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Topological properties of twisted cube [1]

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Abstract

Twisted cube, TQ_n, is derived by changing some connections of hypercube Q_n according to specific rules. Recently, many topological properties of this variation cube are studied. In this paper, we prove that its connectivity is n, its wide diameter and fault diameter are $\lceil n/2 \rceil + 2$. Furthermore, we show that TQ_n is a pancyclic network that is cycles of an arbitrary length at least four.

Keywords: Interconnection networks; Hypercube; Twisted cube; Embedding; Cycle

1. Introduction

Network topology is a crucial factor for interconnection network since it determines the performance of the network. Many interconnection network topologies have been proposed in the literature for the purpose of connecting hundreds or thousands of processing elements [3,4,6,10]. Network topology is always represented by a graph where nodes represent processors and edges represent links between processors. Among these topologies, the binary hypercube, Q_n, is one of the most popular topology. However, Q_n does not make the best use of its hardware in the following sense: given $N = 2^n$ nodes and $nN/2$ links, it is possible to fashion networks with lower diameters than the

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hypercube's diameter n. One of such topologies is called *twisted cube* [7], TQ_n, which is derived by changing the connection of some links of the hypercube according to some specified rules. The diameter of twisted cube topology is $\lceil (n+1)/2 \rceil$, almost a factor of 2 improvement. This is achieved by forfeiting some of the hypercube's high degree of symmetry and redundancy. Recently, many topological properties of this variation cube are studied in the literature [1,2].

In order to evaluate the performance of a network topology, we can consider the following measures: vertex connectivity, diameter, wide diameter, fault diameter, and embedding of cycles. The vertex connectivity (simply abbreviated as connectivity) of a network $G = (V, E)$, denoted by $\kappa(G)$ or κ, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. It follows from Menger's theorem that there always exist κ *internally vertex-disjoint* (abbreviated as *disjoint*) paths between any two vertices. Disjoint paths between a pair of vertices contribute to multipath communication between these two vertices and provide alternative routes in the case of node or link failures. Thus large connectivity is preferred.

Wide diameter and fault diameter were proposed in [5,8]. For any pair of vertices, say u and v, we find κ disjoint paths such that the longest path length of κ disjoint paths is minimum, denoted by $d_\kappa(u,v)$, among all possible choices of κ disjoint paths. The wide diameter is defined as the maximum of $d_\kappa(u,v)$ over all $u, v \in V$. Small wide diameter is preferred since it enables fast multipath communication. Fault diameter estimates the impact on diameter when faults occur, i.e., removal of vertices from G. For a pair of vertices u and v, we find the maximum of shortest path length between u and v over all possible $\kappa - 1$ faults, denoted by $d^{\mathrm{f}}_{\kappa-1}(u,v)$. The $(\kappa - 1)$-fault diameter is the maximum of $d^{\mathrm{f}}_{\kappa-1}(u,v)$ for all $u, v, \in V$, i.e., the maximum transmission delay of $\kappa - 1$ faults. Small $(\kappa - 1)$-fault diameter is also desirable to obtain smaller communication delay when faults occur. Wide diameter and fault diameter of a twisted cube TQ_n are studied in this paper.

An important aspect of TQ_n is its ability of efficiently simulating computations on other networks, which is portability of algorithms from other parallel interconnection structures, such as cycle or tree, to TQ_n. Such simulations can be reduced to graph embedding problem. We also consider the problem of embedding cycles architectures in twisted cubes.

Most of the graph definitions used in this paper are standard (see [9]). Let $G = (V, E)$ be a finite, undirected graph. Throughout this paper, node and vertex are used interchangeably to represent the element of V. Edge and link are used interchangeably to represent the element of E. Let C_i denote the length of a cycle. The *distance* between vertices u and v, denoted by $d_G(u,v)$, is the length of the shortest path from u and v.

The rest of this papers is organized as follows. In Section 2 we discuss some basic topological properties of twisted cubes. The connectivity, fault diameter and wide diameter for twisted cubes of odd dimension are studied in Section 3.

Embedding of cycles into twisted cubes is presented in Section 4. Finally, we make concluding remarks in Section 5.

2. Twisted cube topology and its properties

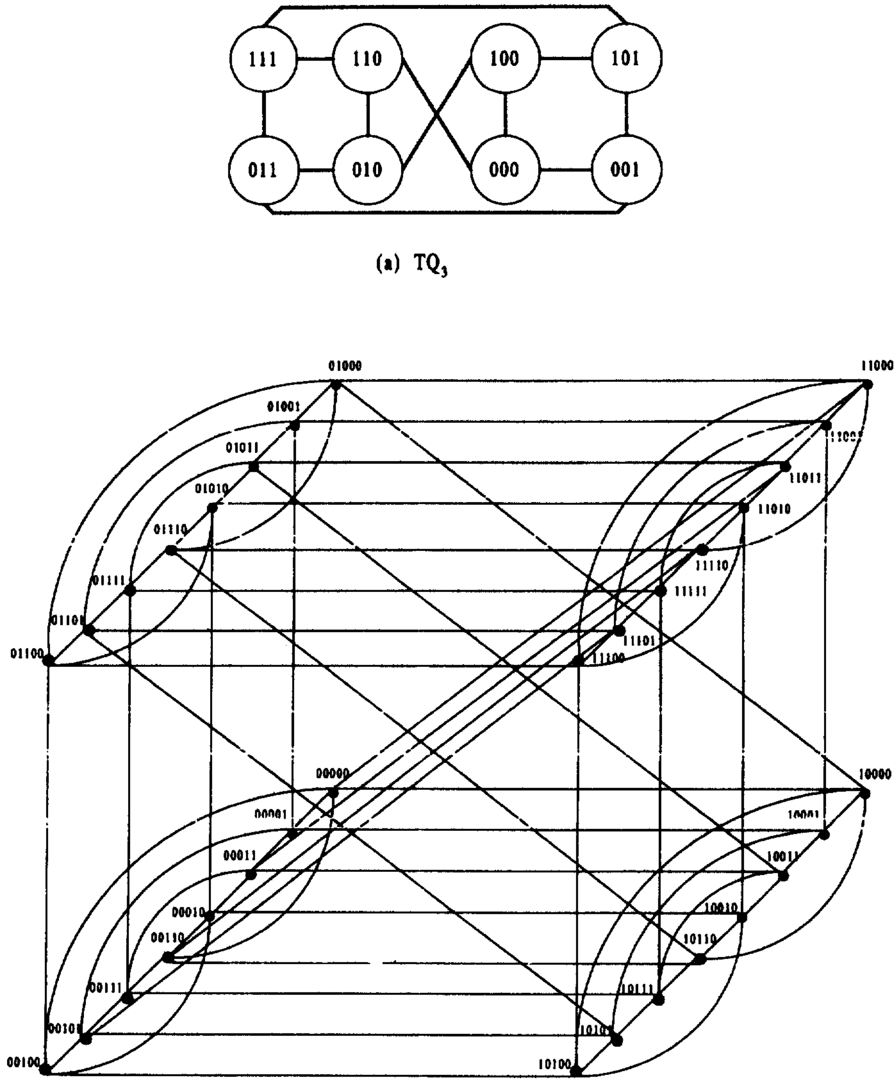
The *n-dimensional hypercube*, Q_n, consists of all the binary n-bit strings as its vertex set and two vertices u and v are adjacent if and only if u differs from v by exactly one bit. Let $u = u_{n-1}u_{n-2}\dots u_0$ and $v = v_{n-1}v_{n-2}\dots v_0$ be two vertices of Q_n. (u, v) is an edge in $E(Q_n)$ of *dimension i* if the ith bit of u is different from that of v. Twisted cube was first defined by Hilbers et al. [7]. A twisted n-cube, denoted by TQ_n, is a variant of n-dimensional hypercube Q_n. TQ_n has the same number of nodes and edges as in Q_n. We restrict the following discussion on TQ_n for the case that n is odd. Let $n = 2m + 1$, to form the twisted cube, we remove some links from the hypercube and replace them with links that span two dimensions in such a manner that the total number of links $(nN/2)$ is conserved. To be precise, let $u = u_{n-1}u_{n-2}\dots u_1u_0$ be any vertex in TQ_n. We define the parity function $P_i(u) = u_i \oplus u_{i-1} \oplus \cdots \oplus u_0$, where $\oplus$ is the exclusive-or operation. If $P_{2j-2}(u) = 0$ for some $1 \leqslant j \leqslant m$, we divert the edge on $(2j-1)$th dimension to node v such that $v_{2j}v_{2j-1} = \bar{u}_{2j}\bar{u}_{2j-1}$ and $v_i = u_i$ for $i \neq 2j$ or $2j-1$. Such diverted edges is called *twisted edges*. TQ_3 and TQ_5 are shown in Fig. 1(a) and (b).

We may formally define the term of twisted cube recursively as follows: A twisted 1-cube, TQ_1, is a complete graph with two vertices, 0 and 1. Let n be an odd integer and $n \geqslant 3$. We decompose vertices of TQ_n into four sets $S^{0,0}, S^{0,1}, S^{1,0}$ and $S^{1,1}$ where $S^{i,j}$ consists of those vertices u with $u_{n-1} = i$ and $u_{n-2} = j$. For each $(i,j) \in \{(0,0), (0,1), (1,0), (1,1)\}$, the induced subgraph of $S^{i,j}$ in TQ_n is isomorphic to TQ_{n-2}. Edges which connect these four subtwisted cubes can be described as follows: Any node $u_{n-1}u_{n-2}\dots u_1u_0$ with $P_{n-3}(u) = 0$ is connected to $\bar{u}_{n-1}\bar{u}_{n-2}u_{n-3}\dots u_0$ and $\bar{u}_{n-1}u_{n-2}u_{n-3}\dots u_0$; and $u_{n-1}\bar{u}_{n-2}u_{n-3}\dots u_0$ and $\bar{u}_{n-1}u_{n-2}u_{n-3}\dots u_0$, if $P_{n-3}(u) = 1$.

The following lemma can be easily obtained from the definition of twisted cubes.

Lemma 1. *Let* $u = u_{n-1}u_{n-2}\dots u_1u_0$ *and* $v = v_{n-1}v_{n-2}\dots v_1v_0$ *be two vertices of* TQ_n *with* $(u, v) \in E(TQ_n)$. *If* $u_{n-1} = v_{n-1}$, $u_{n-2} = v_{n-2}$, *and* $P_{n-3}(u) = P_{n-3}(v)$, *then* $P_{n-5}(u) = P_{n-5}(v)$.

To discuss the wide diameter and the fault diameter of the twisted cube, we need to review the shortest path routing algorithm [1]. Defining the 0th "double bit" of node address u to be the single bit u_0, and the jth "double bit" to be $u_{2j}u_{2j-1}$. Let u, v be any two vertices of TQ_n. We defined the *double Hamming distance* of u and v, denoted by $h_d(u, v)$, to be the number of different double bits between u and v. Obviously, $d_{TQ_n}(u, v) \geqslant h_d(u, v)$.



(a) TQ_3

(b) TQ_5

Fig. 1. TQ_3 and TQ_5.

We can find the shortest path between any two vertices using the algorithm proposed in [1]. Let u and v be two vertices of TQ_n. Let $z = u$. The basic strategy of the algorithm is to recursively find a neighborhood w of z that reduces $h_d(w, v)$. To be precise, the strategy is described as follows.

1. If $z = v$, then the path is determined.
2. Assume that there exist neighbors w of z such that $h_d(w, v) = h_d(z, v) - 1$. Let w' be the such w that differs from z with the largest double bit. Then reset z to be w'.
3. Assume that all the neighborhood w of z satisfy $h_d(w, v) \geqslant h_d(z, v)$. Let j be the smallest index of double bits that z differs from v. Choose w' to be the neighbor of z that differs from z in the $2j$th bit. Then reset z to be w'.

Since the rightmost differing double bit is selected in step 3, the resulting parity change guarantees that all subsequent routing for the message will be by step 2 until the destination is reached. Hence, step 3 is executed at most once for a given message. With this routing algorithm, we have the following theorems.

Theorem 1 [7]. *The diameter of the twisted cube* TQ_n *is* $\lceil (n+1)/2 \rceil$.

Theorem 2. $h_d(u, v) \leqslant d_{TQ_n}(u, v) \leqslant h_d(u, v) + 1$ *for any* u, $v \in V(TQ_n)$.

Lemma 2. *Let* u *and* v *be any two different nodes in the same* $S^{i,j}$ *of* TQ_n *and* L *be any shortest path joining* u to v. *If* $P_{n-3}(z) = 0$ *for all nodes* z *in* L, *then the length of* L *is at most* $\lceil (n-2)/2 \rceil - 1$.

Proof. Write L as $u = u^0, u^1, \ldots, u^k = v$. Suppose that there exists some index i with $0 \leqslant i \leqslant k-1$ such that u^i differs from u^{i+1} in exactly one bit, say t, with $0 \leqslant t \leqslant n-3$. Then $P_{n-3}(u^i) \neq P_{n-3}(u^{i+1})$. This is contradiction to the assumption, i.e., $P_{n-3}(u^i) = 0 = P_{n-3}(u^{i+1})$. Hence each (u^i, u^{i+1}) is either a twisted edge or u^i differs from u^{i+1} in the $\lceil (n-2)/2 \rceil$th double bit. Since both u and v are in the same $S^{i,j}$, the length of L is at most $\lceil (n-2)/2 \rceil$ and $u_{n-1}u_{n-2} = v_{n-1}v_{n-2}$. Suppose that the length of L is $\lceil (n-2)/2 \rceil$. Since each (u^i, u^{i+1}) is a twisted edge, we have $u_0 = v_0 = 0$ and $u_{2j}u_{2j-1} = \bar{v}_{2j}\bar{v}_{2j-1}$ for $1 \leqslant j \leqslant \lceil n/2 \rceil - 2$. Based on the definition of double Hamming distance, we have $h_d(u, v) = \lceil (n-2)/2 \rceil - 1$. Applying the shortest path routing algorithm, we can conclude that $d_{TQ_n}(u, v) = \lceil (n-2)/2 \rceil - 1$. We get a contradiction. Hence the length of L is at most $\lceil (n-2)/2 \rceil - 1$ and the lemma is proved. □

3. Fault diameter, wide diameter, and connectivity

We here formally define wide diameter and fault diameter of an underlying network $G = (V, E)$. For a vertex u in G, the *neighborhood* of u, denoted by $N(u)$, is defined as $\{v \mid (u, v) \in E\}$. Let u and v be two distinct vertices in G, and let $\kappa(G) = \kappa$. Let $C(u, v)$ denote the set of all α disjoint paths between u and v. Each element i of $C(u, v)$ consists of α disjoint paths, and the longest length among these α paths is denoted by $l_i(u, v)$. The number of elements in

$C(u,v)$ is denoted by $|C(u,v)|$. We define $d_\alpha(u,v)$ as the minimum over all l_i, i.e., $d_\alpha(u,v) = \min_{1 \leqslant i \leqslant |C(u,v)|} l_i(u,v)$. We write $d_1(u,v)$ as $d(u,v)$, which means the shortest distance between u and v. $D_\alpha(G)$ is called the α- *diameter* of G and is given by

$$D_\alpha(G) = \max_{u,v \in V} \{d_\alpha(u,v)\}.$$

By definition, $D_\alpha(G) = \infty$ if $\alpha \geqslant \kappa + 1$. We usually write $D_1(G)$ as $D(G)$ and call $D(G)$ simply the *diameter* of G. We are particularly interested in $D_\kappa(G)$. For a positive integer β, $d^{\mathrm{f}}_\beta(u,v)$ is defined as

$$d^{\mathrm{f}}_\beta(u,v) = \max_{\substack{F \subseteq V \\ |F| = \beta}} \{d(u,v) \text{ in } G - F \mid u,v \notin F\}.$$

The *β-fault diameter*, denoted by $D^{\mathrm{f}}_\beta(G)$, is given by

$$D^{\mathrm{f}}_\beta(G) = \max_{u,v \in V} \{d^{\mathrm{f}}_\beta(u,v)\}.$$

If $\beta \geqslant \kappa$, $D^{\mathrm{f}}_\beta(G) = \infty$ by definition. We are in particular interested in $D^{\mathrm{f}}_{\kappa-1}(G)$. Obviously, we have $D(G) \leqslant D^{\mathrm{f}}_{\kappa-1}(G) \leqslant D_\kappa(G)$.

It is known that $\kappa(Q_n) = n$ and $D_n(Q_n) = D^{\mathrm{f}}_{n-1}(Q_n) = n+1$. In this section, we will prove that $D_n(TQ_n) = D^{\mathrm{f}}_{n-1}(TQ_n) = \lceil n/2 \rceil + 2$ for all odd n. With this result, we can conclude that the connectivity of TQ_n is n. A node u of TQ_n, denoted by $u = 0^i 1^{n-i}$, is a binary string of length n with the first i 0's and the last $n-i$ 1's. We first prove the following lemma.

Lemma 3. $D^{f}_{n-1}(TQ_n) \geqslant \lceil n/2 \rceil + 2$, *where n is an odd integer.*

Proof. Let $u = 0^{n-1}1$, $v = 0^2 1^{n-3} 0$, and $u' = 010^{n-3}1$. Assume that the faulty set $F = N(u) - \{u'\}$. Hence $|F| = n-1$. Obviously, any path that joins u to v without traversing any node in F is a path from u through u', then through a neighborhood of u', say u'' ($\neq u$), and then followed by a path joining u'' to v. These u'' are in the set $W = \{010^{n-2}, 1^2 0^{n-3} 1\} \cup \{010^{n-4-j}10^j 1 \mid 0 \leqslant j \leqslant n-4\}$. Obviously, $h_{\mathrm{d}}(x,v) = \lceil n/2 \rceil$ for any $x \in W - \{010^{n-2}\}$. By the shortest path algorithm, we can check that $d^{\mathrm{f}}_{n-1}(y,v) = \lceil n/2 \rceil$ where $y = 010^{n-2}$. Hence the distance between any vertex in W to v in TQ_n is exactly $\lceil n/2 \rceil$. Therefore $D^{\mathrm{f}}_{n-1}(TQ_n) \geqslant \lceil n/2 \rceil + 2$. Hence the lemma is proved. □

A path P: $u = u^0, u^1, \ldots, u^{k-1}, u^k = v$ with $k \geqslant 3$ is called a *twisted path* if $P_{n-3}(u^i) = P_{n-3}(u^j)$ and $P_{n-1}(u^i) = P_{n-1}(u^j)$ for $1 \leqslant i,j \leqslant k-1$. For any node $u = u_{n-1}u_{n-2}\ldots u_1 u_0$ in TQ_n and any $(i,j) \in \{(0,0),(0,1),(1,0),(1,1)\}$, $u^{i,j}$ denotes the node $iju_{n-3}\ldots u_1 u_0$.

Lemma 4. *For any two different vertices u and v in TQ_n, there are n disjoint paths, $L_1, L_2, \ldots, L_n$, joining u to v such that (1) the length of each L_i is at most $\lceil n/2 \rceil + 2$, (2) L_1 is the shortest path joining u to v, and (3) the length of L_i is at most $\lceil n/2 \rceil + 1$ if L_i is a twisted path.*

Proof. The proof is by induction. Obviously, the lemma is true for $n = 1$. For $n \geqslant 3$, assume that such k disjoint paths exist for any two distinct nodes in TQ_k and any odd $k < n$. Now, we consider any two nodes $u = u_{n-1}u_{n-2}\dots u_1u_0$ and $v = v_{n-1}v_{n-2}\dots v_1v_0$ in TQ_n. We discuss the following six cases.

In cases 1 and 2, both u and v satisfy $v_{n-3}\dots v_1v_0 = u_{n-3}\dots u_1u_0$. Without loss of generality, we assume that u is in $S^{0,1}$. Note that the degree of any node in the subgraph of TQ_n induced by $S^{0,1}$ is $n-2$. Let $N(u) \cap S^{0,1} = \{w_3, w_4, \dots, w_n\}$. Since $w_r \neq w_s$ for $3 \leqslant r \neq s \leqslant n$, we have $w_r^{i,j} \neq w_s^{i,j}$ for $(i,j) \in \{(0,0),(0,1),(1,0),(1,1)\}$.

Case 1: $v_{n-1}v_{n-2} \neq u_{n-1}u_{n-2}$ and $P_{n-3}(u) = P_{n-3}(v) = 1$.

Subcase 1.1: $v_{n-1}v_{n-2} = u_{n-1}\bar{u}_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{0,0}$. Let L_1 be the path $u = u^{0,1}, v^{0,0} = v$, and L_2 be the path $u = u^{0,1}, u^{1,1}, u^{1,0}, v^{0,0} = v$. If $P_{n-3}(w_i) = 1$ for $3 \leqslant i \leqslant n$, then set L_i as $u = u^{0,1}, w_i^{0,1}, w_i^{0,0}, v^{0,0} = v$. Obviously, the length of L_i is 3. If $P_{n-3}(w_i) = 0$ for $3 \leqslant i \leqslant n$, then set L_i as $u = u^{0,1}, w_i^{0,1}, w_i^{1,0}, w_i^{0,0}, v^{0,0} = v$. Obviously, the length of L_i is 4. Since $P_{n-1}(w_i^{0,1}) \neq P_{n-1}(w_i^{0,0})$ or $P_{n-1}(w_i^{1,0}) \neq P_{n-1}(w_i^{0,0})$, L_i is not a twisted path. Thus, we have n disjoint paths joining u to v satisfying (1)–(3). See Fig. 2(a) for illustration.

Subcase 1.2: $v_{n-1}v_{n-2} = \bar{u}_{n-1}\bar{u}_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{1,0}$. Let L_1 be the $u = u^{0,1}, u^{0,0}, v^{1,0} = v$, and L_2 be the $u = u^{0,1}, u^{1,1}, v^{1,0} = v$. If $P_{n-3}(w_i) = 1$ for $3 \leqslant i \leqslant n$, then set L_i as $u = u^{0,1}, w_i^{0,1}, w_i^{0,0}, w_i^{1,0}, v^{1,0} = v$. Thus, L_i is not a twisted path and the length of L_i is 4. If $P_{n-3}(w_j) = 0$ for $3 \leqslant j \leqslant n$, then set L_j as $u = u^{0,1}, w_j^{0,1}, w_j^{1,0}, v^{1,0} = v$. Thus, the length of L_j is 3. Obviously, L_j is a twisted path and its length is at most $\lceil n/2 \rceil + 1$.We get n disjoint paths joining u to v satisfying (1)–(3). See Fig. 2(b) for illustration.

Subcase 1.3: $v_{n-1}v_{n-2} = \bar{u}_{n-1}u_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{1,1}$. Let L_1 be the $u = u^{0,1}, v^{1,1} = v$, and L_2 be the $u = u^{0,1}, u^{0,0}, u^{1,0}, v^{1,1} = v$. For each path L_i with $3 \leqslant i \leqslant n$, set L_i to be $u = u^{0,1}, w_i^{0,1}, w_i^{1,1}, v^{1,1} = v$. Thus, the length of L_i is 3. It is observed that none of L_i for $1 \leqslant i \leqslant n$ is a twisted path. We find n disjoint paths joining u to v satisfying (1)–(3). See Fig. 2(c) for illustration.

Case 2: $v_{n-1}v_{n-2} \neq u_{n-1}u_{n-2}$ and $P_{n-3}(u) = P_{n-3}(v) = 0$.

Subcase 2.1: $v_{n-1}v_{n-2} = u_{n-1}\bar{u}_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{0,0}$. Set L_1 as $u = u^{0,1}, u^{1,0}, v^{0,0} = v$, and set L_2 as $u = u^{0,1}, u^{1,1}, v^{0,0} = v$. If $P_{n-3}(w_i) = 1$ for $3 \leqslant i \leqslant n$, then set L_i as $u = u^{0,1}, w_i^{0,1}, w_i^{0,0}, v^{0,0} = v$. Thus, the length of L_i is 3. If $P_{n-3}(w_i) = 0$ for $3 \leqslant i \leqslant n$, then set L_i as $u = u^{0,1}, w_i^{0,1}, w_i^{1,0}, w_i^{0,0}, v^{0,0} = v$. Thus, the length of L_i is 4. We have constructed n disjoint (u,v)-paths satisfying (1)–(3). See Fig. 3(a) for illustration.

Subcase 2.2: $v_{n-1}v_{n-2} = \bar{u}_{n-1}\bar{u}_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{1,0}$. Set L_1 as $u = u^{0,1}, v^{1,0} = v$, and set L_2 as $u = u^{0,1}, u^{1,1}, u^{0,0}, v^{1,0} = v$. If $P_{n-3}(w_i) = 1$ for $3 \leqslant i \leqslant n$, then set L_i as $u = u^{0,1}, w_i^{0,1}, w_i^{0,0}, w_i^{1,0}, v^{1,0} = v$. Thus, the length of L_i is 4. If $P_{n-3}(w_i) = 0$ for $3 \leqslant i \leqslant n$, then set L_i as $u = u^{0,1}, w_i^{0,1}, w_i^{1,0}, v^{1,0} = v$. Thus, the length of L_i is 3. We have found n disjoint (u,v)-paths satisfying (1)–(3). See Fig. 3(b) for illustration.

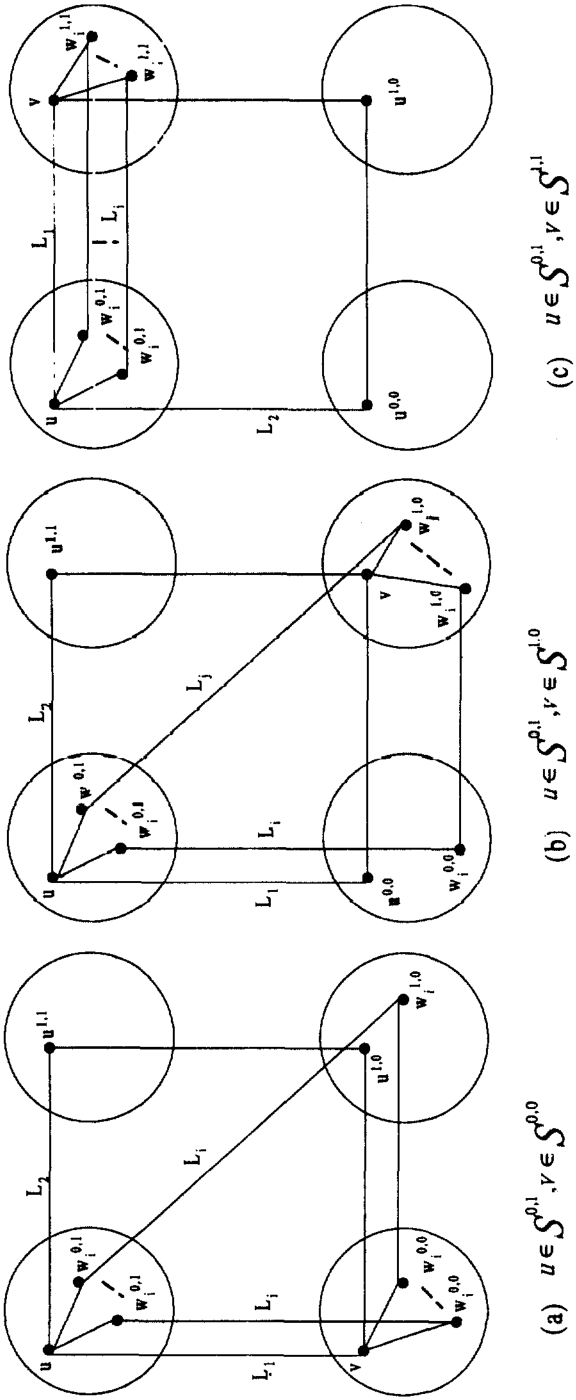


Fig. 2. Relative positions of the source node and the destination node in TQ_n with $P_{n-3}(U) = P_{a-3}(V) = 1$ for case 1.

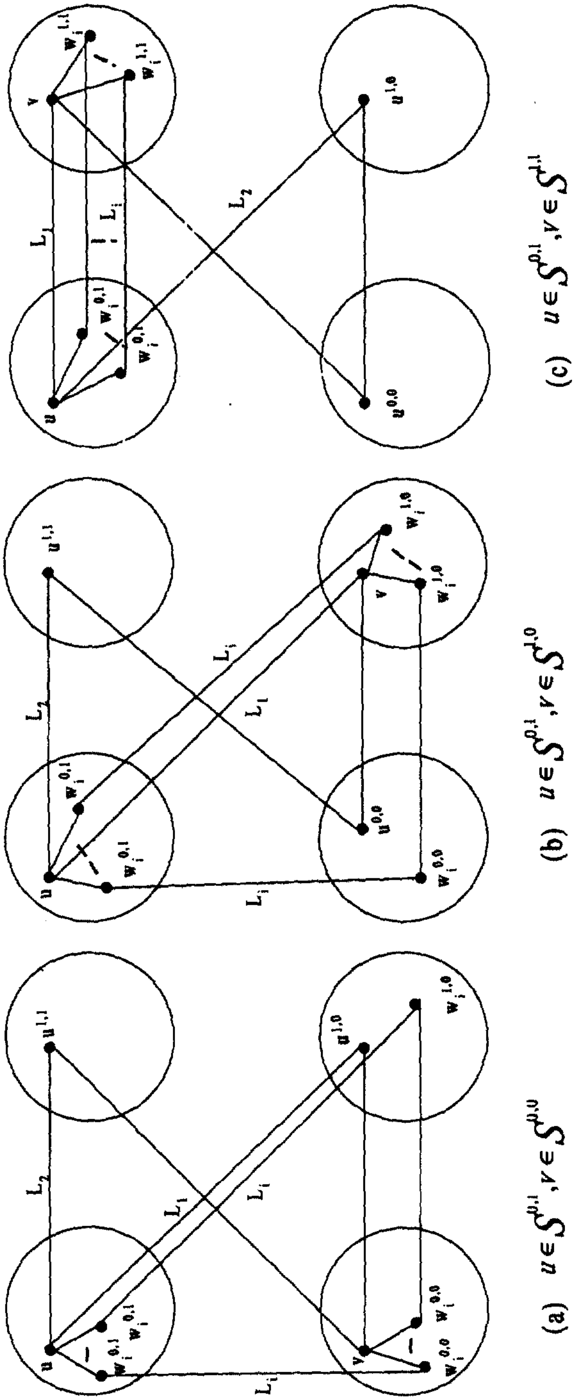


Fig. 3. Relative positions of the source node and the destination node in TQ_n with $P_{n-3}(u) = P_{n-3}(v) = 0$ for case 2.

Subcase 2.3: $v_{n-1}v_{n-2} = \bar{u}_{n-1}u_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{1,1}$. Let L_1 be the path $u = u^{0,1}, v^{1,1} = v$, and L_2 be the path $u = u^{0,1}, u^{1,0}, u^{0,0}, v^{1,1} = v$. For each L_i with $3 \leqslant i \leqslant n$, let L_i be the path $u = u^{0,1}, w_i^{0,1}, w_i^{1,1}, v^{1,1} = v$. Thus, the length of L_i is 3. We have obtained n disjoint (u, v)-paths satisfying (1)–(3). See Fig. 3(c) for illustration.

In cases 3–6, we consider $u_{n-3}u_{n-4} \ldots u_0 \neq v_{n-3}v_{n-4} \ldots v_0$. Since $S^{0,1}$ induces a TQ_{n-2}, by induction there are $n-2$ disjoint paths. Let $L_1^{0,1}, L_2^{0,1}, \ldots, L_{n-2}^{0,1}$ be $n-2$ disjoint paths joining $u^{0,1}$ to $v^{0,1}$ such that (1) the length of each path is at most $\lceil (n-2)/2 \rceil + 2$, (2) $L_1^{0,1}$ is the shortest path joining $u^{0,1}$ to $v^{0,1}$ in $S^{0,1}$, and (3) the length of $L_i^{0,1}$ is at most $\lceil (n-2)/2 \rceil + 1$ if $L_i^{0,1}$ is a twisted path. Hence, the length $L_i^{0,1}$ is at least 2 if $i > 1$. Write $L_i^{0,1}$ as $u = u_{i,0}^{0,1}, u_{i,1}^{0,1}, \ldots, u_{i,k_i}^{0,1} = v^{0,1}$, where k_i is the length of $L_i^{0,1}$. Let $L^{i,j}$ be the corresponding path of $L_1^{0,1}$ in $S^{i,j}$ joining $u^{i,j}$ to $v^{i,j}$. Without loss of generality, we assume that u is in $S^{0,1}$.

Case 3: $u_{n-1}u_{n-2} = v_{n-1}v_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{0,1}$. We simply let $L_i = L_i^{0,1}$ for $1 \leqslant i \leqslant n-2$. We have obtained $n-2$ disjoint (u, v)-paths satisfying (1)–(3). To construct the remaining two disjoint (u, v)-paths L_{n-1} and L_n, we consider the following three subcases.

Subcase 3.1: $P_{n-3}(u) = P_{n-3}(v) = 1$. Let L_{n-1} be the path $u = u^{0,1}$, $u^{0,0} \xrightarrow{L^{0,0}} v^{0,0}$, $v^{0,1} = v$, and L_n be the path $u = u^{0,1}, u^{1,1} \xrightarrow{L^{1,1}} v^{1,1}, v^{0,1} = v$. Since the length of $L^{0,0}$ and $L^{1,1}$ are at most $\lceil (n-2)/2 \rceil$, the length of L_{n-1} and L_n are at most $\lceil (n-2)/2 \rceil + 2$. Note that $\lceil (n-2)/2 \rceil + 2 = \lceil n/2 \rceil + 1$. See Fig. 4(a) for illustration.

Subcase 3.2: $P_{n-3}(u) = P_{n-3}(v) = 0$. Let L_{n-1} be the path $u = u^{0,1}, u^{1,0} \xrightarrow{L^{1,0}} v^{1,0}, v^{0,1} = v$, and let L_n be the path $u = u^{0,1}, u^{1,1} \xrightarrow{L^{1,1}} v^{1,1}, v^{0,1} = v$. Similarly, the length of $L^{1,0}$ is at most $\lceil (n-2)/2 \rceil$. Therfore, the length of L_{n-1} and L_n are at most $\lceil (n-2)/2 \rceil + 2$. See Fig. 4(b) for illustration.

Subcase 3.3: $P_{n-3}(u) \neq P_{n-3}(v)$. Without loss of generality, we assume that $P_{n-3}(u) = 0$ and $P_{n-3}(v) = 1$. Let L_{n-1} be the path $u = u^{0,1}, u^{1,0}, u^{0,0} \xrightarrow{L^{0,0}} v^{0,0}, v^{0,1} = v$, and L_n be the path $u = u^{0,1}, u^{1,1} \xrightarrow{L^{1,1}} v^{1,1}, v^{0,1} = v$. See Fig. 4(c) for illustration. Thus we have obtained n disjoint (u, v)-paths satisfying (1)–(3).

Case 4: $v_{n-1}v_{n-2} = u_{n-1}\bar{u}_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{0,0}$. For those paths $L_i^{0,1}$ with any node $u_{i,j}^{0,1}$ satisfying $P_{n-3}(u_{i,j}^{0,1}) = 1$, where $1 < i \leqslant n-2, 1 \leqslant j < k_i$, set L_i as $u = u_{i,0}^{0,1}, u_{i,1}^{0,1}, \ldots, u_{i,j}^{0,1}, u_{i,j}^{0,0}, u_{i,j+1}^{0,0}, \ldots, u_{i,k_i}^{0,0} = v$. Since $P_{n-1}(u_{i,j}^{0,1}) \neq P_{n-1}(u_{i,j}^{0,0})$, L_i is not a twisted path and its length is at most $\lceil n/2 \rceil + 2$. Others paths $L_t^{0,1}$ where $1 < t \leqslant n-2$ with $P_{n-3}(u_{t,j}^{0,1}) = 0$ for all $1 \leqslant j < k_t$, set L_t as $u = u_{t,0}^{0,1}, u_{t,1}^{0,1}, u_{t,1}^{1,0}, u_{t,1}^{0,0}, u_{t,2}^{0,0}, \ldots, u_{t,k_t}^{0,0} = v$. Since $P_{n-1}(u_{t,j}^{1,0}) \neq P_{n-1}(u_{t,j}^{0,0})$ and by Lemma 2, L_t is not a twisted path and its length is at most $\lceil n/2 \rceil + 2$. We have constructed $n-3$ disjoint (u, v)-paths satisfying (1)–(3). To construct the remaining three disjoint (u, v)-paths L_1, L_{n-1} and L_n, we consider the following three subcases.

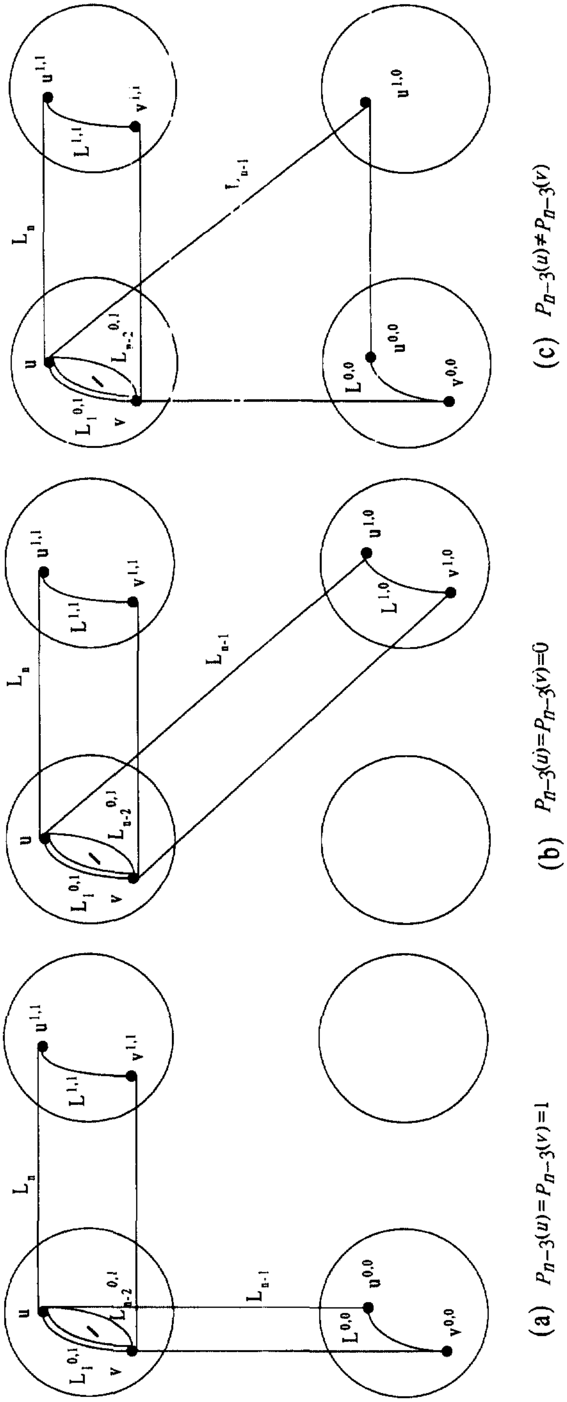
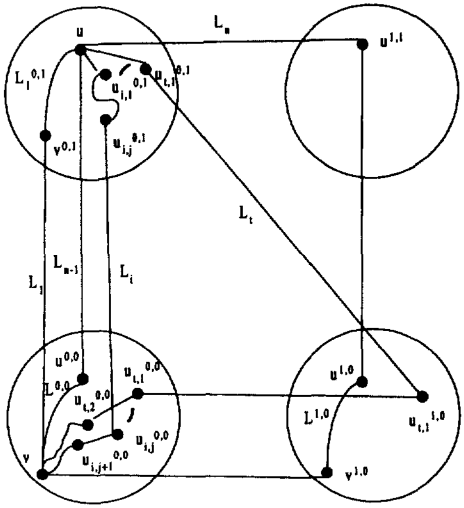
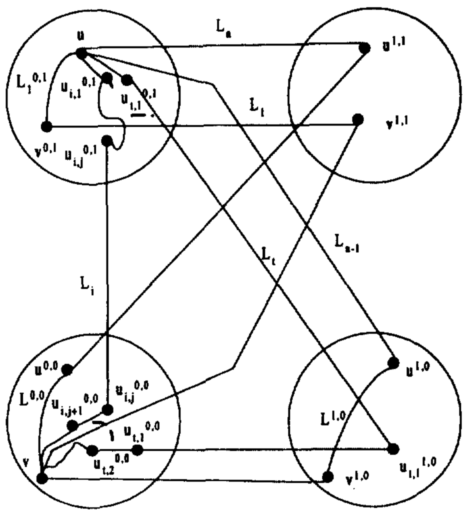


Fig. 4. Relative positions of the source node and the destination node with u, v in $S^{0,1}$ for case 3.

Subcase 4.1: $P_{n-3}(u) = P_{n-3}(v) = 1$. Let L_1 be $u = u^{0,1} \xrightarrow{L_1^{0,1}} v^{0,1}, v^{0,0} = v$, L_{n-1} be $u = u^{0,1}, u^{0,0} \xrightarrow{L^{0,0}} v^{0,0} = v$, and L_n be $u = u^{0,1}, u^{1,1}, u^{1,0} \xrightarrow{L^{1,0}} v^{1,0}, v^{0,0} = v$. See Fig. 5(a) for illustration.

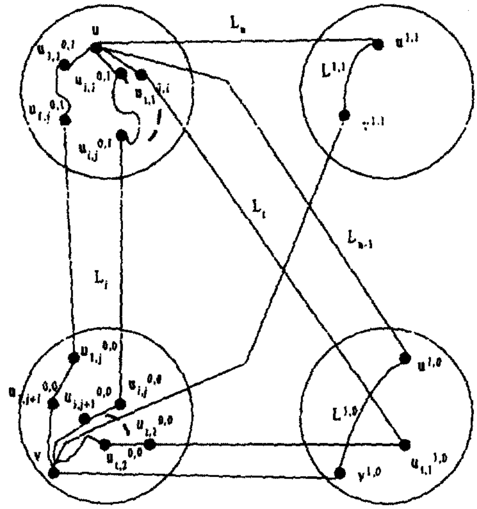


(a) $P_{n-3}(u) = P_{n-3}(v) = 1$

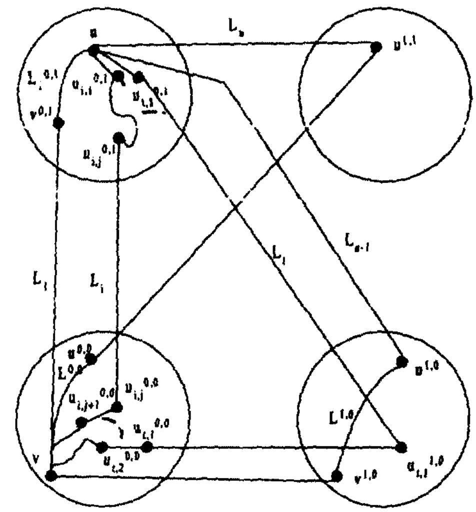


(b) $P_{n-3}(u) = P_{n-3}(v) = 0$

Fig. 5. Relative positions of the source node and the destination node with u in $S^{0,1}$ and v in $S^{0,0}$ for case 4.



(c) $P_{n-3}(u)=P_{n-3}(v)=0$



(d) $P_{n-3}(u)\neq P_{n-3}(v)$

Fig. 5. (Continued)

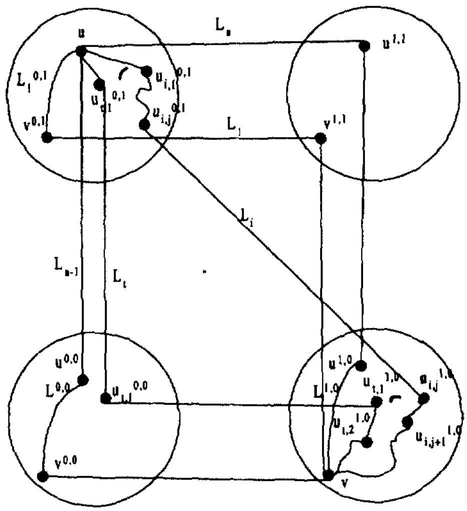
Subcase 4.2: $P_{n-3}(u) = P_{n-3}(v) = 0$. If the length of $L_1^{0,1}$ is 1, or $P_{n-3}(u_{1,j}) = 0$ for all $1 \leqslant j < k_1$, set L_1 as $u = u^{0,1} \stackrel{L_1^{0,1}}{\longrightarrow} v^{0,1}, v^{1,1}, v^{0,0} = v$, set L_{n-1} as $u = u^{0,1}$, $u^{1,0} \stackrel{L^{1,0}}{\longrightarrow} v^{1,0}, v^{0,0} = v$, and set L_n as $u = u^{0,1}, u^{1,1}, u^{0,0} \stackrel{L^{0,0}}{\longrightarrow}, v^{0,0} = v$. If the length of $L_1^{0,1}$ is greater or equal to 2 with any node $u_{1,j}^{0,1}$ satisfying $P_{n-3}(u_{1,j}^{0,1}) = 1$, $1 \leqslant j < k_1$, set L_1 as $u = u_{1,0}^{0,1}, u_{1,1}^{0,1}, \ldots, u_{1,j}^{0,1}, u_{1,j}^{0,0}, u_{1,j+1}^{0,0}, \ldots, v_{1,k_1}^{0,0} = v$, L_{n-1} as $u = u^{0,1}, u^{1,0} \stackrel{L^{1,0}}{\longrightarrow} v^{1,0}, v^{0,0} = v$, and L_n as $u = u^{0,1}, u^{1,1} \stackrel{L^{1,1}}{\longrightarrow} v^{1,1}, v^{0,0} = v$. See Fig. 5(b) and (c) for illustration.

Subcase 4.3: $P_{n-3}(u) \neq P_{n-3}(v)$. Without loss of generality, we assume that $P_{n-3}(u) = 0$ and $P_{n-3}(v) = 1$. Let L_1 be the path $u = u^{0,1} \stackrel{L_1^{0,1}}{\longrightarrow} v^{0,1}, v^{0,0} = v$, L_{n-1} be the path $u = u^{0,1}, u^{1,0} \stackrel{L^{1,0}}{\longrightarrow} v^{1,0}, v^{0,0} = v$, and L_n be the path $u = u^{0,1}$, $u^{1,1}$, $u^{0,0} \stackrel{L^{0,0}}{\longrightarrow} v^{0,0} = v$. See Fig. 5(d) for illustration. Hence we have constructed n disjoint (u, v)-paths satisfying (1)–(3).

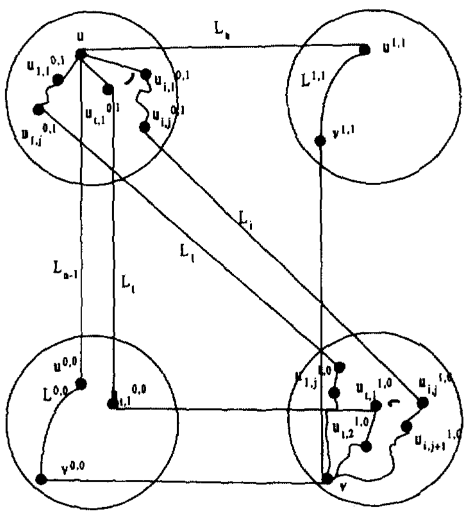
Case 5: $v_{n-1}v_{n-2} = \bar{u}_{n-1}\bar{u}_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{1,0}$. For those paths $L_i^{0,1}$ with any node $u_{i,j}^{0,1}$ satisfying $P_{n-3}(u_{i,j}^{0,1}) = 0$, where $1 < i \leqslant n-2, 1 \leqslant j < k_i$, let L_i be the path $u = u_{i,0}^{0,1}$, $u_{i,1}^{0,1}, \ldots,$ $u_{i,j}^{0,1}, u_{i,j}^{1,0}$, $u_{i,j+1}^{1,0}$, $\ldots, u_{i,k_i}^{1,0} = v$. Obviously, the length of L_i is at most $\lceil n/2 \rceil + 2$. If L_i is a twisted path, $L_i^{0,1}$ is a twisted path in $S^{0,1}$. By induction, the length of $L_i^{0,1}$ is at most $\lceil (n-2)/2 \rceil + 1$. This implies that the length of L_i is at most $\lceil n/2 \rceil + 1$ if L_i is a twisted path. Others paths $L_t^{0,1}$ with $P_{n-3}(u_{t,j}^{0,1}) = 1$ for all $1 \leqslant j < k_t$, set L_t as $u = u_{t,0}^{0,1}, u_{t,1}^{0,1}, u_{t,1}^{0,0}, u_{t,1}^{1,0}, u_{t,2}^{1,0}, \ldots, u_{t,k_t}^{1,0} = v$. It is easy to see that $L_t^{0,1}$ is a twisted path in $S^{0,1}$ with $P_{n-3}(u_{t,j}^{0,1}) = 1$ for all $1 \leqslant j < k_t$. Therefore, the length of L_t is at most $\lceil n/2 \rceil + 2$ and L_t is not a twisted path. We have found $n-3$ disjoint (u, v)-paths satisfying (1)–(3). To construct the remaining three disjoint (u, v)-paths L_1, L_{n-1} and L_n, we consider the following three subcases.

Subcase 5.1: $P_{n-3}(u) = P_{n-3}(v) = 1$. If the length of $L_1^{0,1}$ is 1, or $P_{n-3}(u_{1,j}) = 1$ for all $1 \leqslant j < k_1$, set L_1 as $u = u^{0,1} \stackrel{L_1^{0,1}}{\longrightarrow} v^{0,1}$, $v^{1,1}$, $v^{1,0} = v$, L_{n-1} as $u = u^{0,1}$, $u^{0,0} \stackrel{L^{0,0}}{\longrightarrow} v^{0,0}, v^{1,0} = v$, and L_n as $u = u^{0,1}, u^{1,1}, u^{1,0} \stackrel{L^{1,0}}{\longrightarrow} v^{1,0} = v$. If the length of $L_1^{0,1}$ is greater or equal to 2 with any node $u_{1,j}^{0,1}$ satisfying $P_{n-3}(u_{1,j}^{0,1}) = 0$, $1 \leqslant j < k_1$, let L_1 be the path $u = u_{1,0}^{0,1}, u_{1,1}^{0,1}, \ldots, u_{1,j}^{0,1}, u_{1,j}^{1,0}, u_{1,j+1}^{1,0}, \ldots, v_{1,k_1}^{1,0} = v$, L_{n-1} be the path $u = u^{0,1}, u^{0,0} \stackrel{L^{0,0}}{\longrightarrow} v^{0,0}, v^{1,0} = v$, and L_n be the path $u = u^{0,1}, u^{1,1} \stackrel{L^{1,1}}{\longrightarrow} v^{1,1}, v^{1,0} = v$. See Fig. 6(a) and (b) for illustration.

Subcase 5.2: $P_{n-3}(u) = P_{n-3}(v) = 0$. Let L_1 be the path $u = u^{0,1} \stackrel{L_1^{0,1}}{\longrightarrow} v^{0,1}$, $v^{1,0} = v$, L_{n-1} be the path $u = u^{0,1}, u^{1,0} \stackrel{L^{1,0}}{\longrightarrow} v^{1,0} = v$, and L_n be the path $u = u^{0,1}, u^{1,1}, u^{0,0} \stackrel{L^{0,0}}{\longrightarrow} v^{0,0}, v^{1,0} = v$. If L_n is a twisted path, then $L^{0,0}$ is a shortest path joining $u^{0,0}$ to $v^{0,0}$ satisfying $P_{n-3}(z) = 0$ for all nodes z in $L^{0,0}$. It follows from Lemma 2 that the length of $L^{0,0}$ is at most $\lceil (n-2)/2 \rceil - 1$. Hence, the

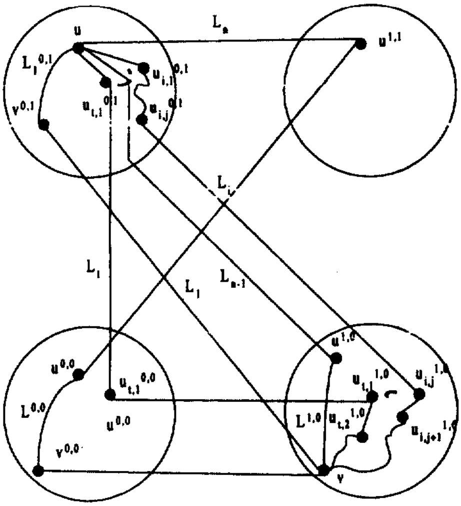


(a) $P_{n-3}(u)=P_{n-3}(v)=1$

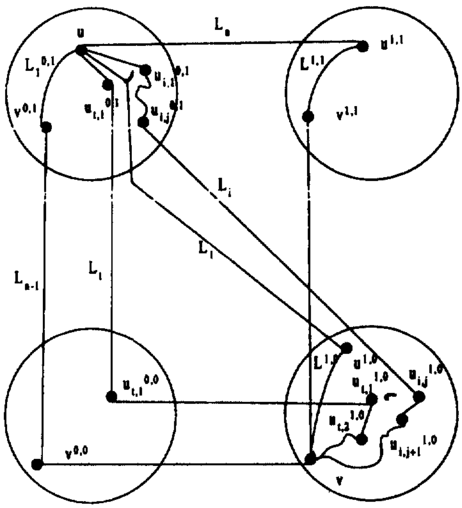


(b) $P_{n-3}(u)=P_{n-3}(v)=1$

Fig. 6. Relative positions of the source node and the destination node with u in $S^{0,1}$ and v in $S^{1,0}$ for case 5.



(c) $P_{n-3}(u)=P_{n-3}(v)=0$



(d) $P_{n-3}(u)\neq P_{n-3}(v)$

Fig. 6. (Continued)

length of L_n is at most $(\lceil (n-2)/2 \rceil - 1) + 3 = \lceil n/2 \rceil + 1$ if L_n is a twisted path.- See Fig. 6(c) for illustration.

Subcase 5.3: $P_{n-3}(u) \neq P_{n-3}(v)$. Without loss of generality, we assume that $P_{n-3}(u) = 0$ and $P_{n-3}(v) = 1$. Let L_1 be the path $u = u^{0,1}, u^{1,0} \xrightarrow{L^{1,0}} v^{1,0} = v$, L_{n-1} be the path $u = u^{0,1} \xrightarrow{L_1^{0,1}} v^{0,1}, v^{0,0}, v^{1,0} = v$, and L_n be the path $u = u^{0,1}, u^{1,1} \xrightarrow{L^{1,1}} v^{1,1}, v^{1,0} = v$. See Fig. 6(d) for illustration. Thus we have found n disjoint (u, v)-paths satisfying (1)–(3).

Case 6: $v_{n-1}v_{n-2} = \bar{u}_{n-1}u_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{1,1}$. Note that the length of $L_i^{0,1}$ is at least two for $1 < i \leqslant n-2$. Set L_i as $u = u_{i,0}^{0,1}, u_{i,1}^{0,1}, u_{i,1}^{1,1}, u_{i,2}^{1,1}, \ldots, u_{i,k_i}^{1,1} = v$ for $1 < i \leqslant n-2$. Since $P_{n-1}(u_{i,1}^{0,1}) \neq P_{n-1}(u_{i,1}^{1,1})$, L_i is not a twisted path and its length is at most $\lceil n/2 \rceil + 2$. We have constructed $n-3$ disjoint (u, v)-paths satisfying (1)–(3). To construct the remaining three disjoint (u, v)-paths L_1, L_{n-1} and L_n, we consider the following three subcases.

Subcase 6.1: $P_{n-3}(u) = P_{n-3}(v) = 1$. Let L_1 be $u = u^{0,1} \xrightarrow{L_1^{0,1}} v^{0,1}, v^{1,1} = v$, L_{n-1} be $u = u^{0,1}, u^{1,1} \xrightarrow{L^{1,1}} v^{1,1} = v$, and L_n be $u = u^{0,1}, u^{0,0}, u^{1,0} \xrightarrow{L^{1,0}} v^{1,0}, v^{1,1} = v$. Since $P_{n-1}(u^{0,0}) \neq P_{n-1}(u^{1,0})$, L_n is not a twisted path and its length is at most $\lceil n/2 \rceil + 2$.See Fig. 7(a) for illustration.

Subcase 6.2: $P_{n-3}(u) = P_{n-3}(v) = 0$. Let L_1 be $u = u^{0,1} \xrightarrow{L_1^{0,1}} v^{0,1}, v^{1,1} = v$, L_{n-1} be $u = u^{0,1}, u^{1,1} \xrightarrow{L^{1,1}} v^{1,1} = v$, and L_n be $u = u^{0,1}, u^{1,0} \xrightarrow{L^{1,0}} v^{1,0}, v^{0,0}, v^{1,1} = v$. See Fig. 7(b) for illustration.

Subcase 6.3: $P_{n-3}(u) \neq P_{n-3}(v)$. Without loss of generality, we assume $P_{n-3}(u) = 0$ and $P_{n-3}(v) = 1$. Let L_1 be $u = u^{0,1} \xrightarrow{L_1^{0,1}} v^{0,1}, v^{1,1} = v$, L_{n-1} be $u = u^{0,1}, u^{1,1} \xrightarrow{L^{1,1}} v^{1,1} = v$, and L_n be $u = u^{0,1}, u^{1,0} \xrightarrow{L^{1,0}} v^{1,0}, v^{1,1} = v$. See Fig. 7(c) for illustration. Hence we have constructed n disjoint (u, v)-paths satisfying (1)–(3). □

The following corollary follows from Lemma 4 and that the degree of each vertex in TQ_n is n.

Corollary 1. *Assume n is an odd integer. The connectivity of* TQ_n, $\kappa(TQ_n)$, *is n, and* $D_\kappa(TQ_n) \leqslant \lceil n/2 \rceil + 2$. *Hence,* TQ_n *is maximal connection.*

The following theorem follows from Lemmas 3 and 4.

Theorem 3. $D_{n-1}^{\mathrm{f}}(TQ_n) = D_n(TQ_n) = \lceil n/2 \rceil + 2$ *if n is odd.*

4. Embedding of cycles

A cycle structure is often used as a connection structure for local area network, for example Token Rings, and can also be used as a control/data flow

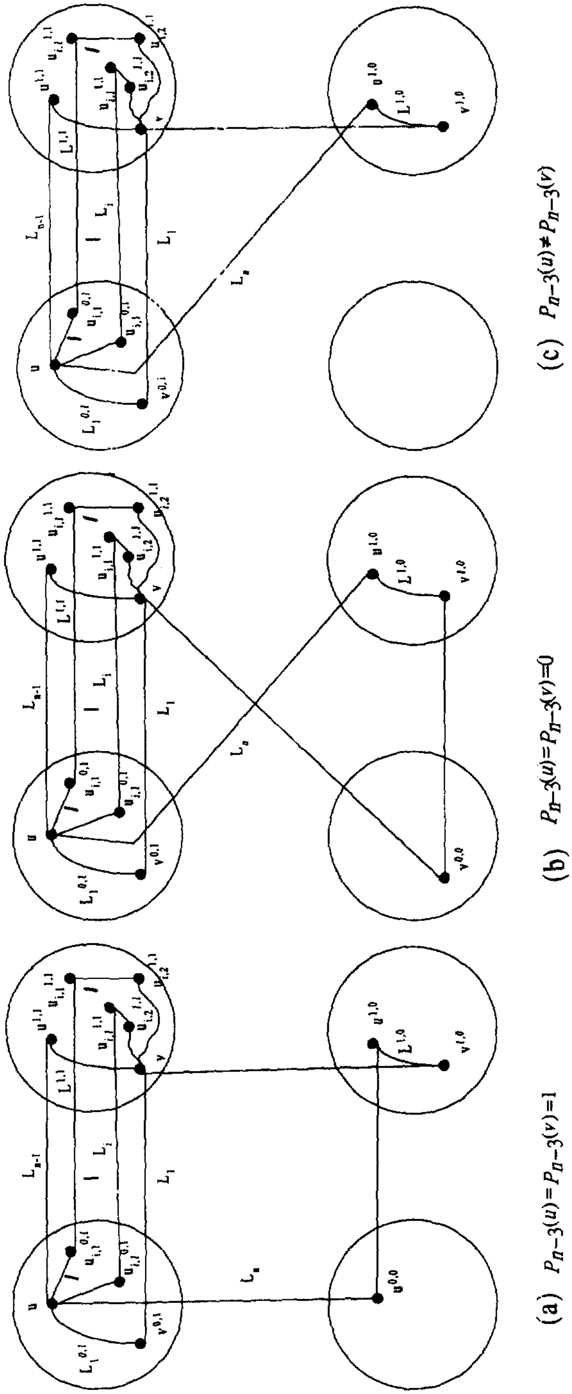


Fig. 7. Relative positions of the source node and the destination node with u in $S^{0,1}$, v in $S^{1,1}$ for case 6.

structure for distributed computations in arbitrary networks. In this section, we will show that TQ_n contains a cycle C_i of length i for all $4 \leqslant i \leqslant 2^n$.

Theorem 4. *Let n be an odd integer and $n \geqslant 3$. For all i with $4 \leqslant i \leqslant 2^n$, there exists a cycle $C_i = \langle u^0, u^1, \ldots u^{i-1}, u^0\rangle$ of length i, where $u^0 = 0^n$, $u^{i-1} = 0^{n-t}1^2 0^{t-2}$ and t is an odd integer with $3 \leqslant t \leqslant n$ such that $2^{t-2} < i \leqslant 2^t$.*

Proof. We prove this lemma by induction. In TQ_3, we have the following cycles:

$$C_4 = \langle 000, 100, 010, 110, 000\rangle,$$
$$C_5 = \langle 000, 001, 011, 010, 110, 000\rangle,$$
$$C_6 = \langle 000, 100, 010, 011, 111, 110, 000\rangle,$$
$$C_7 = \langle 000, 100, 101, 001, 011, 111, 110, 000\rangle,$$
$$C_8 = \langle 000, 001, 011, 010, 100, 101, 111, 110, 000\rangle.$$

With these five cycles, it is easy to see that the lemma is true for $n = 3$. Assume that the Lemma is true for any odd k with $3 \leqslant k < n$.

For $4 \leqslant i \leqslant 2^{n-2}$, by induction there exists a cycle $C_i = \langle u^0, u^1, \ldots u^{i-1}, u^0\rangle$ of length i in TQ_{n-2} where $u^0 = 0^{n-2}, u^{i-1} = 0^{n-t-2}1^2 0^{t-2}$ and t is an odd integer with $3 \leqslant t \leqslant n-2$ such that $2^{t-2} < i \leqslant 2^t$. Since $S^{0,0}$ induces TQ_{n-2}, TQ_n contains a cycle C_i of length i for all $4 \leqslant i \leqslant 2^{n-2}$ in $S^{0,0}$ where $u^0 = 0^n, u^{i-1} = 0^{n-t}1^2 0^{t-2}$ and t is an odd integer with $3 \leqslant t \leqslant n$ such that $2^{t-2} < i \leqslant 2^t$. We first consider $2^{n-2} < i \leqslant 2^{n-1}$. Then there exist two integers a, b such that $a + b = i$ and $2^{n-3} \leqslant a, b \leqslant 2^{n-2}$. By induction, in TQ_{n-2} there exist two cycles $C_a = \langle u^0, \ldots, u^{a-1}, u^0\rangle$ and $C_b = \langle v^0, \ldots, v^{b-1}, v^0\rangle$ with $u^0 = v^0 = 0^{n-2}$ and $u^{a-1} = v^{b-1} = 1^2 0^{n-4}$. Let $C_a^{0,0} = \langle x^0, \ldots, x^{a-1}, x^0\rangle$ denote the corresponding cycle of C_a in $S^{0,0}$ and $C_b^{1,1} = \langle y^0, \ldots, y^{b-1}, y^0\rangle$ denote the corresponding cycle of C_b in $S^{1,1}$. Obviously, $x^0 = 0^n$, $x^{a-1} = 0^2 1^2 0^{n-4}$, $y^0 = 1^2 0^{n-2}$ and $y^{b-1} = 1^4 0^{n-4}$. We define $z^j = x^j$ if $0 \leqslant j \leqslant a-1$ and $z^j = y^{(i-j-1)}$ if $a \leqslant j \leqslant i-1$. It is easy to see that $\langle z^0, z^1, \ldots, z^{i-1}, z^0\rangle$ forms a cycle of length i such that $z^0 = 0^n$ and $z^{i-1} = 1^2 0^{n-2}$.

Now we consider $2^{n-1} < i \leqslant 2^n$. Then there exist four integers a, b, c, d such that $a + b + c + d = i$ and $2^{n-3} \leqslant a, b, c, d \leqslant 2^{n-2}$. By induction, in TQ_{n-2} there exist four cycles $C_a = \langle p^0, \ldots, p^{a-1}, p^0\rangle$, $C_b = \langle q^0, \ldots, q^{b-1}, q^0\rangle$, $C_c = \langle r^0, \ldots, r^{c-1}, r^0\rangle$, and $C_d = \langle s^0, \ldots, s^{d-1}, s^0\rangle$ with $p^0 = q^0 = r^0 = s^0 = 0^{n-2}$ and $p^{a-1} = q^{b-1} = r^{c-1} = s^{d-1} = 1^2 0^{n-4}$. Let $C_a^{0,0} = \langle u^0, \ldots, u^{a-1}, u^0\rangle$, $C_b^{1,0} = \langle v^0, \ldots, v^{b-1}, v^0\rangle$, $C_c^{0,1} = \langle x^0, \ldots, x^{c-1}, x^0\rangle$ and $C_d^{1,1} = \langle y^0, \ldots, y^{d-1}, y^0\rangle$ denote the corresponding cycles of C_a, C_b, C_c, C_d in $S^{0,0}$, $S^{1,0}$, $S^{0,1}$ and $S^{1,1}$, respectively. Obviously, $u^0 = 0^n$, $u^{a-1} = 0^2 1^2 0^{n-4}$, $v^0 = 10^{n-1}$, $v^{b-1} = 101^2 0^{n-4}$, $x^0 = 010^{n-2}$, $x^{c-1} = 01^3 0^{n-4}$, $y^0 = 1^2 0^{n-2}$ and $y^{d-1} = 1^4 0^{n-4}$. We define $z^j = u^j$ if $0 \leqslant j \leqslant a-1$, $z^j = v^{(a+b-j-1)}$ if $a \leqslant j \leqslant a+b-1$, $z^j = x^{(j-a-b)}$ if $a+b \leqslant j \leqslant$

$a+b+c-1$ and $z^j = y^{(i-j-1)}$ if $a+b+c \leqslant j \leqslant i-1$. It is easy to see that $\langle z^0, z^1, \ldots, z^{i-1}, z^0 \rangle$ forms a cycle of length i such that $z^0 = 0^n$ and $z^{i-1} = 1^2 0^{n-2}$. □

Based on the above proof idea, we can easily construct a cycle of arbitrary length. We here illustrate two examples of constructing C_{11} and C_{21} in TQ_5. To construct C_{11} in TQ_5, we find two cycles of length 6 and 5 in TQ_3, where C_6 and C_5 are given by

$$C_6 = \langle 000, 100, 010, 011, 111, 110, 000 \rangle,$$
$$C_5 = \langle 000, 001, 011, 010, 110, 000 \rangle,$$

then $C_6^{0,0}$ and $C_5^{1,1}$ are

$$C_6^{0,0} = \langle 00000, 00100, 00010, 00011, 00111, 00110, 00000 \rangle,$$
$$C_5^{1,1} = \langle 11000, 11001, 11011, 11010, 11110, 11000 \rangle,$$

respectively. We can use $C_6^{0,0}$ and $C_5^{1,1}$ to construct a C_{11} as follows:

$$C_{11} = \langle 00000, 00100, 00010, 00011, 00111, 00110, 11110, 11010, 11011, 11001, 11000, 00000 \rangle.$$

Similarly, we can construct a C_{21} as follows:

$$C_{21} = \langle 00000, 00100, 00010, 00011, 00111, 00110, 10110, 10010, 10011, 10001, 10000, 01000, 01001, 01011, 01010, 01110, 11110, 11010, 11011, 11001, 11000, 00000 \rangle.$$

5. Concluding remarks

This paper studies wide diameter, fault diameter and embedding cycles problems in twisted cubes. We have shown that the twisted cube is to improve the performance of the hypercube. It is known that $D^f_{n-1}(Q_n) = D_\kappa(Q_n) = n+1$. In this paper, we have proven that wide diameter and fault diameter of the twisted cube are about the half of the corresponding parameters of the hypercube. Furthermore, we also proved that the twisted cube is a pancyclic network. Hence, the twisted cube is an attractive topology for interconnection networks.

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