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T-Colorings and T-Edge Spans of Graphs*

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Abstract. Suppose G is a graph and T is a set of non-negative integers that contains 0. A T-coloring of G is an assignment of a non-negative integer $f(x)$ to each vertex x of G such that $|f(x) - f(y)| \notin T$ whenever $xy \in E(G)$. The edge span of a T-coloring f is the maximum value of $|f(x) - f(y)|$ over all edges xy, and the T-edge span of a graph G is the minimum value of the edge span of a T-coloring of G. This paper studies the T-edge span of the dth power C_n^d of the n-cycle C_n for $T = \{0, 1, 2, \ldots, k-1\}$. In particular, we find the exact value of the T-edge span of C_n^d for $n \equiv 0$ or $1 \pmod{d+1}$, and lower and upper bounds for other cases.

1. Introduction

T-colorings were introduced by Hale [3] in connection with the *channel assignment problem* in communications. In this problem, there are n transmitters $x_1, x_2, \ldots, x_n$ situated in a region. We wish to assign to each transmitter x a frequency $f(x)$. Some of the transmitters interfere because of proximity, meteorological, or other reasons. To avoid interference, two interfering transmitters must be assigned frequencies such that the absolute difference of their frequencies does not belong to the forbidden set T of non-negative integers and T contains 0. The objective is to make a frequency assignment that is efficient according to certain criteria, while satisfying the above constraint.

To formulate the channel assignment problem graph-theoretically, we construct a graph G in which $V(G) = \{x_1, x_2, \ldots, x_n\}$, and there is an edge between transmitters x_i and x_j if and only if they interfere. Given graph G and a set T of non-negative integers and T contains 0, a T-*coloring* of G is a function f from $V(G)$ to the set of non-negative integers such that

$$xy \in E(G) \quad \text{implies} \quad |f(x) - f(y)| \notin T.$$

For the case when $T = \{0\}$, T-coloring is the ordinary vertex coloring.

In channel assignments, the objective is to allocate the channels efficiently. From the T-coloring standpoint, three criteria are important for measuring the

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efficiency: first, the *order* of a T-coloring, which is the number of different colors used in f; second, the *span* of f, which is the maximum of $|f(x) - f(y)|$ over all vertices x and y; and third, the *edge span* of f, which is the maximum of $|f(x) - f(y)|$ over all edges xy. Given T and G, the *T-chromatic number* $\chi_T(G)$ is the minimum order of a T-coloring of G, the *T-span* $\mathrm{sp}_T(G)$ is the minimum span of a T-coloring of G, and the *T-edge span* $\mathrm{esp}_T(G)$ is the minimum edge span of a T-coloring of G.

Cozzens and Roberts [1] showed that the T-chromatic number $\chi_T(G)$ is equal to the *chromatic number* $\chi(G)$, which is the minimum number of colors needed to color the vertices of G so that adjacent vertices have different colors. The parameter T-span of graphs has been studied extensively; for a good survey, see [6]; for recent results, see [2, 5, 7]. However, comparing to T-spans, there are relatively fewer known results about T-edge spans of graphs, see [1, 4].

Cozzens and Roberts [1] raised the problem of computing T-edge spans of non-perfect graphs when $T = \{0, 1, 2, \ldots, k-1\}$. Liu [4] studied this problem for odd cycles. In this article, we consider C_n^d, the dth power of the n-cycle C_n. The graph C_n^d has the vertex set $V(C_n^d) = \{v_0, v_1, \ldots, v_{n-1}\}$ and the edge set

$$E(C_n^d) = \bigcup_{0 \le i \le n-1} \{v_i v_j : j = i+1, i+2, \ldots, i+d\},$$

where the index j for v_j is taken modulo n. We find the exact value of $\mathrm{esp}_T(C_n^d)$ for $n \equiv 0$ or $1 \pmod{d+1}$, and lower and upper bounds for other cases.

2. Previous results

In this section, we quote some known results about T-spans and T-edge spans, some of which will be used in Section 3.

The *clique number* $\omega(G)$ of G is the maximum order of a *clique* (*complete graph*), a set of pairwise adjacent vertices. A complete graph of order n is denoted by K_n. The n-cycle is the graph C_n with vertex set $V(C_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E(C_n) = \{v_0v_1, v_1v_2, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_0\}$. Note that C_n^1 is C_n.

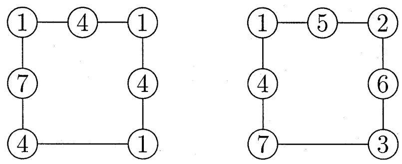
The following are some known results on T-spans and T-edge spans.

Theorem 1. (Cozzens and Roberts [1]) *The following statements hold for all graphs G and sets T.*

(1) $\chi(G) - 1 \le \mathrm{esp}_T(G) \le \mathrm{sp}_T(G)$.
(2) $\mathrm{sp}_T(K_{\omega G)}) \le \mathrm{esp}_T(G) \le \mathrm{sp}_T(G) \le \mathrm{sp}_T(K_{\chi(G)})$.
(3) *If T is $(k-1)$-initial, i.e., $T = \{0, 1, \ldots, k-1\} \cup S$ where S contains no multiple of k, then* $\mathrm{sp}_T(G) = \mathrm{sp}_T(K_{\chi(G)}) = k(\chi(G) - 1)$.

Theorem 2. (Liu [4]) *For any odd cycle C_n and $T = \{0, 1, \ldots, k-1\}$,* $\mathrm{esp}_T(C_n) = \left\lceil \frac{(n+1)k}{n-1} \right\rceil$.

Figure 1 shows an example of C_n with $T = \{0, 1, 2\}$ for which $\chi_T(C_7) = 3 < \mathrm{esp}_T(C_7) = 4 < \mathrm{sp}_T(C_7) = 6$. These values follow from Theorems 1 and 2.



$\chi_T(C_7) = 3$ $\quad$ $\text{esp}_T(C_7) = 4$, $\text{sp}_T(C_7) = 6$

Fig. 1. C_7 with $T = \{0, 1, 2\}$

3. Edge spans for powers of n-cycles

This section gives results for T-edge spans of C_n^d for the $(k-1)$-initial set $T = \{0, 1, 2, \ldots, k-1\}$.

We note that $C_n^d \cong K_n$ for $d \geq \left\lfloor \frac{n}{2} \right\rfloor$ and $\text{esp}_T(K_n) = \text{sp}_T(K_n) = k(n-1)$. Therefore, throughout this article we consider C_n^d only for $d \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ and assume $n = m(d+1) + r$, where $m \geq 2$ and $0 \leq r \leq d$. Our main results are as follows. First, we give an upper bound and a lower bound for $\text{esp}_T(C_n^d)$ (Theorem 4), both of them imply the exact value of $\text{esp}(C_n^d)$ when $r = 0$ (Theorem 5). We then give a better upper bound when $\gcd(n, d+1) = 1$ (Theorem 6) and a better lower bound when $r \geq 1$ (Theorem 7), both of them imply the exact value when $r = 1$ (Theorem 8).

Lemma 3. *If $n = m(d+1) + r$ with $m \geq 2$ and $0 \leq r \leq d$, then $w(C_n^d) = d+1$ and $\chi(C_n^d) = \left\lceil \frac{n}{m} \right\rceil = d + 1 + \left\lceil \frac{r}{m} \right\rceil$.*

Proof. It is easy to see that $\omega(C_n^d) = d+1$ since $d + 1 \leq \left\lfloor \frac{n}{2} \right\rfloor$; and $\chi(C_n^d) \geq \left\lceil \frac{n}{m} \right\rceil$ since any independent set of C_n^d contains at most m vertices. Letting $n_i = \left\lceil \frac{n-i}{m} \right\rceil$, we have

$$n = \sum_{i=0}^{m-1} n_i.$$

Color the n vertices of C_n^d as $1, 2, \ldots, n_0, 1, 2, \ldots, n_1, 1, 2, \ldots, n_2, \ldots, 1, 2, \ldots, n_{m-1}$. This coloring is a proper vertex coloring since each $\frac{n-i}{m} \geq \frac{n-m+1}{m} = d + \frac{r+1}{m}$ and so $\left\lceil \frac{n-i}{m} \right\rceil \geq d+1$. Hence $\chi(C_n^d) \leq \left\lceil \frac{n}{m} \right\rceil$. □

Theorem 4. *If $n = m(d+1) + r$ with $m \geq 2$ and $0 \leq r \leq d$, then $dk \leq \text{esp}_T(C_n^d) \leq \text{sp}_T(C_n^d) = dk + \left\lceil \frac{r}{m} \right\rceil k$.*

Proof. The theorem follows from Theorem 1 and Lemma 3. □

Theorem 5. *If* $n = m(d+1)$ *with* $m \geq 2$, *then* $\mathrm{esp}_T(C_n^d) = \mathrm{sp}_T(C_n^d) = dk$.

Proof. The theorem follows from Theorem 4 as $r = 0$. □

Theorem 6. *Suppose* $n = m(d+1) + r$ *with* $m \geq 2$ *and* $0 \leq r \leq d$. *If* $\gcd(n, d+1) = 1$, *then* $\mathrm{esp}_T(C_n^d) \leq dk + \left\lceil \frac{rk}{m} \right\rceil$.

Proof. Since $\gcd(n, d+1) = 1, d+1$ is a generator of Z_n using modulo n addition, i.e., $j_i \equiv i\ (d+1) \pmod{n}$ for $0 \leq i \leq n-1$ generates each integer in $\{0, 1, \ldots, n-1\}$ exactly once. In other words, we can consider $V(C_n^d)$ as $\{v_{j_0}, v_{j_1}, \ldots, v_{j_{n-1}}\}$. Note that any m circularly consecutive vertices $v_{j_{a+1}}, v_{j_{a+2}}, \ldots, v_{j_{a+m}}$ (with indices $a+p$ considered modulo n) form an independent set in C_n^d. Consequently, $v_{j_a}v_{j_b}$ is not an edge when $0 \leq a < b \leq n-1$ with $1 \leq \min\{b-a, n+a-b\} \leq m-1$.

Now, consider the function f on $V(C_n^d)$ defined by $f(v_{j_i}) = \left\lceil \frac{ik}{m} \right\rceil$ for $0 \leq i \leq n-1$. We claim that f is a T-coloring. For any edge $v_{j_a}v_{j_b}$ with $0 \leq a < b \leq n-1$, according to the preceding discussion, $\min\{b-a, n+a-b\} \geq m$, i.e., $m \leq b-a \leq n-m = md+r$. Then

$$|f(v_{j_a}) - f(v_{j_b})| = \left\lceil \frac{bk}{m} \right\rceil - \left\lceil \frac{ak}{m} \right\rceil \begin{cases} \geq \dfrac{bk}{m} - \dfrac{ak+m-1}{m} \geq k - 1 + \dfrac{1}{m}, \\ \leq \dfrac{bk+m-1}{m} - \dfrac{ak}{m} \leq \dfrac{(md+r)k}{m} + 1 - \dfrac{1}{m}, \end{cases}$$

or

$$|f(v_{j_a}) - f(v_{j_b})| \begin{cases} \geq k, \\ \leq dk + \left\lceil \dfrac{rk}{m} \right\rceil. \end{cases}$$

Therefore, f is a T-coloring of C_n^d and $\mathrm{esp}_T(C_n^d) \leq dk + \left\lceil \frac{rk}{m} \right\rceil$. □

Theorem 7. *If* $n = m(d+1) + r$ *with* $m \geq 2$ *and* $1 \leq r \leq d$, *then* $\mathrm{esp}_T(C_n^d) \geq dk + \left\lceil \frac{k}{m} \right\rceil$.

Proof. Suppose $\mathrm{esp}_T(C_n^d) \leq dk + \left\lceil \frac{k}{m} \right\rceil - 1$. Let f be a T-coloring for which $\mathrm{esp}_T(C_n^d) = \max\{|f(v_i) - f(v_j)| : v_iv_j \in E(C_n^d)\}$. Note that the $m+1$ vertices $v_{i(d+1)}, 0 \leq i \leq m$, are pairwise non-adjacent except for $v_0v_{m(d+1)} \in E(C_n^d)$. Let $\varepsilon_{i,j} = f(v_{i(d+1)}) - f(v_{j(d+1)})$ for $0 \leq i \leq j \leq m$. Then

$$k \leq |\varepsilon_{0,m}| = \left| \sum_{i=0}^{m-1} \varepsilon_{i,i+1} \right| \leq \sum_{i=0}^{m-1} |\varepsilon_{i,i+1}|$$

and so there exists at least one i such that $|\varepsilon_{i,i+1}| \geq \left\lceil \frac{k}{m} \right\rceil$. In other words, the set $U = \{i : |\varepsilon_{i,i+1}| \geq \left\lceil \frac{k}{m} \right\rceil$ and $0 \leq i \leq m-1\}$ is not empty.

For any $i \in U$, the $d+2$ vertices v_j, $i(d+1) \le j \le (i+1)(d+1)$, are pairwise adjacent except that $v_{i(d+1)}$ is not adjacent to $v_{(i+1)(d+1)}$. Sort the $d+2$ values $f(v_j), i(d+1) \le j \le (i+1)(d+1)$, into $b_1 \le b_2 \le \cdots \le b_{d+2}$. If $\{b_1, b_{d+2}\} \ne \{f(v_{i(d+1)}), f(v_{(i+1)(d+1)})\}$, then

$$\mathrm{esp}(C_n^d) \ge b_{d+2} - b_1 = \sum_{j=1}^{d+1}(b_{j+1} - b_j) \ge dk + \left\lceil \frac{k}{m} \right\rceil,$$

a contradiction. Hence, $\{b_1, b_{d+2}\} = \{f(v_{i(d+1)}), f(v_{(i+1)(d+1)})\}$ and

$$|\varepsilon_{i,i+1}| = |f(v_{i(d+1)}) - f(v_{(i+1)(d+1)})| = \sum_{j=1}^{d+1}(b_{j+1} - b_j) \ge (d+1)k.$$

Also,

$$b_{d+2} - b_2 \le \mathrm{esp}(C_n^d) \le dk + \left\lceil \frac{k}{m} \right\rceil - 1,$$

$$b_{d+1} - b_1 \le \mathrm{esp}(C_n^d) \le dk + \left\lceil \frac{k}{m} \right\rceil - 1,$$

$$b_{i+1} - b_i \ge k \text{ for } 2 \le i \le d \text{ and so, } b_{d+1} - b_2 \ge (d-1)k.$$

Then $|\varepsilon_{i,i+1}| = b_{d+2} - b_1 \le (d+1)k + 2\left\lceil \frac{k}{m} \right\rceil - 2$. In conclusion,

$$(d+1)k \le |\varepsilon_{i,i+1}| \le (d+1)k + 2\left\lceil \frac{k}{m} \right\rceil - 2 \quad \text{for all } i \in U.$$

On the other hand, $|\varepsilon_{i,i+1}| \le \left\lceil \frac{k}{m} \right\rceil - 1$ for all $i \notin U$. Let U be the disjoint union of U_1 and U_2 such that $|U_1| \ge |U_2|$ and all $\varepsilon_{i,i+1}$ in U_1 (or U_2) are of the same sign.

For the case $|U_1| > |U_2|$, we have

$$\begin{aligned}
\mathrm{esp}_T(C_n^d) \ge |\varepsilon_{o,m}| &= \left| \sum_{i=0}^{m-1} \varepsilon_{i,i+1} \right| \\
&\ge \sum_{i \in U_1} |\varepsilon_{i,i+1}| - \sum_{i \in U_2} |\varepsilon_{i,i+1}| - \sum_{i \notin U} |\varepsilon_{i,i+1}| \\
&\ge |U_1|(d+1)k - |U_2|\left((d+1)k + 2\left\lceil \frac{k}{m} \right\rceil - 2\right) - (m - |U|)\left(\left\lceil \frac{k}{m} \right\rceil - 1\right) \\
&= (|U_1| - |U_2|)(d+1)k + (|U_1| - |U_2| - m)\left(\left\lceil \frac{k}{m} \right\rceil - 1\right) \\
&\ge (d+1)k + (1-m)\left(\left\lceil \frac{k}{m} \right\rceil - 1\right) \\
&> dk + \left\lceil \frac{k}{m} \right\rceil - 1 \ \left(\text{since } k > m\left(\left\lceil \frac{k}{m} \right\rceil - 1\right)\right),
\end{aligned}$$

a contradiction.

For the case $|U_1| = |U_2|$, say $U_i = \{i_1, i_2, \ldots, i_a\}$ for $i = 1, 2$. Then

$$\begin{aligned} k \le |\varepsilon_{0,m}| &= \left| \sum_{i=0}^{m-1} \varepsilon_{i,i+1} \right| \\ &\le \left| \sum_{j=1}^{a} (\varepsilon_{1_j,1_{j+1}} + \varepsilon_{2_j,2_{j+1}}) \right| + \sum_{i \notin U} |\varepsilon_{i,i+1}| \\ &\le \sum_{j=1}^{a} \left\{ (d+1)k + 2\left\lceil \frac{k}{m} \right\rceil - 2 - (d+1)k \right\} + \sum_{i \notin U} \left\{ \left\lceil \frac{k}{m} \right\rceil - 1 \right\} \\ &= a\left(2\left\lceil \frac{k}{m} \right\rceil - 2 \right) + (m-2a)\left(\left\lceil \frac{k}{m} \right\rceil - 1 \right) \\ &= m\left(\left\lceil \frac{k}{m} \right\rceil - 1 \right) < k, \end{aligned}$$

a contradiction. □

Theorem 8. *If* $n = m(d+1)+1$ *with* $m \ge 2$, *then* $\mathrm{esp}_T(C_n^d) = dk + \left\lceil \frac{k}{m} \right\rceil$.

Proof. The theorem follows from Theorems 6 and 7 and the fact that $\gcd(n, d+1) = 1$. □

Note that Theorem 2 is a special case to the above theorem when n is odd and $d = 1$. For the case where $n \ge 5$ is odd and $d = \frac{n-3}{2}$, we have $r = 1, m = 2$, and C_n^d is isomorphic to the complement $\overline{C_n}$ of C_n. Thus, we have the following result.

Corollary 9. *If* $n \ge 5$ *is odd, then* $\mathrm{esp}_T(\overline{C_n}) = \left\lceil \frac{(n-2)k}{2} \right\rceil$.

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