $x_t^i = j$, there exist vertex-disjoint paths $\bar{q}_i$ from x^i to $\bar{y}^i$ of length at most $2^{t-1} - 1$ in G_j^{t-1}.

For those i such that $\bar{y}^i = y^i$, i.e., $1 \le i \le \overline{w}$, we use $\bar{q}_i$ for q_i, which is of length at most $2^t - 1$. For those i such that $\bar{y}^i \ne y^i$, i.e., $\overline{w}+1 \le i \le w$, let q_i' be a shortest path from $(y_t^i, (x_t^i)^{t-1})$ to y^i in $G_{y_t^i}^{t-1}$. Note that q_i' has a length of at most $2^{t-1} - 1$ and the end vertex of $\bar{q}_i$ is adjacent to the starting vertex of q_i'. Therefore, the concatenation of $\bar{q}_i$ and q_i' yields an x^i-y^i path of length at most $2^t - 1$ in G^t. Also, all the paths q_i are vertex-disjoint. □

Lemma 15. *Suppose $1 \le w \le d$. For any w (not necessarily distinct) vertices $x = (x_t, \ldots, x_1)$, $y^1, \ldots, y^{w-1}$, there exist $d-w$ t-frontiers $y^w, \ldots, y^{d-1}$ in G^t, that are different from $((x_t)^t)$, $x, y^1, \ldots, y^{w-1}$, and there exist $d-1$ vertex-disjoint paths q_i from x to y^i (for $1 \le i \le d-1$) such that each q_i has a length of at most $3 \cdot 2^{t-1} - 1$ (resp., $2^t - 1$) when $1 \le i \le w-1$ (resp., $w \le i \le d-1$).*

Proof: We prove the lemma by induction on t. The proof is trivial for $t = 1$. Suppose the lemma holds for $t-1$ and we consider the case of t. Without loss of generality, we may assume that $y_t^i = x_t$ for $1 \le i \le \overline{w} - 1$ and $y_t^i \ne x_t$ for $\overline{w} \le i \le w-1$. Let $\bar{y}^i = y^i$ for $1 \le i \le \overline{w} - 1$. According to the induction hypothesis, there exist $d - \overline{w}$ $(t-1)$-frontiers $\bar{y}^{\overline{w}}, \bar{y}^{\overline{w}+1}, \ldots, \bar{y}^{d-1}$ in $G_{x_t}^{t-1}$, that are different from $((x_t)^t), x, \bar{y}^1, \ldots, \bar{y}^{\overline{w}-1}$, and $d-1$ vertex-disjoint paths $\bar{q}_i$ from x to $\bar{y}_i$ for $1 \le i \le d-1$, such that each $\bar{q}_i$ has a length of at most $3 \cdot 2^{t-2} - 1$ (resp., $2^{t-1} - 1$) for $1 \le i \le \overline{w} - 1$ (resp., $\overline{w} \le i \le d-1$). Without loss of generality, we may assume $\bar{y}^i = (x_t, (a^i)^{t-1})$, where $0 \le a^i \le d-1$ and $a^i \ne x_t$, for $\overline{w} \le i \le d-1$ and there exists $\overline{w} - 1 \le s \le w-1$, such that $a^i = y_t^i$ for $\overline{w} \le i \le s$, and $a^i \ne y_t^j$ for $s+1 \le i \le d-1$ and $\overline{w} \le j \le w-1$. Define $\hat{y}^i = (a^i, (x_t)^{t-1})$ for $\overline{w} \le i \le s$ and $\hat{y}^i = (y_t^i, (a^i)^{t-1})$ for $s+1 \le i \le w-1$. So, if $y^i \in V_j^{t-1}$ for some i and j, with $\overline{w} \le i \le w-1$ and $0 \le j \le d-1$, then $\hat{y}^i \in V_j^{t-1}$ and $\hat{y}^i$ is a $(t-1)$-frontier of G_j^{t-1}. According to Lemma 14, for all y^i and $\hat{y}^i$ in V_j^{t-1} with $j \ne x_t$, there exist paths $\hat{q}_i$ from $\hat{y}^i$ to y^i of length at most $2^{t-1} - 1$ in G_j^{t-1}.

For those i such that $y_t^i = x_t$, i.e., $1 \le i \le \overline{w} - 1$, we use $\bar{q}_i$ for q_i, which is of length at most $3 \cdot 2^{t-2} - 1$. For those i such that $y_t^i = a^i$, i.e., $\overline{w} \le i \le s$, since the end vertex of $\bar{q}_i$ is adjacent to the starting vertex of $\hat{q}_i$, the concatenation of $\bar{q}_i$ and $\hat{q}_i$ yields an x-y^i path

q_i of length at most $(2^{t-1}-1)+1+(2^{t-1}-1)=2^t-1 \leq 3\cdot 2^{t-1}-1$. For those i such that $x_t \neq y_t^i \neq a^i$, i.e., $s+1 \leq i \leq w-1$, let q_i' be a shortest path from $(a^i,(x_t)^{t-1})$ to $(a^i,(y_t^i)^{t-1})$ in $G_{a^i}^{t-1}$. Note that q_i' has a length of at most $2^{t-1}-1$; the end vertex of $\bar{q}_i$ is adjacent to the starting vertex of q_i'; and the end vertex of q_i' is adjacent to the starting vertex of $\hat{q}_i$. Therefore, the concatenation of $\bar{q}_i$, q_i' and $\hat{q}_i$ yields an x-y^i path q_i of length at most $(2^{t-1}-1)+1+(2^{t-1}-1)+1+(2^{t-1}-1)=3\cdot 2^{t-1}-1$. Finally, for those i for which $w \leq i \leq d-1$, we construct the paths from x to $d-w$ t-frontiers in the following way. Let q_i' be a shortest path from $(a^i,(x_t)^{t-1})$ to $((a^i)^t)$ in $G_{a^i}^{t-1}$. Note that q_i' has a length of at most $2^{t-1}-1$ and the end vertex of $\bar{q}_i$ is adjacent to the starting vertex of q_i'. Therefore, the concatenation of $\bar{q}_i$ and q_i' yields an x-$((a^i)^t)$ path q_i of length at most $(2^{t-1}-1)+1+(2^{t-1}-1)=2^t-1$. Also, all the paths q_i are vertex-disjoint. □

Corollary 16. *If $d \geq 2$ and $t \geq 1$, then $r_{d-1}^*(WK(d,t)) \leq 3\cdot 2^{t-1}-1$.*

Theorem 17.

$$D_w(WK(d,t)) = d_w(WK(d,t)) = r_w(WK(d,t) = r_w^*(WK(d,t)) = \begin{cases} 2^t-1, & \text{for } w=1,\\ 3\cdot 2^{t-1}-1, & \text{for } 2 \leq w \leq d-1. \end{cases}$$

Proof: The case of $w=1$ follows from $d(WK(d,t))=2^t-1$ (see (Chen and Duh, 1994)).

Now assume $2 \leq w \leq d-1$. Let $x=((1)^t)$ and $S=\{(1,(0)^{t-1})\}$. Note that any two subgraphs G_i^{t-1} and G_j^{t-1} of G^t are connected only by the edge $\{(i,(j)^{t-1}),(j,(i)^{t-1})\}$, whose end vertices are $(t-1)$-frontiers of the subgraph they are in. Then any 0-x path q in $WK(d,t)-S$ must pass some $(0,(i)^{t-1})$ with $2 \leq i \leq d-1$, and then in turn, must pass $(i,(0)^{t-1})$, $(i,(1)^{t-1})$ and $(1,(i)^{t-1})$, which are $(t-1)$-frontiers in G_i^{n-1}. Note that Chen and Duh (1994) proved that the distance between any two t-frontiers is 2^t-1 in G^t. Thus, the length of q is at least $(2^{t-1}-1)+1+(2^{t-1}-1)+1+(2^{t-1}-1)=3\cdot 2^{t-1}-1$. So, $D_2(WK(d,t)) \geq 3\cdot 2^{t-1}-1$. This, along with Propositions 1 and 2 and Corollary 16, leads to $D_w(WK(d,t))=d_w(WK(d,t))=r_w(WK(d,t)=r_w^*(WK(d,t))=3\cdot 2^{t-1}-1$ for $2 \leq w \leq d-1$. □

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