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Spectral Characterization of Some Generalized Odd Graphs

Tayuan Huang[1] and Chao-Rong Liu[2]

[1] Department of Applied Mathematics, National Chiao-Tung University, Hsinchu, Taiwan, R.O.C.. e-mail: thuang@math.nctu.edu.tw

[2] Institute of Mathematics, Academia Sinica, Taipei, Taiwan, R.O.C.

Abstract. Suppose G is a connected, k-regular graph such that $Spec(G) = Spec(\Gamma)$ where Γ is a distance-regular graph of diameter d with parameters $a_1 = a_2 = \cdots = a_{d-1} = 0$ and $a_d > 0$; *i.e.*, a generalized odd graph, we show that G must be distance-regular with the same intersection array as that of Γ in terms of the notion of Hoffman Polynomials. Furthermore, G is isomorphic to Γ if Γ is one of the odd polygon C_{2d+1}, the Odd graph O_{d+1}, the folded $(2d+1)$-cube, the coset graph of binary Golay code $(d = 3)$, the Hoffman-Singleton graph $(d = 2)$, the Gewirtz graph $(d = 2)$, the Higman-Sims graph $(d = 2)$, or the second subconstituent of the Higman-Sims graph $(d = 2)$.

1. Introduction

We shall consider only finite undirected graphs without loops and multiple edges. Let $G = (V(G), E(G))$ be a connected, k-regular graph and A an adjacency matrix of G, which is row-indexed, as well as column-indexed by the vertices of G; also let A^i be the usual matrix product of i copies of A, and $A^i(x, y)$ be the entry of A^i at row x and column y. Suppose λ is an eigenvalue of A, then, since A is symmetric, λ is real, and the multiplicity of λ as a root of the characteristic equation $\det(\lambda I - A) = 0$ is equal to the dimension of the eigenspace corresponding to λ. The spectrum of A is also called the *spectrum* of the graph G, denoted by

$$Spec(G) = (k^{m_0}, \theta_1^{m_1}, \ldots, \theta_{s-1}^{m_{s-1}})$$

where $k(=\theta_0) > \theta_1 > \cdots > \theta_{s-1}$ are distinct eigenvalues together with their multiplicities $m_0 = 1, m_1, \ldots,$ and m_{s-1} respectively; refer to [3, 5] for more details.

Now assume Γ is a connected graph with diameter d, let $\Gamma_i(x) = \{y \mid y \in V(\Gamma)$ and $d(x, y) = i\}$, where $V(\Gamma)$ is the vertex set of Γ and $d(x, y)$ is the distance between vertices x and y. Γ is called *distance-regular* if the parameters $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$, $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$ and $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$ depend not on particular vertices x and y we choose, but only on the distance $i = d(x, y)$ between them. It is clear that $c_0 = b_d = 0$, $c_1 = 1$, $b_0 = |\Gamma_1(x)|$ for each $x \in V(\Gamma)$, and $a_i = b_0 - b_i - c_i$. The following array

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & a_3 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & b_3 & \cdots & b_{d-1} & b_d \end{bmatrix}$$

or $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ is called *the intersection array of* Γ. *Generalized odd graphs* are distance-regular graphs of diameter d with parameters $a_1 = a_2 = \cdots = a_{d-1} = 0$, and $a_d > 0$. As shown in Section 2, the odd polygons C_{2d+1}, the Odd graphs O_{d+1}, the folded $(2d+1)$-cubes, the coset graph of binary Golay code, the coset graph of the truncated binary Golay code, the Hoffman-Singleton graph, the Gewirtz graph, the Higman-Sims graph and the second subconstituent of the Higman-Sims graph are examples of generalized odd graphs, refer to [5] for further details.

One can see from the spectrum of a graph whether it is regular and connected, whether it is strongly regular, or whether it is bipartite distance-regular of diameter 3 [5, p. 263], but one can not tell in general its distance-regularity directly. On the other hand, it is known that a connected, regular graph with diameter d has at least $d+1$ distinct eigenvalues, and that distance-regular graphs of diameter d have exactly $d+1$ distinct eigenvalues. The determination of the distance-regularity of those connected regular graphs of diameter d with exactly $d+1$ distinct eigenvalues is an interesting subject. The distance-regularity of some graphs in terms of their spectra have been studied under some additional conditions, for example: with large girth [6], with diameter 3 and $\mu = 1$ [10], with diameter 4 and prescribed λ, k_2 [8], and all graphs with spectra of distance-regular graphs with at most 30 vertices can be found in [11]. Continuing the work in [13], the relationship between distance-regularity and spectra of connected regular graphs will be further studied in this paper. Indeed, we prove

Main Theorem *If G is a connected regular graph which has the same spectrum as that of a generalized odd graph* Γ *of diameter d, then G is distance-regular with the same intersection array as that of* Γ. *Furthermore, G is isomorphic to* Γ *if* Γ *is one of the following: the odd polygon* C_{2d+1}, *the Odd graph* O_{d+1}, *the folded* $(2d+1)$-*cube, the coset graph of the binary Golay code* $(d = 3)$, *the Hoffman-Singleton graph* $(d = 2)$, *the Gewirtz graph* $(d = 2)$, *the Higman-Sims graph* $(d = 2)$, *or the second subconstituent of the Higman-Sims graph* $(d = 2)$.

The main theorem answers affirmatively a question asked by Cvetković [7, p. 36] that whether the Odd graphs can be characterized by their spectra among connected regular graphs. As an easy corollary of a theorem of Tutte [15], we have

Corollary *If G is a connected regular graph which has the same deck of* 1-*vertex-deleted subgraphs as that of* Γ *where* Γ *is a generalized odd graph, then G is a distance-regular graph with the same intersection array as that of* Γ.

One of the significant links between spectra and connectedness, regularity of graphs is the so called *Hoffman Polynomial*. For a connected, k-regular graph G with $Spec(G) = (k, \theta_1^{m-1}, \ldots, \theta_{s-1}^{m_{s-1}})$, let $q(x) = \prod_{i=1}^{s-1}(x - \theta_i)$, then $p(x) =$

$\frac{|V(G)|}{q(k)}q(x)$ is called the *Hoffman Polynomial* of G, which is the unique polynomial of the smallest degree such that $p(A) = J$, where A is an adjacency matrix of G and J is the all-one matrix of order $|V(G)|$. The main theorem is proved in terms of the common Hoffoman polynomial of Γ and G, its coefficients are studied in Section 2. Some systems of linear equations associated with $A^i q(A) = \frac{q(k)}{|V(G)|}k^i J$ are considered in Section 3. Based on the properties among the coefficients of $q(x)$ obtained in Section 2, we show that each of them has a unique solution, which leads to the distance-regularity of G. Lemma 3.1 concerning the non-singularity of those coefficient matrices mentioned above is proved in an algorithmic way. The argument developed in this paper does not work for bipartite distance-regular graphs because Lemma 3.1 is no longer true due to the symmetry (with respect to 0) of their spectra.

2. Hoffman Polynomials of Generalized Odd Graphs

Throughout the rest of this paper, we assume that G is a connected k-regular graph with $Spec(G) = Spec(\Gamma) = (\theta_0^{m_0}, \theta_1^{m_1}, \theta_2^{m_2}, \ldots, \theta_d^{m_d})$ with $\theta_0 = k$, and $m_0 = 1$, where Γ is a generalized odd graph of diameter d with intersection array

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 & \cdots & c_{d-1} & c_d \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_d \\ b_0 & b_1 & b_2 & b_3 & \cdots & b_{d-1} & b_d \end{bmatrix}.$$

Furthermore, let A be an adjacency matrix of G. The common *Hoffman Polynomial* for the graphs Γ and G in terms of their common spectrum is studied in this section, which provides a key step to show the distance-regularity of G in the next section.

Clearly, odd polygons C_{2d+1} are generalized odd graphs of diameter d with intersection array $\{2, 1, \ldots, 1; 1, 1, \ldots, 1\}$. We now recall some other examples, families or sporadic, of generalized odd graphs. The Odd graph O_{d+1} has the d-subsets of $\{1, 2, \ldots, 2d+1\}$ as vertices, and two vertices are adjacent if and only if their corresponding subsets are disjoint. The small Odd graphs are the triangle $K_3 (d = 1)$, and the Petersen graph $(d = 2)$. In general, the Odd graphs O_{d+1} are distance-regular graphs of diameter d with intersection array

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 2 & \cdots & r-1 & r-1 & r \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & r \\ 2r & 2r-1 & 2r-1 & 2r-2 & 2r-2 & \cdots & r+1 & r+1 & 0 \end{bmatrix}$$

for the case $d + 1 = 2r$, and

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 2 & \cdots & r & r \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & r+1 \\ 2r+1 & 2r & 2r & 2r-1 & 2r-1 & \cdots & r+1 & 0 \end{bmatrix}$$

for the case $d + 1 = 2r + 1$.

The eigenvalues of O_{d+1} are the integers $\theta_i = (-1)^i(d+1-i)$ with multiplicities $m_i = \binom{2d+1}{i} - \binom{2d+1}{i-1}$ respectively for $0 \le i \le d$. The Odd graphs are uniquely determined by their intersection arrays, refer to [14] or [5, p. 260], among distance-regular graphs.

Folded $(2d+1)$-cube is the graph defined on the partitions of an $(2d+1)$-set into two subsets, and two partitions being adjacent when their common refinement contains a set of size one. Its intersection array is given by

$$\begin{bmatrix} 0 & 1 & 2 & 3 & \cdots & d-1 & d \\ 0 & 0 & 0 & 0 & \cdots & 0 & d+1 \\ 2d+1 & 2d & 2d-1 & 2d-2 & \cdots & d+2 & 0 \end{bmatrix},$$

and its eigenvalues and multiplicities are $\theta_j = 2d+1-4j$ with $m_j = \binom{2d+1}{2j}$, $j \le d$. The folded $(2d+1)$-cube is also uniquely determined by its intersection array [5, p. 264].

In addition to these families, Moore graphs, *i.e.*, distance-regular graphs with the intersection array

$$\begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & k-1 \\ k & k-1 & k-1 & k-1 & \cdots & k-1 & 0 \end{bmatrix}$$

provide another family of generalized odd graphs, refer to [3] for more details. It is known that the intersection array for a Moore graph with valency $k \ge 3$, and girth $g \ge 5$ is feasible if and only if $g = 5$ and $k \in \{3, 7, 57\}$. The cases $k = 3, 7$ are realized by the Petersen graph and the Hoffman-Singleton graph respectively. The existence of a Moore graph with $k = 57$ remains open, which can not be distance transitive if it exists. Some other sporadic generalized odd graphs are given in the following table, whether these graphs are uniquely determined by their intersection arrays are indicated too.

For $x, y \in V(G)$ at distance i, let

$$|G_j(x) \cap G_1(y)| = \begin{cases} c_i(x,y) & \textit{if } j = i-1, \\ a_i(x,y) & \textit{if } j = i, \\ b_i(x,y) & \textit{if } j = i+1. \end{cases}$$

To show the distance-regularity of G is equivalent to show that all $c_i(x,y)$, $a_i(x,y)$ and $b_i(x,y)$ are functions of $i = d(x,y)$ only, independent of the choice of x and y for all i with $0 \le i \le d$. This can be achieved by showing that some systems of linear equations related to the Hoffman polynomial have unique solutions.

As mentioned before, let A be an adjacency matrix of G which is row-indexed and column-indexed by vertices of G. Since $A^{j+1}(x,y) = (A^jA)(x,y) = \sum_{z \in G_1(y)} A^j(x,z)$, and $G_j(x) \cap G_1(y)$ is empty if $j \ne i-1$, i or $i+1$ whenever $x, y \in V(G)$ at distance i. Lemma 2.1 is obvious, which is included here for later reference.

diameter	intersection array	examples	uniqueness	remarks
$d = 2$	{3,2;1,1}	Petersen graph O_3	Yes	Moore graph
	{7,6;1,1}	Hoffman-Singleton graph	Yes	Moore graph [5, p. 391]
	{57,56;1,1}	?		
	{10,9;1,2}	Gewirtz graph	Yes	[5, p. 372]
	{16,15;1,4}	the second subconstituent of the Higman-Sims graph	Yes	[5, p. 394]
	{22,21;1,6}	the Higman-Sims graph	Yes	[9, p. 933]
$d = 3$	{7,6,6;1,1,2}	?		[5, p. 148]
	{23,22,21;1,2,3}	the coset graph of the binary Golay code	Yes	[5, p. 361]
	{22,21,20;1,2,6}	the coset graph of the truncated binary Golay code	?	[5, p. 362]

Lemma 2.1. *If* $d(x,y) = i$, *then*

$$A^{j+1}(x,y) = \sum_{z \in G_1(y) \cap G_{i-1}(x)} A^j(x,z) + \sum_{z \in G_1(y) \cap G_i(x)} A^j(x,z) + \sum_{z \in G_1(y) \cap G_{i+1}(x)} A^j(x,z)$$

In particular,

$$A^i(x,y) = \sum_{z \in G_1(y) \cap G_{i-1}(x)} A^{i-1}(x,z).$$

Since $A^i(x,y)$ indicates the number of walks of length i in G joining x and y, it follows that the number of closed walks in G of length $2i+1$ is $Tr(A^{2i+1}) = \sum_{j=0}^{d} m_j \theta_j^{2i+1}$ [16, p. 310]. On the other hand, $a_i = 0$ $(i \le d-1)$ for generalized odd graphs, they have no odd cycles of length up to $2d-1$, it follows that $\sum_{j=0}^{d} m_j \theta_j^{2i+1} = 0$, and hence $A^{2i+1}(x,x) = 0$ for all $x \in V(G)$. These observations are summarized in the following.

Lemma 2.2. 1. $A^{2i+1}(x,x) = 0$ *for* $i \le d-1$,
2. $A^{2i+1-j}(x,y) = 0$ *for* $y \in G_j(x)$ *and* $1 \le j \le i$, *and*
3. $a_i(x,y) = 0$ *for all* $y \in G_i(x)$ *and* $i \le d-1$.

We now turn to the explicit expressions for the coefficients of the *Hoffman Polynomial* of Γ and G. Let $m(x) = (x-\theta_0)(x-\theta_1)(x-\theta_2)\cdots(x-\theta_d) = \sum_{i=0}^{d+1} m_i x^i$ be the minimal polynomial of Γ, then the coefficients m_i $(0 \le i \le d+1)$ can be calculated in a combinatorial way in terms of the fact that $m(x) = \det(xI - B)$ [5, p. 128] where

$$B = \begin{bmatrix} 0 & c_1 & & & & 0 \\ b_0 & 0 & c_2 & & & \\ & b_1 & 0 & c_3 & & \\ & & \cdots & \cdots & & \\ & & & \cdots & \cdots & \\ & & & b_{d-2} & 0 & c_d \\ 0 & & & & b_{d-1} & a_d \end{bmatrix}_{(d+1)\times(d+1)}$$

is the intersection matrix of Γ.

Recall that if $M = [a_{i,j}]$ is a tridiagonal matrix of order n, then

$$\det M = \sum_{\sigma} sign(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

where the summation is over all permutations σ which are product of disjoint transformations of the form $(i, i+1)$, since $a_{i,\sigma(i)} = 0$ if $|\sigma(i) - i| \geq 2$ for some i. In the expansion of $\det(xI - B)$ as a sum of products of entries from various rows and columns, as remarked above, $b_i c_{i+1}$ $(0 \leq i \leq d-1)$ always appear in pairs, but $b_i b_{i+1}$ $(0 \leq i \leq d-1)$ does not. It follows that

$$\begin{aligned} &\det(xI - B) \\ &= \sum_{s=0}^{\lfloor \frac{d+1}{2} \rfloor} (-1)^s \left(\left(\sum_{\mathscr{S}_1} \prod_{j=1}^{s} b_{i_j} c_{i_j+1} \right) x^{d+1-2s} + \left(\sum_{\mathscr{S}_2} \prod_{j=1}^{s} b_{i_j} c_{i_j+1} \right) (x - a_d) x^{d-2s} \right) \\ &= \sum_{s=0}^{\lfloor \frac{d+1}{2} \rfloor} (-1)^s \left(\sum_{\mathscr{S}_1 \cup \mathscr{S}_2} \prod_{j=1}^{s} b_{i_j} c_{i_j+1} \right) x^{d+1-2s} + a_d \sum_{s=0}^{\lfloor \frac{d+1}{2} \rfloor} (-1)^{s+1} \left(\sum_{\mathscr{S}_2} \prod_{j=1}^{s} b_{i_j} c_{i_j+1} \right) x^{d-2s} \end{aligned}$$

where $\mathscr{S}_1$, $\mathscr{S}_2$ consist of all s-element subsets $S = \{i_1, i_2, \ldots, i_s\}$ of $\{0, 1, 2, \ldots, d-1\}$ with $i_j + 2 \leq i_{j+1}$ for $j = 1, 2, \ldots, s-1$ and $\{d-1\} \cap S$ is empty or not respectively. The above expressions of those coefficients of $m(x)$ can be transformed into the following way, which is suitable for later computational purpose. Clearly $m_{d+1} = 1$, $m_d = -a_d$, $m_{d-1} = -(b_0 c_1 + b_1 c_2 + \cdots + b_{d-1} c_d)$, and in general

$$\begin{aligned} &m_{d-2s} \\ &= (-1)^{s+1} a_d \left(\sum_{i_1=0}^{d-2s} b_{i_1} c_{i_1+1} \left(\sum_{i_2=i_1+2}^{d-2s+2} b_{i_2} c_{i_2+1} \left(\cdots b_{i_{s-1}} c_{i_{s-1}+1} \left(\sum_{i_s=i_{s-1}+2}^{d-2} b_{i_s} c_{i_s+1} \right) \cdots \right) \right) \right) \end{aligned}$$

for $1 \leq s \leq \lfloor d/2 \rfloor$, and

$$\begin{aligned} &m_{d-2t+1} \\ &= (-1)^t \left(\sum_{i_1=0}^{d-2t+1} b_{i_1} c_{i_1+1} \left(\sum_{i_2=i_1+2}^{d-2t+3} b_{i_2} c_{i_2+1} \left(\cdots b_{i_{t-1}} c_{i_{t-1}+1} \left(\sum_{i_t=i_{t-1}+2}^{d-1} b_{i_t} c_{i_t+1} \right) \cdots \right) \right) \right) \end{aligned}$$

for $1 \leq t \leq \lceil d/2 \rceil$,

Note that a_d ($\neq 0$) occurs in m_{d-2s} for $1 \leq s \leq \lfloor d/2 \rfloor$. Furthermore, let

$$\begin{aligned} q(x) &= m(x)/(x-\theta_0) \\ &= q_d x^d + q_{d-1}x^{d-1} + q_{d-2}x^{d-2} + \cdots + q_2 x^2 + q_1 x^1 + q_0, \end{aligned}$$

then $q_d = 1$, $q_{d-1} = c_d$, $q_{d-2} = c_{d-1}c_d - (b_0 c_1 + \cdots + b_{d-2}c_{d-1})$ and its general expressions are given in the following lemma, which can be checked straightforward in terms of the recurrence relations $q_{i-1} = m_i + kq_i$ where $k = \theta_0 = b_i + c_i$ for $1 \leq i \leq d-1$. Note that c_d occurs in q_{d-2s-1} for $1 \leq s \leq \lceil d/2 \rceil - 1$.

Lemma 2.3. *Let* $q(x) = x^d + c_d x^{d-1} + \sum_{i=0}^{d-2} q_i x^i$, *then*

$$q_{d-2t} = (-1)^t \left(P^d_{d-2t+1} + \sum_{i_1=0}^{d-2t} b_{i_1} c_{i_1+1} \left(P^d_{d-2t+3} + \sum_{i_2=i_1+2}^{d-2t+2} b_{i_2} c_{i_2+1} \left(\cdots \left(P^d_{d-1} + \sum_{i_t=i_{t-1}+2}^{d-2} b_{i_t} c_{i_t+1} \right) \cdots \right)\right)\right)$$

for $1 \leq t \leq \lfloor d/2 \rfloor$, and

$$q_{d-2s-1} = c_d(-1)^s \left(P^{d-1}_{d-2s} + \sum_{i_1=0}^{d-2s-1} b_{i_1} c_{i_1+1} \left(P^{d-1}_{d-2s+2} + \sum_{i_2=i_1+2}^{d-2s+1} b_{i_2} c_{i_2+1} \left(\cdots \left(P^{d-1}_{d-2} + \sum_{i_s=i_{s-1}+2}^{d-3} b_{i_s} c_{i_s+1} \right) \cdots \right)\right)\right)$$

for $1 \leq s \leq \lceil d/2 \rceil - 1$.

where

$$P^l_s = (-1)^{(l-s+1)/2} c_s c_{s+1} \cdots c_{l-1} c_l$$

in case $l - s$ is positive and odd.

The expressions of these coefficients of the polynomial $q(x)$ will be needed in the next section. Let $v = \frac{q(\theta_0)}{|V(G)|}$, then $q(A) = vJ$ where J is the all one matrix of order $|V(G)|$ [12]. Multiplying A^i on both sides of the equation $q(A) = vJ$, and since $AJ = \theta_0 J$, we have $A^i q(A) = \theta_0^i vJ$, $0 \leq i \leq d-1$. The information contained in this system of matrix equations can be translated into a set of systems of linear equations in variables $A^i(x,y)$ and with the coefficients of $q(x)$ as its coefficients. A series of row operations will be performed on these coefficient matrices, which pave a way to show the distance-regularity of G. For this purpose, we define $F_{m,1} = q_{d-2m-1}/c_d$, $S_{m,1} = q_{d-2m}$ and the general terms $F_{m,i}$ and $S_{m,i}$ will be given later in Section 3.

3. Proof of the Main Theorem

Following the same notation used in section 2, we shall show that $c_i(x,y)$, $a_i(x,y)$ and $b_i(x,y)$ are functions of $i = d(x,y)$ only, independent of the choice of x and y,

$0 \le i \le d$, by showing that each system mentioned above has a unique solution. Indeed, the distance structure of the given generalized odd graph Γ provides non-trivial solutions for these systems. We shall show in this section that each of their coefficient matrices is nonsingular, and hence their solutions are unique.

Clearly, $a_i = a_i(x,y) = 0$ whenever x, y are at distance i at most $d-1$ as shown in Lemma 2.2. To determine $c_i(x,y)$, $b_i(x,y) = b_0 - c_i(x,y) - a_i(x,y)$ whenever $x, y \in V(G)$ at distance i, $0 \le i \le d-1$, we shall show the uniqueness of $A^i(x,y)$ with $d(x,y) = i$, $2 \le i \le d-1$ by solving the following systems of linear equations obtained from $q(A) = vJ$,

$$\begin{cases} A^{d-1-i}q(A) = v\theta_0^{d-1-i}J \\ A^{d-2-i}q(A) = v\theta_0^{d-2-i}J \\ \vdots \\ A^2 q(A) = v\theta_0^2 J \\ A^1 q(A) = v\theta_0^1 J \\ A^0 q(A) = v\theta_0^0 J \end{cases} \qquad (*)$$

at entries (x,y) for vertices x and y at distance i, $2 \le i \le d-1$.

As before, θ_0 is also denoted by k. Indeed, for vertices x, y at distance i, $2 \le i \le d-1$, clearly $A^i(x,y) = 0$ if $i < d(x,y)$ or if $i + d(x,y) \le 2d-1$ is odd by Lemma 2.2, the others $A^l(x,y)$ can be regarded as variables. More precisely, for vertices x and y at distance 2, $A(x,y) = A^3(x,y) = A^5(x,y) = \cdots = A^{2d-3}(x,y) = 0$, the above system can be reduced into

$$\left[\begin{array}{cccccccccccc} q_{d-1} & q_{d-3} & q_{d-5} & \cdots & q_5 & q_3 & q_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & q_{d-2} & q_{d-4} & \cdots & q_6 & q_4 & q_2 & q_0 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & q_{d-1} & q_{d-3} & \cdots & q_7 & q_5 & q_3 & q_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & q_{d-2} & \cdots & q_8 & q_6 & q_4 & q_2 & q_0 & 0 & \cdots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \cdots & 0 & q_{d-1} & q_{d-3} & q_{d-5} & q_{d-7} & q_{d-9} & \cdots & q_3 & q_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & q_{d-2} & q_{d-4} & q_{d-6} & q_{d-8} & \cdots & q_4 & q_2 \end{array}\right]$$

$$\times \begin{bmatrix} A^{2d-4}(x,y) \\ A^{2d-6}(x,y) \\ A^{2d-8}(x,y) \\ A^{2d-10}(x,y) \\ \vdots \\ A^4(x,y) \\ A^2(x,y) \end{bmatrix} = v \begin{bmatrix} k^{d-3} \\ k^{d-4} \\ k^{d-5} \\ k^{d-6} \\ \vdots \\ k^1 \\ k^0 \end{bmatrix}$$

whenever d is even; or

$$\left[\begin{array}{ccccccccccccc}
q_{d-1} & q_{d-3} & q_{d-5} & \cdots & q_6 & q_4 & q_2 & q_0 & 0 & 0 & \cdots & 0 & 0 \\
1 & q_{d-2} & q_{d-4} & \cdots & q_7 & q_5 & q_3 & q_1 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & q_{d-1} & q_{d-3} & \cdots & q_8 & q_6 & q_4 & q_2 & q_0 & 0 & \cdots & 0 & 0 \\
0 & 1 & q_{d-2} & \cdots & q_9 & q_7 & q_5 & q_3 & q_1 & 0 & \cdots & 0 & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & 0 & \cdots & 0 & q_{d-1} & q_{d-3} & q_{d-5} & q_{d-7} & q_{d-9} & \cdots & q_2 & q_0 \\
0 & 0 & 0 & \cdots & 0 & 1 & q_{d-2} & q_{d-4} & q_{d-6} & q_{d-8} & \cdots & q_3 & q_1 \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & q_{d-1} & q_{d-3} & q_{d-5} & q_{d-7} & \cdots & q_4 & q_2
\end{array}\right]$$

$$\times \begin{bmatrix} A^{2d-4}(x,y) \\ A^{2d-6}(x,y) \\ A^{2d-8}(x,y) \\ A^{2d-10}(x,y) \\ \vdots \\ A^6(x,y) \\ A^4(x,y) \\ A^2(x,y) \end{bmatrix} = v \begin{bmatrix} k^{d-3} \\ k^{d-4} \\ k^{d-5} \\ k^{d-6} \\ \vdots \\ k^2 \\ k^1 \\ k^0 \end{bmatrix}$$

whenever d is odd.

Similarly for vertices x, y at distance 3, substituting $A(x,y) = A^2(x,y) = A^4(x,y) = A^6(x,y) = \cdots = A^{2d-4}(x,y) = 0$, the above system can also be reduced into

$$\left[\begin{array}{cccccccccccc}
q_{d-1} & q_{d-3} & q_{d-5} & \cdots & q_5 & q_3 & q_1 & 0 & 0 & 0 & \cdots & 0 \\
1 & q_{d-2} & q_{d-4} & \cdots & q_6 & q_4 & q_2 & q_0 & 0 & 0 & \cdots & 0 \\
\hline
0 & q_{d-1} & q_{d-3} & \cdots & q_7 & q_5 & q_3 & q_1 & 0 & 0 & \cdots & 0 \\
0 & 1 & q_{d-2} & \cdots & q_8 & q_6 & q_4 & q_2 & q_0 & 0 & \cdots & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & 0 & \cdots & 0 & q_{d-1} & q_{d-3} & q_{d-5} & q_{d-7} & q_{d-9} & \cdots & q_3
\end{array}\right]$$

$$\times\begin{bmatrix} A^{2d-5}(x,y) \\ A^{2d-7}(x,y) \\ A^{2d-9}(x,y) \\ A^{2d-11}(x,y) \\ \vdots \\ A^{3}(x,y) \end{bmatrix} = v\begin{bmatrix} k^{d-4} \\ k^{d-5} \\ k^{d-6} \\ k^{d-7} \\ \vdots \\ k^{0} \end{bmatrix}$$

whenever d is even; or

$$\left[\begin{array}{cccccccccccc} q_{d-1} & q_{d-3} & q_{d-5} & \cdots & q_6 & q_4 & q_2 & q_0 & 0 & 0 & \cdots & 0 \\ 1 & q_{d-2} & q_{d-4} & \cdots & q_7 & q_5 & q_3 & q_1 & 0 & 0 & \cdots & 0 \\ \hline 0 & q_{d-1} & q_{d-3} & \cdots & q_8 & q_6 & q_4 & q_2 & q_0 & 0 & \cdots & 0 \\ 0 & 1 & q_{d-2} & \cdots & q_9 & q_7 & q_5 & q_3 & q_1 & 0 & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \cdots & 0 & q_{d-1} & q_{d-3} & q_{d-5} & q_{d-7} & q_{d-9} & \cdots & q_2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & q_{d-2} & q_{d-4} & q_{d-6} & q_{d-8} & \cdots & q_3 \end{array}\right]$$

$$\times\begin{bmatrix} A^{2d-5}(x,y) \\ A^{2d-7}(x,y) \\ A^{2d-9}(x,y) \\ A^{2d-11}(x,y) \\ \vdots \\ A^{5}(x,y) \\ A^{3}(x,y) \end{bmatrix} = v\begin{bmatrix} k^{d-4} \\ k^{d-5} \\ k^{d-6} \\ k^{d-7} \\ \vdots \\ k^{1} \\ k^{0} \end{bmatrix}$$

whenever d is odd.

Note that these coefficients matrices for the case $i = 2$ were arranged so that rows $2r+1, 2r+2$ can be obtained from previous two rows by moving one entry to right for $1 \le r \le \lfloor d/2 \rfloor - 2$, except the final row in case of odd d. Note also that the latter two matrices are obtained from the former ones by deleting the last row and the last column respectively. We claim in Lemma 3.1 that the coefficient matrices in both cases can be transformed into upper triangular matrices with entries 1 along their main diagonals, it follows that $A^2(x,y) = c_2(x,y)$ is a constant, say c_2, whenever $x, y \in V(G)$ at distance 2, and $A^3(x,y)$ is also a constant whenever $x, y \in V(G)$ at distance 3. Those processes can be performed successively for all $2 \le i \le d-1$.

Let $E_{d,0}$ be the coefficient matrix given in the case $i = 2$, and $E_{d,i-2}$ be the submatrix obtained from $E_{d,0}$ by deleting the last $i-2$ rows as well as the last $i-2$ columns. The system $(*)$ at entry (x,y) at distance i can be reduced into a system of linear equations

$$E_{d,i-2}\begin{bmatrix} A^{2d-2-i}(x,y) \\ A^{2d-4-i}(x,y) \\ A^{2d-6-i}(x,y) \\ \vdots \\ A^{i+4}(x,y) \\ A^{i+2}(x,y) \\ A^{i}(x,y) \end{bmatrix} = v\begin{bmatrix} k^{d-1-i} \\ k^{d-2-i} \\ k^{d-3-i} \\ \vdots \\ k^{2} \\ k^{1} \\ k^{0} \end{bmatrix}.$$

Note that $E_{d,i-2}$ is a square matrix of order $d-i$. Clearly, these systems have nontrivial solutions through the distance-regularity of the given generalized odd graph Γ. The uniqueness of $A^i(x,y)$ with x, $y \in V(G)$ at distance i follows from the non-singularity of $E_{d,i-2}$ for $2 \le i \le d-1$ as given in Lemma 3.1, which will be proved later.

Lemma 3.1. *The matrices $E_{d,i-2}$ for $2 \le i \le d-1$ are non-singular.*

Corollary 3.2. *$A^i(x,y)$ is a constant for $x, y \in V(G)$ at distance i, $2 \le i \le d-1$. Moreover, $A^d(x,y) = v$ whenever x, $y \in V(G)$ at distance d.*

For vertices x, y and z in $V(G)$ with $d(x,y) = i$ and $d(z,x) = i-1$, both $A^i(x,y)$ and $A^{i-1}(x,z)$ are constants respectively as shown in Corollary 3.2. By Lemma 2.1,

$$A^i(x,y) = \sum_{w \in G_1(y) \cap G_{i-1}(x)} A^{i-1}(x,w) = c_i(x,y)A^{i-1}(x,z),$$

independent of the choice of z, it follows that both $c_i(x,y)$ and $b_i(x,y) = b_0 - c_i(x,y) - a_i(x,y)$ are constants. For vertices $x, y \in V(G)$ at distance d, $A^d(x,y) = v$ is equal to $c_d(x,y)$ multiplied by an absolute constant, hence $c_d(x,y)$ is a constant too, say c_d.

Lemma 3.3. *$c_i(x,y)$, $b_i(x,y)$ are constants, say c_i, b_i, respectively whenever x, $y \in V(G)$ at distance $i \le d-1$. Moreover, $c_d(x,y)$ and hence $a_d(x,y)$ are constants, say c_d, a_d respectively, whenever $d(x,y) = d$.*

Up to this point, combining Lemmas 2.2 and 3.3 we may conclude that G is a distance-regular graph of diameter d with the same intersection array as that of Γ, this proves the first half of the Main Theorem. Those graphs mentioned in the Main Theorem are all generalized odd graphs which are uniquely determined by their intersection arrays as indicated in Section 2. Hence the second half of the Main Theorem follows immediately.

In the rest of this paper, we shall prove Lemma 3.1 in an algorithmic way. Since c_d is a common factor of all entries on the odd rows of $E_{d,0}$ as shown in Lemma 2.3, let M_1 be the matrix obtained from $E_{d,0}$ by factoring out c_d from all entries along the odd rows, and others remain unchanged. Based on the expressions of the coefficients of the polynomial $q(x)$ given in Lemma 2.3, in order to deal with these matrices in a convenient way, let

$$F_{m,i} = (-1)^m \left(P^{d-2i+1}_{d-2i-(2m-2)} + \sum_{i_1=0}^{d-2i-(2m-1)} b_{i_1} c_{i_1+1} \left(P^{d-2i+1}_{d-2i-(2m-4)} + \sum_{i_2=i_1+2}^{d-2i-(2m-3)} b_{i_2} c_{i_2+1} \times \left(\cdots \left(P^{d-2i+1}_{d-2i-2} + \sum_{i_{m-1}=i_{m-2}+2}^{d-2i-3} b_{i_{m-1}} c_{i_{m-1}+1} \left(P^{d-2i+1}_{d-2i} + \sum_{i_m=i_{m-1}+2}^{d-2i-1} b_{i_m} c_{i_m+1} \right) \right) \cdots \right) \right) \right)$$

and

$$S_{m,i} = (-1)^m \left(P^{d-2i+2}_{d-2i-(2m-3)} + \sum_{i_1=0}^{d-2i-(2m-2)} b_{i_1} c_{i_1+1} \left(P^{d-2i+2}_{d-2i-(2m-5)} + \sum_{i_2=i_1+2}^{d-2i-(2m-4)} b_{i_2} c_{i_2+1} \times \left(\cdots \left(P^{d-2i+2}_{d-2i-1} + \sum_{i_{m-1}=i_{m-2}+2}^{d-2i-2} b_{i_{m-1}} c_{i_{m-1}+1} \left(P^{d-2i+2}_{d-2i+1} + \sum_{i_m=i_{m-1}+2}^{d-2i} b_{i_m} c_{i_m+1} \right) \right) \cdots \right) \right) \right)$$

and let $S_{0,j} = F_{0,j} = 1$ for convenience. Note that $c_d F_{m,1} = q_{d-2m-1}$ and $S_{m,1} = q_{d-2m}$. Hence, the matrix M_1 can be expressed as

$$M_1 = \begin{matrix} M_{1,1} \\ M_{1,2} \\ M_{1,3} \\ M_{1,4} \\ \vdots \end{matrix} \begin{bmatrix} 1 & F_{1,1} & F_{2,1} & F_{3,1} & \cdots & F_{d-3,1} \\ 1 & S_{1,1} & S_{2,1} & S_{3,1} & \cdots & S_{d-3,1} \\ 0 & 1 & F_{1,1} & F_{2,1} & \cdots & F_{d-4,1} \\ 0 & 1 & S_{1,1} & S_{2,1} & \cdots & S_{d-4,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}_{(d-2)\times(d-2)}.$$

Before we transform matrices related M_1 into triangular matrices by applying some row-operations over them, the following lemmas are given for computational purpose, which can be proved straightfoward, note again that the conditions $b_0 = c_i + b_i$, $(i \leq d-1)$, and $a_d \neq 0$ play critical roles in Lemma 3.4 and in the following arguments.

Lemma 3.4. 1. *$S_{m,1} - F_{m,1} = (-c_{d-1}a_d)S_{m-1,2}$ for $1 \leq m \leq d-3$, and*

2. *$F_{m,1} - S_{m,2} = (-c_{d-2}b_{d-1})F_{m-1,2}$ for $1 \leq m \leq d-4$.*

Lemma 3.5. *Let $2 \leq j \leq \lceil (d-3)/2 \rceil$,*

1. *$S_{m,j} - F_{m,j} = (-c_{d-2j+1}b_{d-2j+2})S_{m-1,j+1}$ for $1 \leq m \leq (d-2j-1)$, and*
2. *$F_{m,j} - S_{m,j+1} = (-c_{d-2j}b_{d-2j+1})F_{m-1,j+1}$ for $1 \leq m \leq (d-2j-2)$.*

It is worth mentioning here that $c_{d-1}a_d$, $c_{d-2j+1}b_{d-2j+2}$, $c_{d-2j}b_{d-2j+1}$ are common factors of $S_{m,1} - F_{m,1}$, $S_{m,j} - F_{m,j}$, and $F_{m,j} - S_{m,j+1}$ respectively. The purpose of the following steps is to transform M_1 into an upper triangular matrix in terms of row operations. In particular, M_1 is already the trivial matrix [1] if $d = 3$, and steps 1, 2 are enough to reduce M_1 into an upper triangular matrix in case $d = 4, 5$.

Step 1. to get M_2 from M_1 with rows $M_{1,1}, M_{1,2}, \dots, M_{1,d-2}$:

1. since $c_{d-1}a_d$ is a common factor for each entry of the row $-M_{1,1} + M_{1,2}$ by Lemma 3.4, replace

$$\begin{matrix} M_{1,1} \\ M_{1,2} \end{matrix} \begin{bmatrix} 1 & F_{1,1} & F_{2,1} & F_{3,1} & \dots & F_{d-3,1} \\ 1 & S_{1,1} & S_{2,1} & S_{3,1} & \dots & S_{d-3,1} \end{bmatrix}$$

by

$$\begin{matrix} M_{2,1} \\ M_{2,2} \end{matrix} \begin{bmatrix} 1 & F_{1,1} & F_{2,1} & F_{3,1} & \dots & F_{d-3,1} \\ 0 & 1 & S_{1,2} & S_{2,2} & \dots & S_{d-4,2} \end{bmatrix},$$

where $M_{2,1} = M_{1,1}$, and $M_{2,2} = (-1/c_{d-1}a_d)(-M_{1,1} + M_{1,2})$;
2. performing similarly for pairs of rows $M_{1,2i-1}$ and $M_{1,2i}$, which are obtained from rows $M_{1,2i-3}$ and $M_{1,2i-2}$ by shifting right one entry for $i = 2, 3, \dots, \lceil (d-3)/2 \rceil$;
3. if d is odd, then the final row $M_{1,d-2}$ remains unchanged, say $M_{2,d-2}$;
4. let M_2 be the resulting matrix with rows $M_{2,1}, M_{2,2}, \dots, M_{2,d-2}$.

Step 2. to get M_3 from M_2 with rows $M_{2,1}, M_{2,2}, \dots, M_{2,d-2}$:

1. let $M_{3,1} = M_{2,1}$ which remain unchanged;
2. starting from the second row of M_2, since $c_{d-2}b_{d-1}$ is a common factor of the row $-M_{2,2} + M_{2,3}$ by Lemma 3.4, replace

$$\begin{matrix} M_{2,2} \\ M_{2,3} \end{matrix} \begin{bmatrix} 0 & 1 & S_{1,2} & S_{2,2} & \dots & S_{d-4,2} \\ 0 & 1 & F_{1,1} & F_{2,1} & \dots & F_{d-4,1} \end{bmatrix}$$

by

$$\begin{matrix} M_{3,2} \\ M_{3,3} \end{matrix} \begin{bmatrix} 0 & 1 & S_{1,2} & S_{1,2} & \dots & S_{d-4,2} \\ 0 & 0 & 1 & F_{1,2} & \dots & F_{d-5,2} \end{bmatrix},$$

where $M_{3,2} = M_{2,2}$, and $M_{3,3} = (-1/c_{d-2}b_{d-1})(-M_{2,2} + M_{2,3})$;
3. performing similarly for pairs of rows $M_{2,2i}$ and $M_{2,2i+1}$ where $i = 2, 3, \dots, \lfloor (d-3)/2 \rfloor$;
4. if d is even, then the final row $M_{2,d-2}$ remains unchanged, say $M_{3,d-2}$;
5. let M_3 be the resulting matrix with rows $M_{3,1}, M_{3,2}, \dots, M_{3,(d-2)}$.

The above two steps can be done in pairs recursively as follows for $2 \le j \le \left\lceil \frac{d-3}{2} \right\rceil$, but step $d-2$ is skipped in case d is even.

Step 2j-1. to get M_{2j} from M_{2j-1} with rows $M_{2j-1,1}, M_{2j-1,2}, \dots, M_{2j-1,d-2}$:

1. let $M_{2j,i} = M_{2j-1,i}$, for $1 \le i \le 2j-2$, which remain unchanged;
2. starting from the $(2j-1)$-th row, since $c_{d-2j+1}b_{d-2j+2}$ is a common factor for each entry of the row $-M_{2j-1,2j-1} + M_{2j-1,2j}$ by Lemma 3.5, replace

$$\begin{matrix} M_{2j-1,2j-1} \\ M_{2j-1,2j} \end{matrix} \begin{bmatrix} 0 & \dots & 0 & 1 & F_{1,j} & F_{2,j} & F_{3,j} & \dots & F_{d-2j-1,j} \\ 0 & \dots & 0 & 1 & S_{1,j} & S_{2,j} & S_{3,j} & \dots & S_{d-2j-1,j} \end{bmatrix}$$

by

$$\begin{matrix} M_{2j,2j-1} \\ M_{2j,2j} \end{matrix} \begin{bmatrix} 0 & \dots & 0 & 1 & F_{1,j} & F_{2,j} & F_{3,j} & \dots & F_{d-2j-1,j} \\ 0 & \dots & 0 & 0 & 1 & S_{1,j+1} & S_{2,j+1} & \dots & S_{d-2j-2,j+1} \end{bmatrix},$$

where

$$M_{2j,2j-1} = M_{2j-1,2j-1} \text{ and}$$
$$M_{2j,2j} = (-1/c_{d-2j+1}b_{d-2j+2})(-M_{2j-1,2j-1} + M_{2j-1,2j}).$$

Note that the first $2j-2$ columns consists of entries 0 only;
3. performing similarly for pairs of rows $M_{2j-1,2i-1}$, and $M_{2j-1,2i}$ for $j+1 \leq i \leq \lceil (d-3)/2 \rceil$;
4. if d is odd then the final row $M_{2j-1,d-2}$ remains unchanged, say $M_{2j,d-2}$;
5. let M_{2j} be the resulting matrix with rows $M_{2j,1}, M_{2j,2}, \dots, M_{2j,d-2}$.

Step 2j. to get M_{2j+1} from M_{2j} with rows $M_{2j,1}, M_{2j,2}, \dots, M_{2j,d-2}$:

1. let $M_{2j+1,i} = M_{2j,i}$ for $1 \leq i \leq 2j-1$ remain unchanged;
2. starting from the $2j$-th row, since $c_{d-2j}b_{d-2j+1}$ is a common factor for each entry of the row $-M_{2j,2j} + M_{2j,2j+1}$ by Lemma 3.5, replace

$$\begin{matrix} M_{2j,2j} \\ M_{2j,2j+1} \end{matrix} \begin{bmatrix} 0 & \dots & 0 & 1 & S_{1,j+1} & S_{2,j+1} & S_{3,j+1} & \dots & S_{d-2j-2,j+1} \\ 0 & \dots & 0 & 1 & F_{1,j} & F_{2,j} & F_{3,j} & \dots & F_{d-2j-2,j} \end{bmatrix}$$

by

$$\begin{matrix} M_{2j+1,2j} \\ M_{2j+1,2j+1} \end{matrix} \begin{bmatrix} 0 & \dots & 0 & 1 & S_{1,j+1} & S_{2,j+1} & S_{3,j+1} & \dots & S_{d-2j-2,j+1} \\ 0 & \dots & 0 & 0 & 1 & F_{1,j+1} & F_{2,j+1} & \dots & F_{d-2j-3,j+1} \end{bmatrix}$$

where

$$M_{2j+1,2j} = M_{2j,2j} \text{ and}$$
$$M_{2j+1,2j+1} = (-1/c_{d-2j}b_{d-2j+1})(-M_{2j,2j} + M_{2j,2j+1}).$$

Note that the first $2j-1$ columns consist of entries 0 only;
3. performing similarly for pairs of rows $M_{2j,2i}$, and $M_{2j,2i+1}$ for $j+1 \leq i \leq \lfloor (d-3)/2 \rfloor$;
4. if d is even, then the final row $M_{2j,d-2}$ remains unchanged, say $M_{2j+1,d-2}$;
5. let M_{2j+1} be the resulting matrix with rows $M_{2j+1,1}, M_{2j+1,2}, \dots, M_{2j+1,d-2}$.

After steps $1, 2, \dots, d-3$, an upper triangular matrix with 1 along its main diagonal is obtained, hence $\det M_1$, $\det E_{d,0}$, and $\det E_{d,i}$ are all non-zero. This completes the proof of Lemma 3.1 and hence the main theorem.

Remark. The above argument does not work for bipartite distance-regular graphs of diameter $d \geq 4$. Since $a_d = 0$, $S_{m,1} = F_{m,1}$ for all $m \leq d-3$ by Lemma 3.4, it follows that M_1, and hence $E_{d,0}$, $E_{d,i}$ for all $i \leq d-3$ are all singular.

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