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ON SINGULAR NONLINEAR H^∞ CONTROL: A STATE-SPACE APPROACH*

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Abstract. A simple state-space approach for the four-block singular nonlinear H^∞ control problem is proposed in this paper. This approach combines a (J, J')-lossless and a class of conjugate (J, J')-expansive systems to yield a family of nonlinear H^∞ output feedback controllers. The singular nonlinear H^∞ control problem is thus transformed into a simple lossless network problem that is easy to deal with in a network-theory context.

1. Introduction

The singular H^∞ control problem has been widely studied in linear systems. Khargonekar, Petersen, and Zhou [9], [11], [19] derived the solvable conditions for this problem in terms of a family of algebraic Riccati euqations parametrized by a positive constant ε. Alternatively, Stoorvogel et al. [14], [15] investigated this problem by using quadratic matrix inequalities corresponding to Riccati equations.

As in the singular nonlinear H^∞ control problem, Maas and Van der Schaft [10] extended the results from [9], [11], [19] to show that under assumptions this problem can be solved by a state feedback that leads to an L_2-gain less than or equal to a prescribed bound γ for the closed-loop system. Maas and Van der Schaft also used the worst-case certainty equivalence principle to find a nonlinear output feedback controller.

In this paper the singular nonlinear H^∞ control problem is reformulated in terms of the chain-scattering matrix description. An alternative approach that offers a family of controllers solving this singular problem is then developed. This approach is based on the classical network theory, which combines the traditional (J, J')-lossless system with a class of nonlinear conjugate (J, J')-expansive systems. As will be seen, the controller thus obtained is straightforward and provides a deeper insight into the synthesis of the controllers.

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2. Notation and preliminary information

Throughout this paper, $dom(Ric)$ denotes a Hamiltonian matrix with no eigenvalues on the $j\omega$-axis. $\mathbf{R}^n$ denotes n-dimensional Euclidean space. The chain-scattering matrix description is abbreviated as CSMD. We now proceed to define the L_2-gain of a nonlinear system.

We are given a nonlinear system of the form

$$\Sigma := \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

with $x \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, and $y(t) \in \mathbf{R}^p$. Furthermore, let $\mathcal{U}_\delta$ denote the set of piecewise continuous functions $u : \mathbf{R} \to \mathbf{R}^m$ satisfying $\|u(t)\| < \delta$ for all $t \in \mathbf{R}$. This gives the following definition for L_2-gain of the system Σ [6], [8], [18].

Definition 1. If Σ has a locally asymptotically stable equilibrium at $x = 0$ and is dissipative with respect to a supply rate $s = \|u(t)\|^2 - \|y(t)\|^2$, then for each $\epsilon > 0$ there exists a storage function $E(x)$ (with $E(x) \geq 0$ and $E(0) = 0$) and $\delta(\epsilon) > 0$ such that for each $u(\cdot) \in \mathcal{U}_{\delta(\epsilon)}$ the response $y(\cdot)$ of Σ to $u(\cdot)$ from the initial state $x(0) = 0$ satisfies

$$E(x(t_1)) - E(x(t_0)) \leq \int_{t_0}^{t_1} (\gamma^2\|u(t)\|^2 - \|y(t)\|^2)\, dt$$

for all $t_1 > t_0 > 0$. Therefore, Σ has an L_2-gain which is less than or equal to γ.

2.1. The singular nonlinear affine H^∞ control problem.

Consider the following smooth (C^∞) singular nonlinear affine H^∞ framework:

$$P := \begin{cases} \dot{x} = A(x) + B_1(x)w + B_2(x)\binom{u_1}{u_2} \\ z = C_1(x) \\ y = C_2(x) + \check{D}(x)w \end{cases}, \tag{1}$$

where $z(t) \in \mathbf{R}^{p_1}$, $y(t) \in \mathbf{R}^{p_2}$, $w(t) \in \mathbf{R}^{m_1}$, and $u(t) \in \mathbf{R}^{m_2}$ are the error, observation, disturbance, and control input, respectively. The states $x = (x_1, x_2, \ldots, x_n)$ are local coordinates for a state-space manifold M defined in a neighborhood Ω of the origin in $\mathbf{R}^n$. Assume that $x = 0$, an equilibrium point, and also that $A(0) = 0$, $C_1(0) = 0$, and $C_2(0) = 0$.

The singular nonlinear affine H^∞ control problem is then modeled so as to choose a controller K that connects the observation vector y to u such that K locally, asymptotically stabilizes the closed-loop system in a neighborhood Ω of the origin. Furthermore, the closed-loop system with a local L_2-gain is less than or equal to a prescribed number γ. Figure 1 shows a general setup for this singular nonlinear affine H^∞ control system.

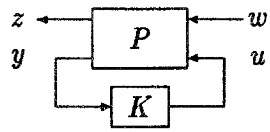


Figure 1. General setup.

For simplicity, and yet without any loss of generality of the derivations in subsequent sections, we make the following assumptions.

Assumption 1. $(A(x), B_1(x))$ and $(A(x), B_2(x))$ are locally stabilizable and $(C_1(x), A(x))$ and $(C_2(x), A(x))$ are locally detectable in a neighborhood Ω of the origin.

Assumption 2. $\breve{D}(x)\breve{D}^T(x) = I_{p_2}$.

Furthermore, if one considers the *linear version* of this setup, Desoer and Vidyasagar [2] have shown that the stability of the closed-loop system also guarantees that

$$\int_0^\infty \|u(t)\|^2\, dt \;\le\; k \int_0^\infty \|w(t)\|^2\, dt,$$

where k is a constant and $0 < k < \infty$. For this reason, in the present singular nonlinear problem, it is further assumed that the stabilizing feedback $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}$ not only ensures the L_2-gain of the closed-loop system is less than or equal to γ but satisfies the following assumption.

Assumption 3. For all $x \in M$ there exists a constant $k > 0$ such that

$$\int_0^T \|u_2^*(t)\|^2\, dt \;\le\; k \int_0^T \|w(t)\|^2\, dt, \tag{2}$$

where $T \ge 0$ and $w \in L_2[0, T]$.

Now an extra signal $\bar{z} = \varepsilon u_2$ is introduced into P to obtain a modified system $\bar{P}$ given by

$$\bar{P} := \begin{cases} \dot{x} = & A(x) + B_1(x)w + B_2(x)\binom{u_1}{u_2} \\ \binom{\bar{z}}{z} = & \binom{\varepsilon u_2}{C_1(x)} \\ y = & C_2(x) + \breve{D}(x)w \end{cases} . \tag{3}$$

Figure 2 shows the setup of this system.

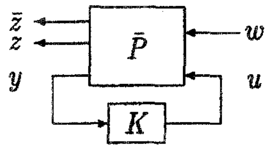


Figure 2. Setup of modified system.

Theorem 1. *The following two conditions are equivalent.*

(i) *The closed-loop system P with stabilizing feedback u^* has L_2-gain less than or equal to γ.*

(ii) *Suppose that ε is sufficiently small; then the closed-loop system $\bar{P}$ with stabilizing feedback u^* has L_2-gain less than or equal to γ.*

Proof. (i) $\Rightarrow$ (ii) Suppose that the stabilizing feedback u^* is already such that the L_2-gain of the closed-loop system P is less than or equal to γ. From Definition 1, this implies that for every $x \in M$ there exist two functions $U(x)$ (with $0 \le U(x) < \infty$ and $U(0) = 0$) and $\bar{U}(x)$ (with $0 \le \bar{U}(x) < \infty$ and $\bar{U}(0) = 0$), and $\delta > 0$ such that

$$\begin{aligned} &\int_0^T \|z(t)\|^2\,dt + U(x) \le \int_0^T \gamma^2 \|w(t)\|^2\,dt \\ \text{or}\quad &\int_0^T \|z(t)\|^2\,dt \le \int_0^T \gamma^2 \|w(t)\|^2\,dt + \bar{U}(x) \\ \text{or}\quad &\int_0^T \|z(t)\|^2\,dt \le (\gamma^2 - \delta)\int_0^T \|w(t)\|^2\,dt + \bar{U}(x) \end{aligned} \tag{4}$$

for all $T \ge 0$ and $w \in L_2[0, T]$.

We multiply both sides of equation (2) by ε^2 and let $\varepsilon > 0$ with ε sufficiently small so that $\delta - \varepsilon^2 k = \mu > 0$. From equation (4), one can find a function $\widehat{U}(x)$ such that

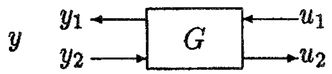
$$\int_0^T (\|z(t)\|^2 + \varepsilon^2 \|u_2^*(t)\|^2)\,dt \le (\gamma^2 - \mu)\int_0^T \|w(t)\|^2\,dt + \widehat{U}(x),$$

for all $T \ge 0$ and $w \in L_2[0, T]$. This implies that the stabilizing feedback u^* also provides an L_2-gain of an closed-loop system $\bar{P}$ that is less than or equal to γ.

(ii) $\Rightarrow$ (i) Because $\int_0^T (\|z(t)\|^2 + \varepsilon^2 \|u_2^*(t)\|^2)\,dt \ge \int_0^T \|z(t)\|^2\,dt$, this is naturally true. □

3. The CSMD and (J, J')-systems

A nonlinear C^∞ chain-scattering system G is given as



$$G := \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) + d(x)u \end{cases} \tag{5}$$

with $y_1 \in \mathbf{R}^p$, $y_2 \in \mathbf{R}^q$, $u_1 \in \mathbf{R}^m$, and $u_2 \in \mathbf{R}^n$. Without a loss of generality, one assumes that G has an equilibrium point at $x = 0$ with $c(0) = 0$.

Definition 2. The controllability function $W(x)$ and observability function $V(x)$ of the nonlinear system G are defined as

$$W(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} \|u_1\|^2 - \|u_2\|^2 dt = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0, x(0)=x_0}} \frac{1}{2}\int_{-\infty}^{0} u^T J' u \, dt$$

and

$$\begin{aligned} V(x_0) &= \frac{1}{2}\int_{-\infty}^{0} \|y_1\|^2 - \|y_2\|^2 dt \\ &= \frac{1}{2}\int_{-\infty}^{0} y^T J y \, dt, \quad x(0) = x_0, \quad u(t) \equiv 0, \quad 0 \le t < \infty, \end{aligned}$$

where $J = \text{diag}\{I_p, -I_q\}$ and $J' = \text{diag}\{I_m, -I_n\}$.

This definition results from the facts that the value of the controllability function at x_0 is the minimum of control energy required to reach state x_0 and the value of the observability function at x_0 is the amount of output energy generated by x_0. Furthermore, from equation (5), the u consists of input element u_1 and output element u_2, and the y consists of output element y_1 and input element y_2. Particularly if system G is asymptotically stable and if x_0 is reachable from 0, then the preceding two functions will be finite.

Theorem 2. *Suppose that $a(x)$ is asymptotically stable on a neighborhood Ω of the origin, and the smooth function $W(x)$ on Ω (with $W(x) \ge 0$ and $W(0) = 0$) is the solution of*

$$\begin{aligned} &W_x(x)a(x) + \frac{1}{2}W_x(x)b(x)J'b^T(x)W_x^T(x) = 0 \\ \Longrightarrow\ &a(x) + \frac{1}{2}b(x)J'b^T(x)W_x^T(x) = 0 \end{aligned}$$

such that $-(a(x)+b(x)J'b^T(x)W_x^T(x))$ is asymptotically stable on Ω; then $W(x)$ is the unique smooth solution. Furthermore, for all $x \in \Omega$, $V(x)$ (with $V(x) \ge 0$ and $V(0) = 0$) is the unique smooth solution of $V_x(x)a(x) + \frac{1}{2}c^T(x)Jc(x) = 0$.

Proof. The proof is quite similar to that of Theorem 3.2 in [12] and is thus omitted. □

The following two theorems are derived from Theorem 2; they are extensions of the (I, I') case [13] applied to the chain-scattering setting. Also, Theorem 3 is a modification from the (I, I')-lossless system [18] to the (J, J')-lossless system.

Theorem 3. *System G with $d^T(x)Jd(x) = J'$, which is reachable from 0, is called (J, J')-lossless if and only if there exists a differentiable function $V(x)$ such that*

(i) $V_x(x)a(x) + \frac{1}{2}c^T(x)Jc(x) = 0$;

(ii) $V_x(x)b(x) + c^T(x)Jd(x) = 0$;

(iii) $V(x) \geq 0$, $V(0) = 0$.

Proof. From the lossless property, there exists a smooth storage function $V(x)$ such that

$$V(x(t_1)) - V(x(t_0)) = \frac{1}{2}\int_{t_0}^{t_1} (u^T(t)J'u(t) - y^T(t)Jy(t))\,dt \geq 0,$$

where $V(x) \geq 0$, $V(0) = 0$, and $t_0 \leq t_1$.

Differentiating both sides, one obtains

$$V_x(x)[a(x) + b(x)u] = \frac{1}{2}u^T J'u - \frac{1}{2}[c(x) + d(x)u]^T J[c(x) + d(x)u].$$

Direct computation yields (i) through (iii). □

Theorem 4. *System G with $d(x)J'd^T(x) = J$, which is reachable from 0, is called conjugate (J, J')-lossless if and only if there exists a differentiable function $W(x)$ such that*

(i) $a(x) + \frac{1}{2}b(x)J'b^T(x)W_x^T(x) = 0$;

(ii) $c(x) + d(x)J'b^T(x)W_x^T(x) = 0$;

(iii) $W(x) \geq 0$, $W(0) = 0$.

Proof. Because it is a direct extension of the co-inner matrix in (I, I') case [13] to the (J, J') system, the proof is omitted. □

Corollary 1. *Let G be a conjugate (J, J')-lossless system. Then*

$$\frac{1}{2}\int_{t_0}^{t_1} (u^T(t)J'u(t) - y^T(t)Jy(t))\,dt \geq 0.$$

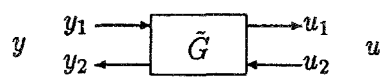
Proof. From the lossless property, there must exist a storage function $W(x)$ such that

$$W(x(t_1)) - W(x(t_0)) = \frac{1}{2}\int_{t_0}^{t_1} (u^T(t)J'u(t) - y^T(t)Jy(t))\, dt \geq 0,$$

where $W(x) \geq 0$ and $W(0) = 0$. □

The following theorem is an extension of the linear conjugate (J, J')-expansive system [7] to the nonlinear setting. One might confuse the conjugate (J, J')-expansive system with the preceding conjugate (J, J')-lossless system; however, the J-expansive has been defined in [3]. For clarity, our current study also uses this definition. As will be seen, the conjugate (J, J')-expansive system is indeed a conjugate $(-J, -J')$-lossless system with the chain-scattering setting contrary to the conjugate (J, J')-lossless system.

This nonlinear C^{∞} chain-scattering system $\tilde{G}$ is given as



$$\tilde{G} := \begin{cases} \dot{x} = a(x) + b(x)u \\ y = c(x) + d(x)u \end{cases}.$$

Theorem 5. *System $\tilde{G}$ with $d(x)J'd^T(x) = J$, which is reachable from 0, is called conjugate (J, J')-expansive iff there exists a differentiable function $W(x)$ such that*

(i) $-a(x) + \frac{1}{2}b(x)J'b^T(x)W_x^T(x) = 0$;

(ii) $-c(x) + d(x)J'b^T(x)W_x^T(x) = 0$;

(iii) $W(x) \geq 0,\ W(0) = 0$.

Proof. Replacing (J, J') in Theorem 4 by $(-J, -J')$ leads immediately to this result. □

Corollary 2. *Let $\tilde{G}$ be a conjugate (J, J')-expansive system. Then*

$$\frac{1}{2}\int_{t_0}^{t_1} (y^T(t)Jy(t) - u^T(t)J'u(t))\ dt \geq 0.$$

Proof. Replacing (J, J') in Corollary 1 by $(-J, -J')$ gives this result. □

4. The state-space formulas for deriving H^{∞} controllers

We now propose an alternative method for designing nonlinear H^{∞} controllers. This method is based on the combination of the chain-scattering matrix description (CSMD) with the (J, J')-lossless and conjugate (J, J')-expansive properties.

From equation (1), let $P = NM^{-1}$ be a normalized right-coprime factorization, as in linear system theory, thus giving

$$N := \begin{cases} \dot{x} &= A(x) + B(x)F(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} \bar{z} \\ z \\ y \end{bmatrix} &= C(x) + D(x)F(x) + D(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases}$$

$$M := \begin{cases} \dot{x} &= A(x) + B(x)F(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} w \\ u \end{bmatrix} &= F(x) + U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases}$$

where

$$B(x) = [\, B_1(x) \quad B_2(x) \,], \quad D(x) = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \\ \check{D}(x) & 0 \end{bmatrix}, \quad C(x) = \begin{bmatrix} 0 \\ C_1(x) \\ C_2(x) \end{bmatrix},$$

$$F(x) = \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix}, \quad U_a(x) = \begin{bmatrix} U_{a_{11}}(x) & U_{a_{12}}(x) \\ U_{a_{21}}(x) & U_{a_{22}}(x) \end{bmatrix}.$$

One further defines G_1 and G_2 as

$$\mathrm{G}_1 := \begin{cases} \dot{x} &= A(x) + B(x)F(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} \bar{z} \\ z \\ w \end{bmatrix} &= \begin{bmatrix} 0 \\ C_1(x) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \\ I & 0 \end{bmatrix}\begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} + \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \\ I & 0 \end{bmatrix} U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix}, \end{cases}$$

$$\mathrm{G}_2 := \begin{cases} \dot{x} &= A(x) + B(x)F(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} u \\ y \end{bmatrix} &= \begin{bmatrix} 0 \\ C_2(x) \end{bmatrix} + \begin{bmatrix} 0 & I \\ \check{D}(x) & 0 \end{bmatrix}\begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} + \begin{bmatrix} 0 & I \\ \check{D}(x) & 0 \end{bmatrix} U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix}. \end{cases} \tag{6}$$

Obviously, the singular nonlinear H^{∞} setup of Figure 2 is thus transformed into the CSMD as shown in Figure 3.

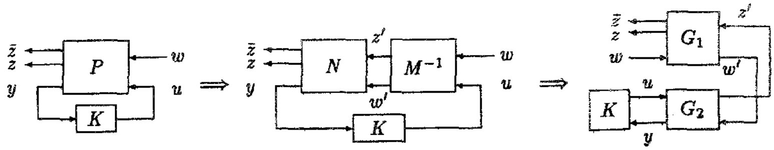


Figure 3. Transformation of closed-loop system from modified system to CSMD.

4.1. The state feedback gain $F(x)$.

To derive the state feedback gain, first one rewrites G_1 in equation (6) as

$$G_1 := \begin{cases} \dot{x} &= \hat{A}(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} \bar{z} \\ z \\ w \end{bmatrix} &= \hat{C}(x) + \hat{D}(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases}.$$

From Assumptions 1 through 3 and Theorem 3, G_1 will be a (J, J')-lossless system if

(i) one chooses $U_a(x) = \begin{bmatrix} 0 & I \\ \frac{1}{\varepsilon}I & 0 \end{bmatrix}$ such that $U_a(x)^T\hat{D}(x)^T J\hat{D}(x)U_a(x) = J'$.

(ii) there exists a C^2 nonnegative differentiable function $V(x)$ (with $V(0) = 0$) that is locally defined in a neighborhood of the origin and that satisfies the Hamilton-Jacobi equation

$$V_x(x)\hat{A}(x) + \frac{1}{2}\hat{C}(x)J\hat{C}(x) = 0$$

such that

$$V_x(x)B(x)U_a(x) + \hat{C}^T(x)J\hat{D}(x)U_a(x) = 0. \tag{7}$$

From [1], [16], [17] and equation (6), one knows that, if the Jacobian matrix of the Hamiltonian flow associated with $G_1^* J G_1$ (with $z' = 0$ and $w' = 0$) at equilibrium belongs to $dom(Ric)$, then the corresponding Riccati equation has a solution. This also implies that there is a $V(x)$ so that $\hat{A}(x) = A(x) + B(x)F(x)$ is locally asymptotically stable in the neighborhood of the origin.

By direct computation, we obtain such a Jacobian matrix as

$$\begin{bmatrix} A & B_1B_1^T - \frac{1}{\varepsilon^2}B_2B_2^T \\ -C_1{}^T C_1 & -A^T \end{bmatrix},$$

and the corresponding Riccati equation is given by

$$A^T X + XA + XB_1B_1^T X - \frac{1}{\varepsilon^2}XB_2B_2^T X + C_1^T C_1 = 0. \tag{8}$$

From Hewer [5], the existence of a solution to this game Riccati equation is related to the standard filter Riccati equation (FARE) $AZ + ZA^T - ZC_1^T C_1 Z + \frac{1}{\varepsilon^2}B_2B_2^T = 0$ and the bounded real Riccati equation (BRRE) $(A - ZC_1^T C_1)W + W(A - C_1^T C_1 Z)^T - WC_1^T C_1 W - B_1B_1^T = 0$. However, from Lemma 3 in Doyle et al. [4], because Assumption A provides the existence of a solution to the FARE and BRRE, the solution for equation (8) thus exists.

Now, we derive the *state feedback gain* $F(x)$ from equation (7), which gives us:

$$V_x(x)B(x)U_a(x) + \hat{C}^T(x)J\hat{D}(x)U_a(x) = 0;$$

$$\Longrightarrow V_x(x)B(x) + \hat{C}^T(x)J\hat{D}(x) = 0;$$

$$\Longrightarrow V_x(x)B(x) + \left(\begin{bmatrix} 0 \\ C_1(x) \\ 0 \end{bmatrix} + \hat{D}(x)F(x)\right)^T J\hat{D}(x) = 0;$$

$$\Longrightarrow V_x(x)B(x) + [0 \quad C_1^T(x) \quad 0]\,J\hat{D}(x) + F^T(x)\hat{D}^T(x)J\hat{D}(x) = 0;$$

$$\Longrightarrow B^T(x)V_x^T(x) + \hat{D}^T(x)J\begin{bmatrix} 0 \\ C_1(x) \\ 0 \end{bmatrix} + \hat{D}^T(x)J\hat{D}(x)F(x) = 0;$$

$$\Longrightarrow F(x) = -R^{-1}(x)B^T(x)V_x^T(x);$$

$$\Longrightarrow \begin{bmatrix} F_1(x) \\ F_2(x) \end{bmatrix} = \begin{bmatrix} B_1^T(x)V_x^T(x) \\ -\frac{1}{\varepsilon^2}B_2^T(x)V_x^T(x) \end{bmatrix},$$

where $R(x) = \hat{D}^T(x)J\hat{D}(x) = \begin{bmatrix} -I_{m_1} & 0 \\ 0 & \varepsilon^2 I \end{bmatrix}$ and $\hat{D}(x) = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \\ I & 0 \end{bmatrix}$.

4.2. Local disturbance attenuation by measurement feedback.

In nonlinear systems, state x of the plant is difficult to measure directly from output y. The actual message for designing the output-feedback controller is thus difficult to obtain. Hence, it is natural to replace x by some estimate ξ provided by proper auxiliary dynamics. One then seeks an appropriate nonlinear system $\tilde{\Pi}$ constructed by this estimate state ξ such that $\tilde{\Pi}G_2$ satisfies the conjugate (J, J')-expansive properties.

First, rewrite G_2 as:

$$G_2 := \begin{cases} \dot{x} & = \hat{A}(x) + B(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \\ \begin{bmatrix} u \\ y \end{bmatrix} & = \tilde{C}(x) + \tilde{D}(x)U_a(x)\begin{bmatrix} z' \\ w' \end{bmatrix} \end{cases}$$

and define the system $\tilde{\Pi}$ given by

$$\tilde{\Pi} := \begin{cases} \dot{\xi} & = \hat{A}(\xi) + H(\xi)\tilde{C}(\xi) + H(\xi)\begin{bmatrix} u \\ y \end{bmatrix} \\ \begin{bmatrix} v \\ \sigma \end{bmatrix} & = U_z(\xi)\tilde{C}(\xi) + U_z(\xi)\begin{bmatrix} u \\ y \end{bmatrix} \end{cases},$$

where ξ is an estimate of x,

$$H(\cdot)=[\,H_1(\cdot)\quad H_2(\cdot)\,],\quad \tilde{C}(\cdot)=\begin{bmatrix}\tilde{C}_1(\cdot)\\ \tilde{C}_2(\cdot)\end{bmatrix}=\begin{bmatrix}F_2(\cdot)\\ C_2(\cdot)+\breve{D}(\cdot)F_1(\cdot)\end{bmatrix},$$

and $v=L(\sigma)$ ($L(\sigma)$ is a free stable system with $\|L(\sigma)\|_{L_2}\le 1$).

The state-space representation of $\tilde{\Pi}G_2$ is therefore given by [

$$\tilde{\Pi}G_2 := \begin{cases}\begin{bmatrix}\dot{x}\\ \dot{\xi}\end{bmatrix}=\begin{bmatrix}\hat{A}(x)\\ \hat{A}(\xi)+H(\xi)\tilde{C}(\xi)+H(\xi)\tilde{C}(x)\end{bmatrix}+\begin{bmatrix}B(x)U_a(x)\\ H(\xi)\tilde{D}(x)U_a(x)\end{bmatrix}\begin{bmatrix}z'\\ w'\end{bmatrix}\\ \begin{bmatrix}v\\ \sigma\end{bmatrix}=U_z(\xi)\left[\tilde{C}(x)+\tilde{C}(\xi)\right]+U_z(\xi)\tilde{D}(x)U_a(x)\begin{bmatrix}z'\\ w'\end{bmatrix}\end{cases}$$

The block diagram of this closed-loop system can be illustrated as in Figure 4.

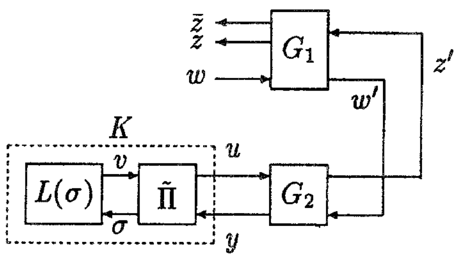


Figure 4. CSMD for over all closed-loop system.

Rewrite $\tilde{\Pi}G_2$ as

$$\tilde{\Pi}G_2 := \begin{cases}\dot{x}_e = A_e(x_e)+B_e(x_e)\begin{bmatrix}z'\\ w'\end{bmatrix}\\ \begin{bmatrix}v\\ \sigma\end{bmatrix} = C_e(x_e)+D_e(x_e)\begin{bmatrix}z'\\ w'\end{bmatrix}\end{cases} \tag{9}$$

Remark 1. From Assumptions 1 through 3 and Theorem 5, $\tilde{\Pi}G_2$ will be a conjugate (J, J')-expansive system if

(i) one chooses $U_z(x)=\begin{bmatrix}\varepsilon I & 0\\ 0 & I\end{bmatrix}$ such that

$$D_e(x_e)JD_e^T(x_e)=U_z(x)\tilde{D}(x)U_a(x)JU_a^T(x)\tilde{D}^T(x)U_z^T(x)=J$$

(ii) there exists a C^2 nonnegative function $W(x_e)=Q(x-\xi)=(x-\xi)^TS(x-\xi)$ (with $W(0)=0$ and $S\ge 0$) that is locally defined in a neighborhood of $(x,\xi)=(0,0)$ and that satisfies the Hamilton-Jacobi equation $W_{x_e}A_e(x_e)-\frac{1}{2}W_{x_e}B_e(x_e)JB_e^T(x_e)W_{x_e}^T=0$ such that

$$C(x_e)-D_e(x_e)JB_e^T(x_e)W_{x_e}^T=0. \tag{10}$$

From [1], [16], and [17], one has that, if the Jacobian matrix of the Hamiltonian flow associated with $\tilde{\pi}G_2JG_2^*\tilde{\pi}^*$ (with the input being zero) at equilibrium belongs to $dom(Ric)$, then there is a $W(x_e) = Q(x-\xi)$ so that $A_e(x_e)$ is locally, asymptotically stable in a neighborhood of $(x, \xi) = (0, 0)$. However, Hong and Teng [7] had shown that such a Jacobian matrix belongs to $dom(Ric)$ and has a solution indicated by $\widehat{Z}$. By direct computation, one can verify that $Q(x-\xi) = (x-\xi)^T\widehat{Z}^{-1}(x-\xi)$ and $\widehat{Z}^{-1} = \frac{1}{2}\left[\frac{\partial^2 Q}{\partial x^2}\right]_{x=0}$.

4.2.1. The measurement feedback gain $H(\xi)$. Because $W(x_e) = Q(x-\xi) = (x-\xi)^T\widehat{Z}^{-1}(x-\xi)$, and then $W_{x_e} = [\ W_x(x_e)\ \ W_\xi(x_e)\ ] = [\ Q_x\ \ -Q_x\ ]$, one has the following derivation for the *measurement feedback gain* $H(\xi)$. Equation (10) gives

$$
\begin{aligned}
&-C(x_e) + D_e(x_e)JB_e^T(x_e)\begin{bmatrix} Q_x^T \\ -Q_x^T \end{bmatrix} = 0 \\
\Longrightarrow\ & -(\tilde{C}(x) + \tilde{C}(\xi)) + \tilde{D}(x)U_a(x)JU_a^T(x)(B^T(x) - \tilde{D}^T(x)H^T(\xi))Q_x^T = 0, \\
\Longrightarrow\ & -(\tilde{C}(x) + \tilde{C}(\xi))Q_x^{-T} + \tilde{D}(x)U_a(x)JU_a^T(x)(B^T(x) - \tilde{D}^T(x)H^T(\xi)) = 0), \\
\Longrightarrow\ & -Q_x^{-1}(\tilde{C}^T(x) + \tilde{C}^T(\xi)) + B(x)U_a(x)JU_a^T(x)\tilde{D}^T(x) \\
& - H(\xi)\tilde{D}(x)U_a(x)JU_a^T(x)\tilde{D}^T(x) = 0, \\
\Longrightarrow\ & H(\xi)\tilde{D}(x)U_a(x)JU_a^T(x)\tilde{D}^T \\
& = -Q_x^{-1}(\tilde{C}^T(x) + \tilde{C}^T(\xi)) + B(x)U_a(x)JU_a^T(x)\tilde{D}^T(x), \\
\Longrightarrow\ & [\ H_1(\xi)\ \ H_2(\xi)\ ] \\
& = [\ Q_x^{-1}(\tilde{C}_1^T(x) + \tilde{C}_1^T(\xi)) - B_2(x)\quad Q_x^{-1}(\tilde{C}_2^T(x) + \tilde{C}_2^T(\xi)) + B_1(x)\breve{D}^T(x)\ ]\tilde{R}^{-1},
\end{aligned}
$$

where $\tilde{R} = \tilde{D}(x)U_a(x)JU_a^T(x)\tilde{D}^T = \begin{bmatrix} \frac{1}{\varepsilon^2}I & 0 \\ 0 & -\breve{D}(x)\breve{D}^T(x) \end{bmatrix} = \begin{bmatrix} \frac{1}{\varepsilon^2}I & 0 \\ 0 & -I \end{bmatrix}$.

Remark 2. Because $v = L(\sigma)$, where $\|L(\sigma)\|_{L_2} \leq 1$, and $\tilde{\Pi}G_2$ is conjugate (J, J')-expansive, it immediately follows from Corollary 2 that

$$\int_{t_0}^{t_1} (\|v(t)\|^2 - \|\sigma(t)\|^2)\,dt \leq 0 \Longrightarrow \int_{t_0}^{t_1} (\|z'(t)\|^2 - \|w'(t)\|^2)\,dt \leq 0.$$

Furthermore, by the property in Theorem 3, having G_1 be (J, J')-lossless implies that

$$
\begin{aligned}
&\int_{t_0}^{t_1} (\|z'(t)\|^2 - \|w'(t)\|^2)\,dt \leq 0 \\
\Longrightarrow &\int_{t_0}^{t_1} (\|\bar{z}(t)\|^2 + \|z(t)\|^2 - \|w(t)\|^2)\,dt \leq 0,
\end{aligned}
$$

and then, from Theorem 1, this also means that the original closed-loop system P has L_2-gain less than or equal to γ ($\gamma = 1$).

5. Conclusion

This paper has defined the nonlinear conjugate (J, J')-lossless and nonlinear conjugate (J, J')-expansive systems. Using these (J, J') systems, we have obtained a family of state-space controllers for the singular four-block local nonlinear H^{∞} output-feedback control problem. As this paper has proposed, our approach has transformed this problem into a simple lossless network problem, which provides deeper insight into the synthesis of the controllers.

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