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Localizing Combinatorial Properties for Partitions on Block Graphs

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Abstract. We extend the study on partition properties from the set partition to the graph partition, especially for the class of connected block graphs which includes trees. We introduce seventeen partition properties and determine their inter-relations. The notions of k-consistency and k-sortability were studied in the set partition to localize the properties, i.e., a global property can be verified through checking local conditions. We carry on these studies for partitions on connected block graphs. In particular, we completely determine the consistency for all the seventeen properties.

Keywords: partition, block graph, tree, penetration, nested partition, consecutive partition, order consecutive partition

1. Introduction

Many optimization problems can be reduced to choosing a partition of a set $\{1, 2, \ldots, n\}$ into p parts so as to maximize an objective function. Since the total number of such partitions is exponential in n, it will be helpful to identify a smaller class of partitions which contains an optimal one. Such a class is often characterized by a partition property, e.g. consecutiveness, order-consecutiveness, nestedness, full nestedness, and so on. The sizes of the four classes characterized by these properties have been shown to be polynomial in n.

It is usually easier to verify a partition property locally. For example, for some objective function, it can be shown that, for any partition, if one rearranges the elements in a pair of parts in such a way that every element in one part is greater than any element in the other part (this is the consecutiveness property), then the objective function does not decrease. The question is: Can we use this "local" result on consecutiveness to infer the existence of a consecutive optimal partition? While the answer is in the affirmative for this example, our objective is to treat partition properties in a general framework and find "local" conditions which guarantee the existence of an optimal partition having a desired property.

A p-partition is a partition consisting of p parts. Let Q be a partition property. For a given p-partition π and a k-subpartition K ($k \leq p$), by "*Q-sorting K*" is meant to rearrange the elements in the k parts of K such that the resulting k-subpartition has property Q. Note that there can be more than one way to Q-sort K.

All the partition properties we study are hereditary: if a partition π has property Q, then every subpartition of π has property Q. A property Q is called *k-consistent* if Q holds for π whenever Q holds for all k-subpartitions of π. Let P be a class of partitions. We say that *P satisfies Q k-locally* if for every $\pi \in P$ and for every k-subpartition K of π, there exists a partition in P which is obtained from π by Q-sorting K. We say that *P weakly satisfies Q k-locally* if for every $\pi \in P$ with at least one k-subpartition not having the property Q, there exists a partition in P obtained from π by Q-sorting a k-subpartition that does not have Q. A property Q is called *k-sortable* if there exists a partition in P having Q whenever P satisfies Q k-locally. A k-consistent property Q is called *strongly k-sortable* if there exists a partition in P having Q whenever P weakly satisfies Q k-locally. Clearly, strong k-sortability implies k-sortability. It is also known (Hwang et al., 1996) that k-sortability implies k-consistency.

Note that k-consistency and k-sortability are defined independently of any objective function, and are purely combinatorial notions. k-Sortability is often not easy to prove or disprove. Since checking k-consistency is much easier, one may disprove k-sortability by invalidating k-consistency.

Hwang and Chang (1998) extended the set partition problem to the graph partition setting, and viewed the former as a special case of the latter when the graph is a path. They defined consecutiveness, order-consecutiveness and nestedness for the graph partition and obtained some enumeration results. In this paper, we study k-consistency and (strong) k-sortability for the above three partition properties and their variations on block graphs.

We conclude this section by briefly indicating how the notion of sortability can be used in an optimal partition problem (the reader is referred to (Hwang et al., 1996) for more details). Suppose that Q is a k-sortable property. Also, suppose that the objective function f has the following property R: for a p-partition π and a k-subpartition K, there exists a partition π' obtained from π by Q-sorting K such that $f(\pi') \geq f(\pi)$. Then it follows that the class of optimal p-partitions satisfies Q k-locally. By k-sortability of Q, there exists an optimal p-partition having Q. In the special case that f is additive, the property R is equivalent to the existence of an optimal k-partition having Q. So, if Q is k-sortable, the task of proving the existence of an optimal p-partition having Q is reduced to establishing the existence of an optimal k-partition having Q. The latter is often a relatively simple task if k is small.

2. Properties of the graph partition

For a given connected graph $G(V, E)$, consider the problem of partitioning a subset V' of V into p nonempty parts. For a subset S of V denote by $\langle S \rangle$ the subgraph induced by S. A *hull* of S is a minimal superset S' of S such that $\langle S' \rangle$ is connected. In general, S may have more than one hull. In this paper, we will only be concerned with a class of graphs, called block graphs, in which every subset has a unique hull (see Lemma 1 below, also see (Jamison, 1981)). We will denote the unique hull of S by $H(S)$.

A *cutpoint* of a graph is one whose removal increases the number of components. A *nonseparable graph* is connected, nontrivial and has no cutpoints. A *block* of a graph is a maximal nonseparable subgraph. If we take the blocks of a graph G as vertices, then the intersection graph (with an edge for each pair of intersecting blocks) is called *block graph* of G. It is well known (Harary, 1972) that a graph G is the block graph of some graph if and only if every block of G is complete. In particular, trees are connected block graphs.

A path is *chordless* if no two vertices nonconsecutive on the path are adjacent in the graph.

Lemma 1. *For any connected graph $G(V, E)$ the following are equivalent*:

(1) *G is a block graph.*

(2) *For any two vertices x and y there exists a unique x-y chordless path.*

(3) *Every subset S of V has a unique hull.*

Proof:

(1) ⇒ (2). Suppose, to the contrary, that there exist two distinct chordless paths p and q connecting some vertices x and y. By minimizing $|p| + |q|$, over all such pairs (x, y), we may assume that p and q are disjoint. So x and y are nonadjacent vertices in a block of G, contradicting assumption (1).

(2) ⇒ (3). Let C be a hull of S. Consider the set S' consisting of all vertices in the unique chordless x-y path for all x, y in S. Then S' is a superset of S and $\langle S'\rangle$ is connected. Since C contains S and $\langle C\rangle$ is connected, for all x, y in S the unique chordless x-y path is in $\langle C\rangle$, i.e., S' is a subset of C. By the definition of a hull, $C = S'$.

(3) ⇒ (1). If G is not a block graph, then there exists a block containing two nonadjacent vertices. Since a block has no cutpoint, there must exist two distinct chordless paths connecting these two vertices. But each such path is a hull of the two vertices, contradicting assumption (3). □

Consider a connected block graph $G(V, E)$ and a partition π of $V' \subseteq V$. Let CP denote the set of all cutpoints of G.

Lemma 2. *Suppose that A is a part of π. Then*

$$H(A)\backslash A \subseteq CP.$$

Proof: From the proof of Lemma 1 $\langle H(A)\rangle$ is the union of all chordless paths connecting pairs of vertices of A. Lemma 2 then follows from the fact that all internal vertices on the unique chordless path from x to y are cutpoints. □

For a given connected block graph $G(V, E)$ and a partition π of $V' \subseteq V$, part A is said to *penetrate* part B, written $A \to B$, if $A \cap H(B) \neq \emptyset$. The penetration is called *inclusive*, written $A \hookrightarrow B$, if $A \subseteq H(B)$. The *penetration graph* $g(\pi)$ of π is a digraph with parts as nodes and penetrations as links. A partition π is called *nested* (N) if $g(\pi)$ is acyclic; π is called *transitivity-nested* (T) if $g(\pi)$ is a partial order. Two special cases of T are: fully nested (F), when the partial order is linear; and consecutive (C), when $g(\pi)$ has no link. A partition is called *inclusive* (I) if $A \to B$ implies $A \hookrightarrow B$; and called *disjoint* (D) if A and B not penetrating each other implies $H(A) \cap H(B) = \emptyset$. Note that transitivity could be separated from nestedness to become an independent property, but this property is essentially captured by I in the current context. It should also be noted if $V' = V$, then C always implies D.

Lemma 3. *Inclusive penetration defines a partial order on the parts of a partition.*

Proof: It is easily seen that $A \subseteq H(B)$ implies $H(A) \subseteq H(B)$. Hence inclusive penetration is transitive. It suffices to prove that inclusive penetration is asymmetric. Suppose to the contrary that $A \subseteq H(B)$ and $B \subseteq H(A)$. Then $H(A) = H(B)$. Clearly, there exists a vertex $x \in A$ which is not a cutpoint of $\langle H(A) \rangle$. Since $A \cap B = \emptyset$, $x \notin B$. Hence $\langle H(B) \backslash x \rangle$ is still connected and contains B, contradicting the definition of $H(B)$. □

Corollary 1. *I implies T.*

The properties can be combined, but some of the combinations are redundant. For example, inclusiveness, inclusive nestedness and inclusive transitivity-nestedness are the same thing and we will only use the first term. Furthermore, the definition of consecutiveness is not affected by the notion of inclusiveness, so inclusive consecutiveness is not needed. On the other hand, inclusive full nestedness (*IF*) is a legitimate property. The notion of disjointness does not affect the definitions of F and *IF*. But we do have *disjoint nestedness* (*DN*), *disjoint transitivity-nestedness* (*DT*), *disjoint inclusive* (*DI*) and *disjoint consecutiveness* (*DC*).

We now define some properties which treat the partition as an ordered partition. A partition is called *order-consecutive* (O) if the parts can be labeled $V_1, \ldots, V_p$ such that $V_j \nrightarrow \bigcup_{i=1}^{k} V_i$ for all $j > k$, $k = 1, \ldots, p-1$. Similarly, we define *inclusive order-consecutiveness* (*IO*), *disjoint order-consecutiveness* (DO) and *disjoint inclusive order-consecutiveness* (DIO).

Lemma 4. *Suppose that $\langle \bigcup_{i \in I} H(V_i) \rangle$ is connected. Then $\bigcup_{i \in I} H(V_i) = H(\bigcup_{i \in I} V_i)$.*

Proof: Since $H(\bigcup_{i \in I} V_i)$ contains $H(V_i)$ for every $i \in I$, it contains $\bigcup_{i \in I} H(V_i)$. On the other hand, V_i is a subset of $H(V_i)$, hence $\bigcup_{i \in I} V_i$ is a subset of $\bigcup_{i \in I} H(V_i)$. Furthermore, $\langle \bigcup_{i \in I} H(V_i) \rangle$ is connected by assumption. It follows that $H(\bigcup_{i \in I} V_i) \subseteq \bigcup_{i \in I} H(V_i)$. □

Next we extend a characterization result for order-consecutiveness on the path (Hwang et al., 1996) to the block graph. However, the original proof cannot be easily adapted. A new approach is required.

Lemma 5. *An N p-partition is O if and only if there do not exist four parts V_1, V_2, V_3, V_4, all distinct except possibly $V_3 = V_4$, such that $V_1 \to V_3$, $V_2 \to V_4$, $V_3 \to (V_1 \cup V_2)$ and $V_4 \to (V_1 \cup V_2)$.*

Proof:

(i) The "only if" part. Suppose that an O partition contains four parts V_1, V_2, V_3, V_4 as characterized in Lemma 5. Then the ordering of parts must observe: V_1 before V_3, V_2 before V_4, V_3 before either V_1 or V_2, hence before V_2, and V_4 before either V_1 or V_2, hence before V_1. It is easily verified that the conditions are inconsistent.

(ii) The "if" part. Lemma 5 is trivially true for $p = 1$ and 2. We prove the general case by induction on $p \geq 3$. Let π denote an N p-partition of V'. It suffices to prove the existence of a part A of π such that $A \nrightarrow V' \backslash A$. Since by induction, $\pi \backslash A$ is an O $(p-1)$-partition of $V' \backslash A$, it follows from $A \nrightarrow V' \backslash A$ that π is an O p-partition.

Since $g(\pi)$ is acyclic, there exists a minimal part B (a part having no inlink in $g(\pi)$) in π. We consider two cases:

Case 1. B is the only minimal part. By induction, there exists a part A in $\pi \backslash B$ such that $A \nrightarrow V' \backslash (A \cup B)$. Since $p \geq 3$, there exist parts other than A and B. Furthermore, since B is the unique minimal part, all these other parts must be penetrated by some parts, and one of them, say C, must be penetrated by B. Thus $\langle H(B) \cup H(V' \backslash (A \cup B)) \rangle$ is connected since $H(V' \backslash (A \cup B))$ contains $H(C)$ which contains a point of B. By Lemma 4, $H(B) \cup H(V' \backslash (A \cup B)) = H(V' \backslash A)$. Since $A \cap H(B) = \emptyset$ and $A \cap H(V' \backslash (A \cap B)) = \emptyset$, it follows $A \cap H(V' \backslash A) = \emptyset$, i.e., $A \nrightarrow V' \backslash A$ and A is the part we look for.

Case 2. There exist two minimal parts B_1 and B_2. By induction, there exists a part A_i, $i \in \{1, 2\}$, in $\pi \backslash B_i$ such that $A_i \nrightarrow V' \backslash (A_i \cup B_i)$. If $A_i \nrightarrow V' \backslash A_i$ for either i, then we are done. So assume $A_i \rightarrow V' \backslash A_i$ for both i.

A vertex $v \in (A_i \cap H(V' \backslash A_i))$ is a cutpoint of $\langle H(V' \backslash A_i) \rangle$ (cf. Lemma 2). Note that B_i and $V' \backslash (A_i \cup B_i)$ are in two different components of $\langle H(V' \backslash A_i) \rangle \backslash v$, (otherwise $H(V' \backslash A_i) = H(B_i \cup (V' \backslash (A_i \cup B_i))) \subset H(V' \backslash A)_i) \backslash v$, a contradiction). Let C_i denote the component containing $V' \backslash (A_i \cup B_i)$. Since C_i is connected to v, which is in $H(A)$, $\langle C_i \cup H(A_i) \rangle$ is a connected graph containing all vertices in $V' \backslash B_i$ but no vertex of B_i. If $B_i \nrightarrow A_i$, then B_i is the part we look for (i.e. $B_i \nrightarrow V' \backslash B_i$). Thus we may assume $B_i \rightarrow A_i$ for both i. It is easily verified that B_1, B_2, A_1, A_2 are distinct except perhaps $A_1 = A_2$. By the 4-part condition of Lemma 5, $A_i \nrightarrow (B_1 \cup B_2)$ for at least one i. Let it be A_1. So we have $A_1 \nrightarrow B_1 \cup B_2$ and $A_1 \nrightarrow V' \backslash (A_1 \cup B_1)$. But $\langle H(B_1 \cup B_2) \cup H(V' \backslash (A_1 \cup B_1)) \rangle$ is a connected graph since B_2 is in both $B_1 \cup B_2$ and $V' \backslash (A_1 \cup B_1)$. By Lemma 4,

$$H(B_1 \cup B_2) \cup H(V' \backslash (A_1 \cup B_1)) = H(V' \backslash A_1).$$

It follows $A_1 \nrightarrow V' \backslash A_1$ and A_1 is the part we look for. □

Corollary 2. *Lemma* 5 *remains valid if N and O are replaced αN and αO, respectively, where $\alpha \in \{D, I, DI\}$.*

Corollary 3. *C implies O.*

Let $\mathcal{B}$ denote the set of properties $\{F, C, T, O, N, D, I\}$.

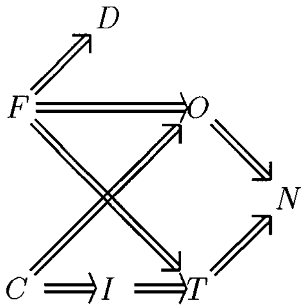

Figure 1. Relations among members of $\mathcal{B}$.

Theorem 1. *The relations among members of $\mathcal{B}$ are characterized in Figure* 1 ($x \Rightarrow y$ *means x implies y*).

Proof: $F \Rightarrow T$, $C \Rightarrow I$, $F \Rightarrow D$, $T \Rightarrow N$ and $O \Rightarrow N$ are obvious. $I \Rightarrow T$ was given by Corollary 1 and $C \Rightarrow O$ by Corollary 3. We prove the relation $F \Rightarrow O$. Let π be an F partition with parts $V_1, \ldots, V_p$ such that $V_i \rightarrow V_j$ for $i < j$. Suppose to the contrary that the 4-part condition in Lemma 5 is violated, i.e., there exist distinct i, j, k, l (except possibly $k = l$) such that $V_i \rightarrow V_k$, $V_j \rightarrow V_l$, $V_k \rightarrow V_i \cup V_j$ and $V_l \rightarrow V_i \cup V_j$. Then $i < k$ and $j < l$. Without loss of generality, assume $k \leq l$. Then $i < l$. So V_l penetrates neither V_i nor V_j; or equivalently, $V_l \cap (H(V_i) \cup H(V_j)) = \emptyset$. But $\langle H(V_i) \cup H(V_j) \rangle$ is a connected graph; hence $H(V_i) \cup H(V_j) = H(V_i \cup V_j)$ by Lemma 4. It follows $V_l \cap H(V_i \cup V_j) = \emptyset$, contradicting the assumption $V_l \rightarrow V_i \cup V_j$.

Next we prove the nonimplication part. It suffices to prove $F \not\Rightarrow I$, $D \not\Rightarrow N$, $O \not\Rightarrow T$, $C \not\Rightarrow D$ and $I \not\Rightarrow O$. Figure 2 shows these examples. □

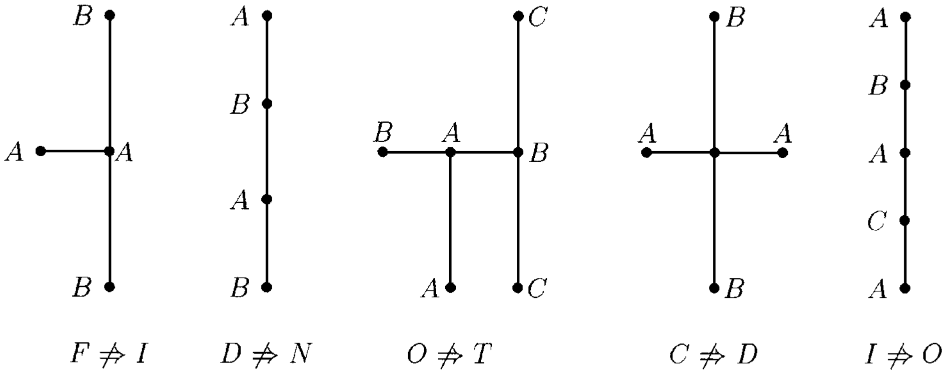

Figure 2. Examples of nonimplications.

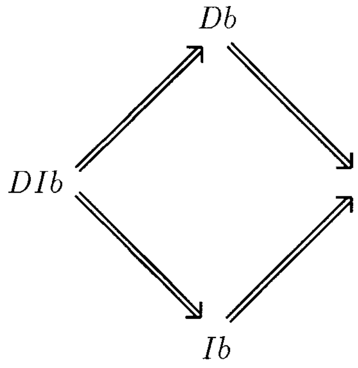

Figure 3. The associated relations.

The relations shown in Figure 3 hold for any $b \in B \setminus \{I, D\}$. Combining Figures 1 and 3, we obtain the relations on the 17 properties. It is easily verified that all valid implications are included in Figure 4.

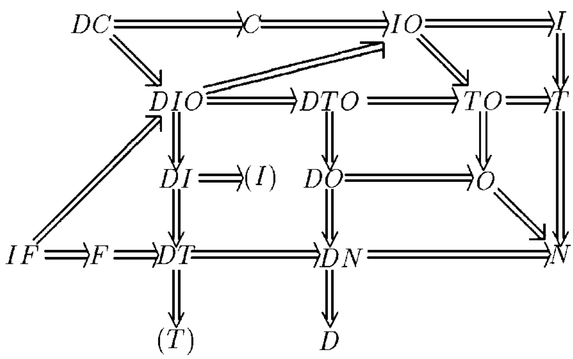

Figure 4. Relations among members of extended B .

3. Consistency

Clearly, k-consistency implies k'-consistency for any $k' > k$. We now give some results on the minimum consistency index of a property Q, which is defined as $\inf\{k : Q$ is k-consistent$\}$, the infimum of the empty set being ∞.

Theorem 2.
(i) *The minimum consistency index for I, D, DI, C, DC, and IF is* 2.
(ii) *The minimum consistency index for T, DT and F is* 3.

(iii) *The minimum consistency index for IO, TO, DIO and DTO is* 4.
(iv) *The minimum consistency index for* N, *DN*, *O and DO is* ∞.

Proof:

(i) The results for I, D, *DI*, *C* and *DC* follow directly from their definitions. Suppose that *IF* holds for any 2-subpartition of a partition π. Then $g(\pi)$ contains no 2-cycle. For *IF* the links in $g(\pi)$ are of the $\hookrightarrow$ type. Suppose there exists a cycle $V_1 \hookrightarrow V_2 \hookrightarrow \cdots \hookrightarrow V_k \hookrightarrow V_1$ in $g(\pi)$ for $k \geq 3$. Since $\hookrightarrow$ is transitive, the cycle implies $V_1 \hookrightarrow V_k \hookrightarrow V_1$, contradicting the assumption of no 2-cycle. Therefore, $g(\pi)$ is a partial order. Also the partial order is linear since every pair of nodes has a link and no cycle exists.

(ii) The partition in figure 5 is not any of T, *DT* and F. But deleting any part, the remaining 2-partition is T and *DT* and F (in a trivial way). Hence these properties are not 2-consistent. We now show that they are 3-consistent. Suppose to the contrary that $g(\pi)$ contains a cycle $V_1 \to V_2 \to \cdots \to V_k \to V_1$ for $k \geq 4$. By assumption penetration must be transitive on any three parts. Thus $V_1 \to V_2 \to V_3$ implies $V_1 \to V_3$. Using the same arguement, eventually we obtain $V_1 \to V_{k-1} \to V_k \to V_1$, contradicting the assumption that penetration is acyclic on three parts.

(iii) It was shown (Hwang et al., to appear) that *IO*, *TO*, *DIO* and *DTO* have the minimum consistency index 4 for the path. Hence their minimum consistency indices are at least 4 for the connected block graph. Let π be a partition where every 4-subpartition K of π is *IO*. Then K is I and satisfies the 4-part condition of Lemma 5. Clearly, π also satisfies the 4-part condition. Since I is 4-consistent (indeed, 2-consistent), π is I. Since $I \Rightarrow N$, by Corollary 2, π is *IO*. The arguments for *TO*, *DIO* and *DTO* are analogous.

(iv) We now show that N, *DN*, *O* and *DO* are not k-consistent for any $k > 1$. Consider the example in Figure 5. Any two parts have the specified properties, but not all three parts. This example can be easily generalized to $k > 2$ by replacing the triangle in the

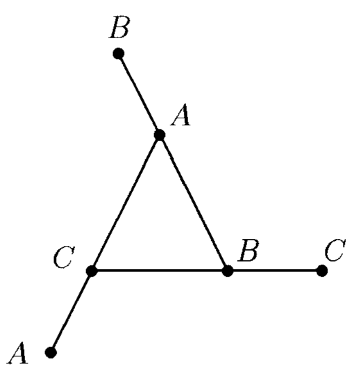

Figure 5. A counterexample against 2-consistency.

middle with a complete graph of $k+1$ vertices, resulting in a connected block graph of $2(k+1)$ vertices. □

4. Sortability

It appears that k-sortability does not imply, nor is implied by, k'-sortability for $k' > k$. However, we are still interested in the minimum k since the smaller k is, the easier it is to check the k-local condition. For strong sortability we have the following result.

Theorem 3. *Let Q be an hereditary property. Suppose that for every connected block graph G (with vertex-set V), we can construct another connected block graph $\tilde{G}$ (with vertex-set $\tilde{V}$) containing G so that if K is a k-subpartition of G satisfying Q, then $K \cup (\tilde{V} \backslash V)$ is a $(k+1)$-subpartition also satisfying Q. Then for the class of connected block graphs, Q is strongly k'-sortable implies that Q is strongly k-sortable for $k' > k$.*

Proof: It suffices to prove that if Q is not strongly k-sortable, then it is not strongly $(k+1)$-sortable. Suppose that Q is not strongly k-sortable. Then there exists a connected block graph G and a class P of partitions of $V' \subseteq V$ which weakly satisfies Q k-locally, but does not satisfy Q. Consider the connected block graph $\tilde{G}$ as specified in Theorem 3 and let $\tilde{P}$ be the class of partitions of $V' \cup (\tilde{V} \backslash V)$ by adding $\tilde{V} \backslash V$ as a part to every partition in P. It is easily verified that $\tilde{P}$ weakly satisfies Q $(k+1)$-locally. But $\tilde{P}$ cannot satisfy Q for if $\tilde{\pi}$ were a partition in $\tilde{P}$ satisfying Q, then since Q is hereditary, $\tilde{\pi} \backslash (\tilde{V} \backslash V)$ would be a partition in P satisfying Q, a contradiction. Therefore Q is not strongly $(k+1)$-sortable. □

Corollary 4. *Strong k'-sortability implies strong k-sortability for all properties studied in this paper whenever $k' > k$.*

Proof: For the properties F and IF, let $\tilde{G}$ be obtained from G by adding a new edge (plus a new vertex) to every vertex of G. For all other properties, let $\tilde{G}$ be obtained from G by adding a new edge (plus a new vertex) to an arbitrary vertex of G. □

Logically one should study the maximum k for which Q is strongly k-sortable. As a practical matter, it suffices to know if Q is strongly 2-sortable since it is easier to check the 2-local condition in most applications. It should be noted that Theorem 3 and Corollary 4 are stated in reference to the class of connected block graphs. For a subclass of connected block graphs, these results may or may not hold.

Theorem 4. *DC is strongly 2-sortable.*

Proof: For a partition π and a part A, let $\text{size}(A) = |H(A)| - 1$ and let $\text{size}(\pi)$ denote the sum of $\text{size}(A)$ over all A in π. Let P be a class of partitions such that P weakly satisfies DC 2-locally. Choose a $\pi \in P$. If π is not DC, find a $\pi' \in P$ obtained from π by DC-sorting a 2-subpartition. If π' is still not DC, then find another $\pi'' \in P$ obtained from π' by DC-sorting a 2-subpartition. We show that at each sorting step, $\text{size}(\pi)$ is decreasing. Hence this sorting process must end, which implies that at the end we obtain a *DC* partition.

Let A and B be two parts which are not a *DC* 2-subpartition and are *DC*-sorted into A' and B'. Since $H(A) \cap H(B) \neq \emptyset$, $\langle H(A) \cup H(B) \rangle$ is a connected graph containing $A \cup B$. By Lemma 4, $H(A \cup B) = H(A) \cup H(B)$. Hence

$$\text{size}(A \cup B) = |H(A) \cup H(B)| - 1.$$

Therefore,

$$\begin{aligned}\text{size}(A) + \text{size}(B) &= |H(A)| - 1 + |H(B)| - 1 \\ &\geq |H(A) \cup H(B)| + 1 - 2 \\ &= \text{size}(A \cup B).\end{aligned}$$

On the other hand, $H(A')$ and $H(B')$ are disjoint subsets of $H(A' \cup B') = H(A \cup B)$. Thus

$$\begin{aligned}\text{size}(A') + \text{size}(B') &= |H(A')| - 1 + |H(B')| - 1 \\ &\leq |H(A \cup B)| - 2 \\ &= \text{size}(A \cup B) - 1 \\ &< \text{size}(A) + \text{size}(B).\end{aligned}$$

□

Suppose that $G(V, E)$ is a connected block graph and P is a class of partitions of $V' \subseteq V$.

Lemma 6. *If P satisfies C 2-locally, then there exists a $\pi \in P$ and a part $A \in \pi$ such that A and $V' \backslash A$ do not penetrate each other.*

Proof: We prove the lemma by induction on the size of V. Recall that *CP* is the set of cutpoints of G. Define $CP' = CP \cap V'$.

Case 1. $CP' = \emptyset$. By Lemma 2, for every part A of a partition $\pi \in P$,

$$H(A) \cap (V' \backslash A) \subseteq CP' = \emptyset$$

and

$$H(V' \backslash A) \cap A \subseteq CP' = \emptyset.$$

Case 2. $CP' \neq \emptyset$. Choose any vertex $v \in V$ not a cutpoint and also choose a vertex $u \in CP'$ farthest away from v. Then every component of $\langle V \backslash u \rangle$ other than the one containing v contains no point in CP' (or u would not be the farthest point in CP'). Let U be the vertex-set of such a component. Without loss of generality, we will always label the part containing u (of a partition π) as $V_1 = V_1(\pi)$. Let $\tilde{\pi} = \{\tilde{V}_1, \ldots, \tilde{V}_p\}$ be a partition in P such that U intersects a minimum number of parts other than $\tilde{V}_1$. (Note that $u \in \tilde{V}_1$.)

Suppose that U contains vertices in some $\tilde{V}_i \neq \tilde{V}_1$. By the 2-local condition, there exists a partition $\pi' \in P$ which can be obtained from $\tilde{\pi}$ by C-sorting $\tilde{V}_1$ and $\tilde{V}_i$ into V_1' and V_i' ($\pi' = \tilde{\pi}$ is allowed). Since $\tilde{\pi}$ minimizes the number of parts (other than $\tilde{V}_1$) that intersect U, $V_i' \cap U \neq \emptyset$. Necessarily $V_i' \subset U$ (or V_1' would penetrate V_i'). By Lemma 2,

$$H(V_i') \cap (V' \backslash V_i') \subseteq U \cap CP' = \emptyset$$

and

$$H(V' \backslash V_i') \cap V_i' \subseteq U \cap CP' = \emptyset,$$

i.e., Lemma 6 is proved by taking V_i' as part A (of π').

Therefore it suffices to consider the case that $U \cap V' \subseteq \tilde{V}_1$. Define $G^* = \langle V \backslash U \rangle$ (which remains a connected block graph), $V^* = V \backslash U$, $V'^* = V' \backslash U$ and

$$P^* = \{\pi^* = \pi(V'^*) : \pi \in P, U \cap V' \subseteq V_1 = V_1(\pi)\},$$

where $\pi(V'^*)$ denotes the restriction of π to V'^*. Note that P^* is nonempty since $\tilde{\pi}(V'^*) \in P^*$. We claim that either P^* satisfies C 2-locally or there exist a $\pi \in P$ and an $A \in \pi$ such that A and $V' \backslash A$ do not penetrate each other. Let $\pi(V'^*) = \{V_1^*, \ldots, V_p^*\} \in P^*$. Consider $i \neq j$. If neither i nor j is 1, then $V_i^* = V_i$ and $V_j^* = V_j$. By the 2-local condition on P, there exists a $\pi' \in P$ which can be obtained from π by C-sorting V_i and V_j. Hence $\pi'(V'^*) \in P^*$ and $\pi'(V'^*)$ can be obtained from $\pi(V'^*)$ by C-sorting V_i^* and V_j^*. If $i = 1$, then $V_1^* = V_1 \backslash U$ and $V_j^* = V_j$. By the 2-local condition on P, there exists a $\pi' \in P$ which can be obtained from π by C-sorting V_1 and V_j such that V_1' and V_j' do not penetrate each other. Since $u \in V_1'$, either $V_j' \subseteq U$ or $V_j' \cap U = \emptyset$. If $V_j' \subseteq U$, then V_j' and $V' \backslash V_j$ do not penetrate each other (see the argument in the preceding paragraph). If $V_j' \cap U = \emptyset$, then $U \cap V' \subseteq V_1'$. Hence $\pi'(V'^*) \in P^*$ and $\pi'(V'^*)$ can be obtained from $\pi(V'^*)$ by C-sorting V_1^* and V_j^* into $V_1' \backslash U$ and V_j' which do not penetrate each other. This proves the claim.

So we may assume that P^* satisfies C 2-locally. Since $|V^*| < |V|$, Lemma 6 holds for G^* and V'^* by induction. Therefore, there exists a $\pi(V'^*) \in P^*$ and a part $A \in \pi(V'^*)$ such that A and $V'^* \backslash A$ do not penetrate each other. We can write $A = V_i^*$ for some i. Note that $V_i^* = V_i$ for $i \neq 1$. It is not difficult to see that V_i and $V' \backslash V_i$ do not penetrate each other. □

Theorem 5. *C is 2-sortable.*

Proof: We prove Theorem 4 by induction on $|V'|$. Suppose that P satisfies C 2-locally. By Lemma 6 the set $P_A = \{\pi \in P : A \in \pi, A \nrightarrow V' \backslash A, V' \backslash A \nrightarrow A\}$ is not empty for some A. Define $P_A^* = \{\pi^* = \pi(V' \backslash A) : \pi \in P_A\}$. Then P_A^* satisfies C 2-locally. By induction there exists a C partition π' (in P_A^*) of $V' \backslash A$. Then $\pi = \pi' \cup A$ is a C partition in P. □

Unlike consecutiveness, we have only negative results on the sortability of nestedness.

We show that I is not strongly 2-sortable by giving a tree T and a class P of partitions which weakly satisfies I 2-locally, but does not contain a I partition. Let P consist of the following four partitions (a vertex is labeled by the part it belongs):

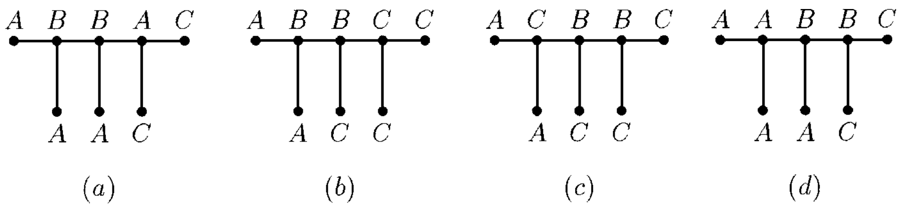

Figure 6. A class of 4 partitions.

In Figure 6(a), A and C are not I, but figure 6(b) gives a partition obtained by I-sorting A and C. Similarly, partition 6(c) (6(d), 6(a)) is obtained from partition 6(b) (6(c), 6(d)) by I-sorting B and C (A and C, A and B). So P weakly satisfies I 2-locally, but none of the partitions in P is I. This is also an example against *DI* being strongly 2-sortable.

Note that figure 6 does not give an example against 2-sortability. Partition 6(b) has two non-I pairs, (A, B) and (B, C). But P does not contain a partition obtainable from partition 6(b) by I-sorting A and B. Hence, P does not satisfy I 2-locally.

It has been shown (Hwang et al., 1996) that *IF*(F) is not 2-sortable for the path. Hence, it is not 2-sortable for the connected block graph in general. Furthermore, since k-sortability implies k-consistency, *DO*, O, *DN* and N are not k-sortable for any $k > 1$.

5. Conclusion

We summarize what is known about consistency and sortability for the connected block graph in Table 1. The entries give the minimum k for which the property in the column is k-consistent or k-sortable, and answer yes (Y) or no (N) to strong 2-sortability.

Table 1. A summary on the connected block graph.

	IF	*DC*	*DIO*	*C*	*F*	*DI*	*IO*	*DT*	*I*	*DO*	*T*	*DN*	*O*	*N*	*D*	*TO*	*DTO*
Consistency	2	2	4	2	3	2	4	3	2	∞	3	∞	∞	∞	2	4	4
Sortability	>2	2	>3	2	>2		>3	>2		∞	>2	∞	∞	∞		>3	>3
Strong 2-sort	N	Y	N		N	N	N	N	N	N	N	N	N	N		N	N

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