IEEE TRANSACTIONS ON RELIABILITY, VOL. 47, NO. 4, 1998 DECEMBER

A Fast Reliability-Algorithm for the Circular Consecutive-Weighted-k-out-of-n:F System

Jen-Chun Chang, *Member IEEE*
National Chiao Tung University, Hsinchu

Rong-Jaye Chen, *Member IEEE*
National Chiao Tung University, Hsinchu

Frank K. Hwang
National Chiao Tung University, Hsinchu

Key Words — Consecutive-k-out-of-n:F system, Consecutive weighted-k-out-of-n:F system, Computation complexity.

Summary & Conclusions — An $O(T \cdot n)$ algorithm is presented for the circular consecutive-weighted-k-out-of-n:F system, where $T \leq \min[n, \lceil (k - w_{\max})/w_{\min} \rceil + 1]$; $w_{\max}$, $w_{\min}$ are the maximum, minimum weights of all components. This algorithm is simpler and more efficient than the Wu & Chen $O(\min[n, k] \cdot n)$ algorithm. When all weights are unity, this algorithm is simpler than other $O(k \cdot n)$ published algorithms.

1. INTRODUCTION

Acronyms

- Ckn:F consecutive-k-out-of-n:F system
- CCH the algorithm in this paper
- iff if and only if

Notation

- i component index; $i = 1, 2, \ldots, n$ unless otherwise stated
- w_i weight of component i, w_i is a positive integer
- $w_{\min}$ minimum of all w_i
- $w_{\max}$ maximum of all components; $w_{\max} = w_1$, see assumption #2
- p_i, q_i [reliability, unreliability] of component i
- $R_L(i, j)$ reliability of the linear weighted Ckn:F consisting of components $i, i+1, \ldots, j$
- $R_C(i, j)$ reliability of the circular weighted Ckn:F consisting of components $i, i+1, \ldots, j$
- B_i event: the functioning component with the minimum index is component i
- S_{CW} event: the circular weighted Ckn:F (consisting of components $1, 2, \ldots, n$) functions
- T $\max\left[i : \sum_{j=1}^{i-1} w_j < k,\ 1 \leq i \leq n+1\right]$

A Ckn:F [1 – 5] consists of n components arranged in a line or a circle; each component has its own reliability such that the system fails iff some k consecutive components fail. A weighted-Ckn:F [6] consists of n components; each component has its own reliability and positive-integer-weight such that the system fails iff the total weight of some consecutive failed components is at least k. When all weights are unity, the weighted Ckn:F is the ordinary Ckn:F.

This paper proposes an equation which expresses the circular (weighted & ordinary) Ckn:F reliability in fewer reliability terms of smaller linear systems than other published equations. For example, there are:

- $2k - 1$ linear reliability terms in the equation of [7],
- $k + 1$ linear reliability terms (and one circular reliability term) in the equation of [8],
- $O(k^2)$ linear reliability terms in the equation of [9],
- only k linear reliability terms in (1).

Based on (1) we give CCH for both the ordinary circular Ckn:F and weighted circular Ckn:F. For the ordinary circular Ckn:F, the CCH is simpler than other $O(k \cdot n)$ published algorithms [7 – 9]. For the weighted Ckn:F, the CCH takes $O(T \cdot n)$ time,

$T \leq \min[n, \lceil (k - w_{\max})/w_{\min} \rceil + 1]$,

which is simpler and more efficient than the $O(\min[n, k] \cdot n)$ algorithm [6].

Assumptions

1. Component i has its own w_i, p_i, q_i, for $i = 1, 2, \ldots, n$.
2. The ordering of components is arranged so that $w_1 = \max_i[w_i]$, $i = 1, \ldots, n$. This can be done by re-indexing components in the circular Ckn:F.
3. Each component is either failed or not-failed. All components are mutually s-independent.
4. Components $1, 2, \ldots, n$ are arranged in a circle in that order.
5. For $k > n$, $p_k = 0$, $w_k = w_{k-n}$. ◂

2. MAIN RESULT

Lemmas 1 & 2 support main theorem (theorem 1). Lemma 1 shows that if any B_i occurs, the circular weighted Ckn:F becomes a linear weighted Ckn:F. Lemma 2 shows that if some B_i occurs with $i > T$, then the circular weighted Ckn:F is sure to fail. All proofs are in the appendix.

0018-9529/98/$10.00 ©1998 IEEE

Lemma 1
$\Pr\{S_{CW} \mid B_i\} = R_L(i+1, n+i-1)$, for $i = 1, 2, \ldots, n$. ◄

Lemma 2
$\Pr\{S_{CW} \mid B_i\} = 0$, for $i > T$. ◄

The focus of this paper is theorem 1, which expresses the circular weighted Ckn:F reliability in terms of linear reliabilities.

Theorem 1

$$R_C(1,n) = \begin{cases} 1, & T > n, \\ \sum_{i=1}^{T} \left[R_L(i+1, n+i-1) \cdot \left(\prod_{j=1}^{i-1} q_j \right) \cdot p_i \right], & T \le n. \end{cases} \quad (1)$$

CCH is based on theorem 1.

Algorithm CCH

1. Compute T.
2. If $T > n$, then $R_C(1,n) = 1$. STOP.
3. Compute $R_L(i+1, n+i-1)$, for $i = 1, 2, \ldots, T$.
4. Compute $R_C(1,n)$ by the equation in theorem 3.

STOP ◄

Lemma 3 gives an upper bound of T; it is useful in the time complexity analysis of CCH.

Lemma 3
$T \le \lceil (k - w_{\max})/w_{\min} \rceil + 1$. ◄

Theorem 2
CCH takes $O(T \cdot n)$ time, where:
$T \le \lceil (k - w_{\max})/w_{\min} \rceil + 1$,
for a weighted circular system. ◄

For an ordinary circular system, all $w_i = 1$ and $T = k$. The worst-case time complexity of CCH is $O(k \cdot n)$.

APPENDIX

A.1 Proof of Lemma 1

If B_i occurs, there is no possibility that k consecutive components including component i all fail. So we can remove component i and take the remaining system as linear. Components $1, 2, \ldots, i-1$ are re-indexed as components $n+1, n+2, \ldots, n+i-1$ and they all fail.

A.2 Proof of Lemma 2

If B_i occurs, components $1, 2, \ldots, i-1$ all fail. By definition of T, $\sum_{j=1}^{i-1} w_j \ge k$, the system cannot function, and so S_{CW} can not occur.

A.3 Proof of Theorem 1

· If $T > n$, then $T = n+1$, $\sum_{i=1}^{n} w_i < k$, and the system cannot fail even when all components fail. $R_C(1,n) = 1$.

· If $T \le n$, then $\sum_{i=1}^{n} w_i \ge k$, and the system must contain at least one functioning component in order to function. Thus at least one of $B_1, B_2, \ldots, B_n$ must occur. The $B_1, B_2, \ldots, B_n$ are mutually exclusive.

$$\begin{aligned} R_C(1,n) &= \Pr\{S_{CW}\} \\ &= \sum_{i=1}^{n} \Pr\{S_{CW} \cap B_i\} \\ &= \sum_{i=1}^{n} \Pr\{S_{CW} \mid B_i\} \cdot \Pr\{B_i\} \\ &= \sum_{i=1}^{T} \Pr\{S_{CW} \mid B_i\} \cdot \Pr\{B_i\} \\ &= \sum_{i=1}^{T} R_L(i+1, n+i-1) \cdot \left[\prod_{j=1}^{i-1} q_j \right] \cdot p_i \end{aligned}$$

A.4 Proof of Lemma 3

By definition of T,

$$\begin{aligned} &\sum_{i=1}^{T-1} w_i < k \\ &\Rightarrow w_{\max} + \sum_{i=2}^{T-1} w_i < k \\ &\Rightarrow w_{\max} + (T-2) \cdot w_{\min} < k \\ &\Rightarrow T - 2 < (k - w_{\max})/w_{\min} \\ &\Rightarrow T - 2 < \lceil (k - w_{\max})/w_{\min} \rceil - 1 \\ &\Rightarrow T \le \lceil (k - w_{\max})/w_{\min} \rceil + 1 \end{aligned}$$

A.5 Proof of Theorem 2

For a weighted circular system,

· step 1 takes $O(n)$ time,
· step 2 takes $O(1)$ time,
· step 3 takes $O(T \cdot n)$ time,
· step 4 takes $O(T)$ time.

IF $T > n$, THEN steps 3 & 4 are not executed and the total time complexity is $O(n)$; ELSE steps 3 & 4 are executed; ENDIF.

The worst-case time complexity is $O(T \cdot n)$, where $T \le \lceil (k - w_{\max})/w_{\min} \rceil + 1$ by lemma 3.

REFERENCES

[1] I. Antonopoulou, S. Papastavridis, "Fast recursive algorithm to evaluate the reliability of a circular consecutive-k-out-of-n:F system", *IEEE Trans. Reliability*, vol R-36, 1987 Apr, pp 83 – 84.

[2] D.T. Chiang, S.C. Niu, "Reliability of consecutive-k-out-of-n:F system", *IEEE Trans. Reliability*, vol R-30, 1981 Apr, pp 87 – 89.

[3] C. Derman, G. Lieberman, S. Ross, "On the consecutive-k-out-of-n:F system", *IEEE Trans. Reliability*, vol R-31, 1982 Apr, pp 57 – 63.

[4] F.K. Hwang, "Fast solutions for consecutive-k-out-of-n:F system", *IEEE Trans. Reliability*, vol R-31, 1982 Dec, pp 447 – 448.

[5] J.G. Shanthikumar, "Recursive algorithm to evaluate the reliability of a consecutive-k-out-of-n:F system", *IEEE Trans. Reliability*, vol R-31, 1982 Dec, pp 442 – 443.

[6] J.S. Wu, R.J. Chen, "Efficient algorithms for k-out-of-n & consecutive-weighted-k-out-of-n:F systems", *IEEE Trans. Reliability*, vol 43, 1994 Dec, pp 650 – 655.

[7] J.S. Wu, R.J. Chen, "An $O(k \cdot n)$ algorithm for a circular consecutive-k-out-of-n:F system", *IEEE Trans. Reliability*, vol 41, 1992 Jun, pp 303 – 305.

[8] F.K. Hwang, "An $O(n \cdot k)$-time algorithm for computing the reliability of a circular consecutive-k-out-of-n:F system", *IEEE Trans. Reliability*, vol 42, 1993 Mar, pp 161 – 162.

[9] J.S. Wu, R.J. Chen, "Efficient algorithm for reliability of a circular consecutive-k-out-of-n:F system", *IEEE Trans. Reliability*, vol 42, 1993 Mar, pp 163 – 164.

AUTHORS

Jen-Chun Chang; Dept. of Computer Science and Information Eng'g; National ChiaoTung Univ; 1001 Ta Hsueh Rd; Hsinchu 30050 TAIWAN – R.O.C.

Internet (e-mail): jcchang@csie.nctu.edu.tw

Jen-Chun Chang was born (1967) in I-Lan, Taiwan. He received his BS (1989) and MS (1991) in Computer Science and Information Engineering from National Taiwan University, Taipei. He is a PhD student in the Department of Computer Science and Information Engineering at National Chiao Tung University, Hsinchu. He is a member of IEEE. His research interests include network reliability, algorithm design, and theory of computation.

Dr. Rong-Jaye Chen; Dept. of Computer Science and Information Eng'g; National ChiaoTung Univ; 1001 Ta Hsueh Rd; Hsinchu 30050 TAIWAN – R.O.C.

Internet (e-mail): rjchen@csie.nctu.edu.tw

Rong-Jaye Chen was born (1952) in Taiwan. He received his BS (1977) in Mathematics from National Tsing Hua University, and his PhD (1987) in Computer Science from University of Wisconsin-Madison. He is a professor in the Department of Computer Science and Information Engineering at National Chiao Tung University, Hsinchu, and is a member of IEEE. His research interests include algorithm design, network reliability, and cryptology.

Dr. Frank K. Hwang; Dept. of Applied Mathematics; National ChiaoTung Univ; 1001 Ta Hsueh Rd; Hsinchu 30050 TAIWAN – R.O.C.

Internet (e-mail): fkhwang@math.nctu.edu.tw

Frank K. Hwang (born 1940 Aug 24) obtained his PhD (1968) from North Carolina State University. Before joining Chiaotung University in 1996, he had worked at Bell Labs since 1967. He has published 300 papers, and (co)authored 4 books: *Introduction to Operations Research* (1970, in Chinese), *The Steiner Tree Problem* 1992, *Combinatorial Group Testing and Applications* 1993, and *The Mathematical Theory of Nonblocking Switching Networks* 1993. He has edited several books and been granted 11 patents.

Manuscript TR98-079 received: 1998 May 22;
revised: 1998 October 28
Responsible editor: W. Kuo
Publisher Item Identifier S 0018-9529(98)09969-2