A New Class of Optimal Biorthogonal Subband Coder

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***Abstract*— In this letter, we introduce a novel two-channel biorthogonal subband coder. The new coder employs the ladder structure (or so-called wavelet with lifting scheme). The coding gain of the biorthogonal coder is always greater than or equal to unity, and it can be expressed in closed form. Moreover the proposed coder has a very low complexity and the perfect reconstruction property is structurally preserved. Given any filter order, the optimal solution can be found by the well-known Levinson fast algorithm. For both AR(1) process and MA(1) process, the proposed biorthogonal coder with 2 taps has a higher coding than the optimal orthonormal coder with infinite number of taps.**

***Index Terms*— Compression, optimal coder, subband coding, wavelet coding.**

I. Introduction

RECENTLY, there has been considerably interest in applying the ladder structure to subband coding. Fig. 1 shows a simple two-channel filterbank (FB) that uses only one ladder. In the absence of the quantizers, such a biorthogonal system always has the perfect reconstruction [that is, $y(n) = x(n)$ for all possible $x(n)$], regardless of the choice of $P(z)$. In other words, the FB is structurally perfect reconstruction. The analysis filters $H_i(z)$ and synthesis filters $F_i(z)$ are, respectively, $H_0(z) = 1 - z^{-1}P(z^2)$, $H_1(z) = z^{-1}$, $F_0(z) = 1$, $F_1(z) = z + P(z^2)$. The implementation and design of the biorthogonal system involve only $P(z)$, hence the design and computational cost is very low. Note that such a FB can never be orthonormal unless $P(z)$ is zero. The ladder structure has been applied to lossless coding of images, and satisfactory coding results can be obtained, as demonstrated in [1]–[3]. In the case of lossy compression, like most biorthogonal coder, the coding gain $\mathcal{CG}$ of the ladder structure FB is not guaranteed to be greater than unity.

On the other hand, the class of orthonormal FB is known to have coding gain $\mathcal{CG} \geq 1$. There has been a lot of interest in finding the optimal orthonormal FB that yields a maximum coding gain for a given input statistics [4]–[6]. It is shown in [4] that the analysis and synthesis filters of an optimal orthonormal FB are the ideal filters that satisfy the majorization and decorrelation properties. The optimal finite impulse response (FIR) case is solved in [5] and [6]. In this paper, we propose a new class of the FIR biorthogonal FB by modifying the ladder structure. The new coder has all the advantages of the ladder based FB. The optimal biorthogonal coder can be solved using Levinson recursion. Furthermore it is guaranteed that the new coder has coding gain $\mathcal{CG} \geq 1$. For both autoregressive (AR) process and moving average (MA) process of order one, the proposed biorthogonal coder with 2 taps has a higher coding gain than *any* optimal orthonormal FB (with any number of taps).

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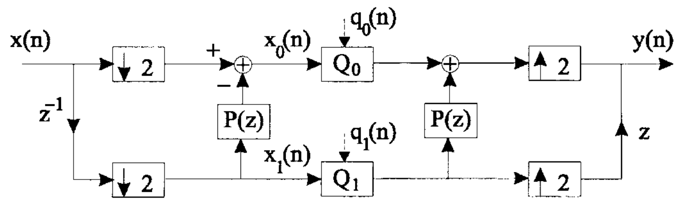

Fig. 1. Conventional subband coder using ladder.

1) Noise Model and Bit Rate: In this work, we make some commonly used assumptions on the quantizers. Assume that the quantizers are scalar uniform quantizers and can be modeled as an additive noise source. Therefore, $\hat{x}(n) = x(n) + q(n)$ (as indicated by the dashed line in Fig. 1), where $x(n)$ and $\hat{x}(n)$ are, respectively, the input and output of the quantizers. We assume that for a b-bit quantizer, the variance of quantization noise $q(n)$ satisfies:

$$\sigma_q^2 = c2^{-2b}\sigma_x^2 \tag{1}$$

where σ_x^2 is the variance of the input $x(n)$ and c is some constant depending only on the statistics of $x(n)$. Assume that b_0 and b_1 are the number of bits assigned to Q_0 and Q_1, respectively. The average bit rate in this case is $b = 1/2(b_0 + b_1)$.

A. The Traditional Subband Coder

In a traditional subband coder, quantizers are placed directly after the subband signals $x_i(n)$ as shown in Fig. 1. The output noise contains contribution from both $q_0(n)$ and $q_1(n)$. It is not difficult to see that the noise gain for $q_0(n)$ is unity while $q_1(n)$ is amplified by $P(z)$. One can verify that under the assumption that $q_1(n)$ is white and uncorrelated with $q_0(n)$, the average variance of output error is given by

$$\sigma_{q_{\text{out}}}^2 = 0.5(\sigma_{q_0}^2 + \sigma_{q_1}^2 + \sigma_{q_1}^2\mathcal{E}_p) \tag{2}$$

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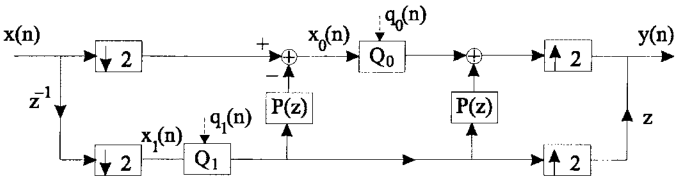

Fig. 2. New subband coder using ladder.

where $\mathcal{E}_p = \int_0^{2\pi} |P(e^{j\omega})|^2 (d\omega/2\pi)$ is the energy of the filter $P(z)$. It represents the noise amplification of the quantizers Q_1. Due to the noise amplification, it is not guaranteed that the coding gain $\mathcal{CG} \geq 1$. In the following, we will show how to eliminate the noise amplification by judiciously placing the quantizers Q_1.

II. A Novel Biorthogonal Subband Coder

Consider Fig. 1. Note that the input to $P(z)$ at the analysis end is $x(2n-1)$, while the input to $P(z)$ at the synthesis end is the quantized version of $x(2n-1)$. It is this mismatch that causes the amplification of $q_1(n)$. To avoid this mismatch, we can simply move the quantizers Q_1 to the left, as shown in Fig. 2. It is not difficult to verify that in this case, the average variance of output error is given by

$$\sigma^2_{q_{\text{out}}} = 0.5(\sigma^2_{q_0} + \sigma^2_{q_1}). \tag{3}$$

The above equation is valid for any additive noise source. We *do not make any assumptions* on $q_0(n)$ and $q_1(n)$. That means, the noise gain is *always* one even though the FB is never orthonormal. If the quantizers used are scalar quantizers that satisfy (1), then (3) can be rewritten as:

$$\sigma^2_{q_{\text{out}}} = 0.5c(2^{-2b_0}\sigma^2_{x_0} + 2^{-2b_1}\sigma^2_x) \tag{4}$$

where we have used the fact that $\sigma^2_{x_1} = \sigma^2_x$. Applying the arithmetic mean geometric mean inequality to the above equation, we get

$$\sigma^2_{q_{\text{out}}} \geq c\,2^{-2b}[\sigma^2_{x_0}\sigma^2_x]^{1/2} \tag{5}$$

with equality if and only if the bits are allocated as:

$$b_i = b + \tfrac{1}{2}\log\sigma^2_{x_i} - \tfrac{1}{2}\log[\sigma^2_{x_0}\sigma^2_x]^{1/2}. \tag{6}$$

If we define the coding gain of the coder as the ratio of the error variance in direct quantization [as in (1)] over that of the coder, $\sigma^2_{q_{\text{out}}}$. Then under the optimal bit allocation (6), the coding gain can be written as:

$$\mathcal{CG} = \frac{\sigma^2_x}{[\sigma^2_{x_0}\sigma^2_x]^{1/2}} = \sqrt{\frac{\sigma^2_x}{\sigma^2_{x_0}}}. \tag{7}$$

A. Optimal Biorthogonal Coder

From (7), the coding gain $\mathcal{CG}$ is maximized if $\sigma^2_{x_0}$ is minimized. The optimal solution of $P(z)$ such that $\sigma^2_{x_0}$ is minimized can be obtained from linear prediction theory. To see this, let $P(z)$ be an FIR filter of the form

$$P(z) = \sum_{n=-N}^{N-1} p(n)z^{-n}. \tag{8}$$

Then the optimal solution is precisely the optimal predictor of $x(2n)$ based on the observations of $x(2n-2N+1)$, $x(2n-2N+3)$, $\cdots$, $x(2n+2N-1)$. Noncausal predictor can be used here since we are predicting the even samples from the odd samples. A causal implementation of such a system is always possible by inserting enough delays at appropriate places in Fig. 2. Let $x(n)$ be a real-valued wide sense stationary process with autocorrelation coefficients $r(k)$. Then the optimal $p(n)$ that minimizes $\sigma^2_{x_0}$ is the solution of the following equation

$$\begin{pmatrix} r(0) & r(2) & r(4) & \cdots & r(4N-2) \\ r(2) & r(0) & r(2) & \cdots & r(4N-4) \\ r(4) & r(2) & r(0) & \cdots & r(4N-6) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r(4N-2) & r(4N-4) & r(4N-6) & \cdots & r(0) \end{pmatrix} \cdot \begin{pmatrix} p(-N) \\ p(-N+1) \\ p(-N+2) \\ \vdots \\ p(N-1) \end{pmatrix} = \begin{pmatrix} r(2N-1) \\ r(2N-3) \\ \vdots \\ r(1) \\ r(1) \\ \vdots \\ r(2N-1) \end{pmatrix}. \tag{9}$$

The above equation can be solved in $\mathcal{O}(N^2)$ by using the Levinson fast algorithm. And the prediction gain

$$G_p = \frac{\sigma^2_x}{\sigma^2_{x_0}} \geq 1. \tag{10}$$

The above inequality follows from the linear prediction theory. The prediction gain is unity if and only if all the observations are uncorrelated to the target of prediction $x(2n)$. The following theorem summarizes the results we have so far.

Theorem 1: Consider the subband coder in Fig. 2, where $p(n)$ is as in (8). The coding gain of the coder is maximized when $p(n)$ is chosen as the optimal prediction filter obtained from (9). The maximum coding gain $\mathcal{CG}_{\max}$ is given by

$$\mathcal{CG}_{\max} = \sqrt{G_p} \geq 1 \tag{11}$$

where G_p is the prediction gain in (10). The coding gain is always greater than or equal to unity, with equality if and only if the autocorrelation coefficients of $x(n)$ satisfy $r(2k+1) = 0$ for $0 \leq k \leq N-1$.

III. Merits of the Proposed Biorthogonal Coder

The biorthogonal coder in Fig. 2 enjoys many advantages. In the following, we list some of its advantages.

1) *Structurally Perfect Reconstruction*: Similar to the orthonormal FB, the proposed biorthogonal FB has a structurally perfect reconstruction implementation, as in Fig. 2.
2) *Equal Step Size Rule*: From (3) and (4), we see that the average output noise variance $\sigma^2_{q_{\text{out}}}$ is minimized when

the two quantizers have the same noise variance. The noise variances $\sigma_{q_i}^2$ and the quantization step size Δ_i are related as $\sigma_{q_i}^2 = const * \Delta_i^2$. Therefore we conclude that the coder continues to be optimal if the stepsizes of the quantizers are equal.

3) *Unity Noise Gain*: The synthesis bank does not amplify the quantization noise. Hence, the optimal biorthogonal coder has a coding gain $\mathcal{CG} \geq 1$. Moreover one can show [7] that a finite-order optimal biorthogonal coder has a unity gain if and only if the optimal orthonormal coder [4] of the same order has a unity gain.

4) *Simple Design*: The design of the optimal biorthogonal coder is simple. Unlike the optimal orthonormal coder, no constrained optimization and no spectral factorization is needed. Optimal biorthogonal coder can be obtained by using Levinson algorithm.

5) *Low Complexity*: To implement the analysis or synthesis bank, we need only one filter $P(z)$. Moreover the optimal $P(z)$ has *linear-phase.* To see this, note that the vector on the right hand side of (9) is symmetric. It is not difficult to show that the optimal predictor $p(n)$ obtained from (9) is symmetric, i.e., $p(n) = p(-n-1)$. Therefore the complexity of the biorthogonal coder is roughly a quarter of that of an orthonormal coder of the same order.

6) *Coding Gain Increases with N*: It is well known that the prediction gain G_p is a non decreasing function of N. Hence, the coding gain increases when the filter order increases.

7) *A Very Low Delay Coder*: Let the filter $P(z)$ be a causal FIR filter of the form $\sum_{n=0}^{N-1} p(n)z^{-n}$. Then regardless of the filter length N, the coder in Fig. 2 has a delay of only one sample [i.e., $y(n) = x(n-1)$]! The optimal solution of $P(z)$ is the given by the optimal causal predictor of $x(2n)$ based on the observations of $x(2n-2N+1), x(2n-2N+3), \cdots, x(2n-1)$.

8) *Integration of Lossy and Lossless Coder*: Let the input $x(n)$ be a discrete amplitude signal with step size Δ_x. Suppose the output of $P(z)$ is quantized using step size Δ_x. Then lossless coding can be obtained from Fig. 2 by setting the step sizes of the quantizers $\Delta_0 = \Delta_1 = \Delta_x$. By varying the stepsizes of the quantizers, we can get both lossy and lossless compression with the same structure.

9) *Application to Finite Length Input Signal*: In a conventional subband coder, when the input has finite length L, the total number of samples in the subband increases due to linear convolution with the analysis filters unless these filters have length 2. Therefore periodic extension is used to solve this problem. In the proposed biorthogonal coder in Fig. 2, to reconstruct the output signal we need only to retain L samples in the subbands ($\lfloor (L+1)/2 \rfloor$ samples of $x_0(n)$ and $\lfloor L/2 \rfloor$ samples of $x_1(n)$ where $\lfloor a \rfloor$ denotes the largest integer $\leq a$). No periodic extension is needed!

Example 1—AR(1) Inputs: Let the input be an AR(1) process with $r(k) = \rho^{|k|}$ for $0 < \rho < 1$. One can verify [7] that the coding gain is optimized when $P(z) = (\rho + \rho z)/(1+\rho^2)$.

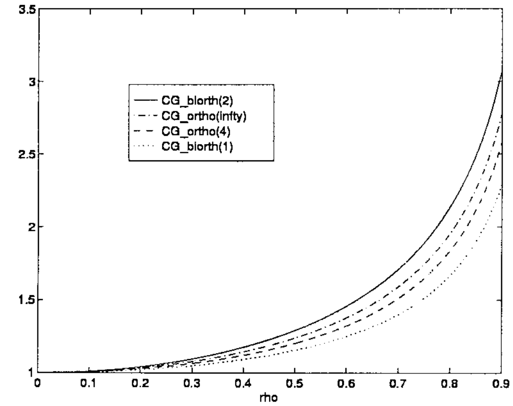

Fig. 3. Coding gain comparison for AR(1) process.

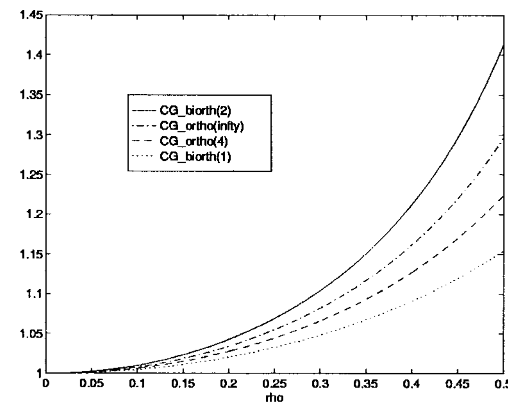

Fig. 4. Coding gain comparison for MA(1) process.

In this case, the optimal coding gain has the closed form expression $\mathcal{CG}_{biorth}(2) = \sqrt{(1+\rho^2)/(1-\rho^2)}$, where the index 2 indicates that the predictor has 2 taps. If we use a one-tap predictor, the optimal predictor $P(z) = \rho$ and the gain is $\mathcal{CG}_{biorth}(1) = 1/\sqrt{1-\rho^2}$. These gains are shown in Fig. 3. For comparison, we also show the gains for optimal orthonormal coders (with infinite taps and four taps). It was shown in [4]–[6] that the coding gains are, respectively, $\mathcal{CG}_{ortho}(\infty) = 1/\sqrt{1-(16/\pi^2)(\tan^{-1}\rho)^2}$ and $\mathcal{CG}_{ortho}(4) = \sqrt{(1+1/3\rho^2)/(1-\rho^2)}$.

Example 2—MA(1) Inputs: Let the input be an MA(1) process with $r(0) = 1$, $r(\pm 1) = \rho$ for $0 < \rho < 0.5$, and $r(k) = 0$ for all the other k. The coding gain for the four cases considered in Example 1 are, respectively [4]–[7], $\mathcal{CG}_{biorth}(2) = 1/\sqrt{1-2\rho^2}$, $\mathcal{CG}_{ortho}(\infty) = 1/\sqrt{1-(16/\pi^2)\rho^2}$, $\mathcal{CG}_{ortho}(4) = 1/\sqrt{1-(4/3)\rho^2}$ and $\mathcal{CG}_{biorth}(1) = 1/\sqrt{1-\rho^2}$. It is not difficult to verify that $\mathcal{CG}_{biorth}(2) > \mathcal{CG}_{ortho}(\infty) > \mathcal{CG}_{ortho}(4) > \mathcal{CG}_{biorth}(1)$. These gains are shown in Fig. 4.

IV. Concluding Remarks

In this work, we have derived a number of properties of the novel biorthogonal coder in Fig. 2. We showed that the biorthogonal coder has many advantages enjoyed by the orthonormal coder but it has a lower design and computational cost. Many of its features make the biorthogonal coder a valuable and attractive alternative to orthonormal coder. The following extensions are possible for the new system [7].

1) For a more general biorthogonal FB with more than one ladder, the structure can be modified so that the noise gain is still unity.
2) Using a tree structure, one can obtain a wavelet-type decomposition. This tree structure extension continues to have many of the properties listed in Section III.
3) We can also generalize the idea to the uniform M-channel case. The M-channel biorthogonal transform coder (a transform coder has a constant polyphase matrix) has the same coding gain as the Karhunen–Loeve transform (KLT), but it has a much lower design and implementation cost than the KLT. In the special case of AR(1) process, the optimal biorthogonal transform coder has a closed form expression, and no optimization is required.

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