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New generalization of process capability index C_{pk}

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SUMMARY *The process capability index* C_{pk} *has been widely used in manufacturing industry to provide numerical measures of process potential and performance. As noted by many quality control researchers and practitioners,* C_{pk} *is yield-based and is independent of the target T. This fails to account for process centering with symmetric tolerances, and presents an even greater problem with asymmetric tolerances. To overcome the problem, several generalizations of* C_{pk} *have been proposed to handle processes with asymmetric tolerances. Unfortunately, these generalizations understate or overstate the process capability in many cases, so reflect the process potential and performance inaccurately. In this paper, we first introduce a new index* C''_{pk}*, which is shown to be superior to the existing generalizations of* C_{pk}*. We then investigate the statistical properties of the natural estimator of* C''_{pk}*, assuming that the process is normally distributed.*

1 Introduction

Process capability indices (PCIs), the purpose of which is to provide numerical measures of whether or not the ability of a manufacturing process meets a preset level of production tolerance, have received considerable research attention in recent years. Examples include Boyles (1991, 1994), Chan *et al.* (1988), Choi and Owen (1990), Franklin and Wasserman (1992), Johnson *et al.* (1994), Kane (1986), Kushler and Hurley (1992), Pearn and Chen (1996), Pearn *et al.* (1992) and many others. Most research work, however, has focused on developing and investigating PCIs for processes with symmetric tolerances.

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A process is said to have a symmetric tolerance if the target value T is the midpoint of the specification interval (LSL, USL), i.e. $T = M = (\text{USL} + \text{LSL})/2$, where USL and LSL are the upper and the lower specification limits. Among various capability indices that have been introduced, C_{pk} defined as

$$C_{pk} = \frac{d - |\mu - M|}{3\sigma} \tag{1}$$

where μ is the process mean, σ is the process standard deviation and $d = (\text{USL} - \text{LSL})/2$, has been the most widely used index in manufacturing industry, providing unitless measures of process potential and performance. However, as noted by many quality control researchers and practitioners, C_{pk} is essentially a measure of process yield and is independent of T. In fact, we can calculate the process yield as

$$2\Phi(3C_{pk}) - 1 < \%\text{Yield} < \Phi(3C_{pk})$$

if the process is normally distributed, where $\Phi(\cdot)$ is the cumulative function for the standard normal distribution. Consequently, C_{pk} fails to account for process centering with symmetric tolerances, and encounters an even greater problem with asymmetric tolerances.

We consider the following example with asymmetric tolerance (LSL, T, USL), where $T = [3(\text{USL}) + (\text{LSL})]/4$. For processes A and B with $\mu_A = T$, $\mu_B = \text{LSL} + d/2$ and $\sigma_A = \sigma_B = d/6$, both result in $C_{pk} = 1$. The expected proportions non-conforming are the same for both processes (approximately 0.135%). We note that process A is on target, but process B is far away from its target. In fact, we have $\mu_A - \mu_B = T - \mu_B = d$. Clearly, C_{pk} fails to distinguish between on-target and off-target processes in this case.

To overcome the problem, several generalizations of C_{pk}—including $C_{pk}^{\star}$, C'_{pk} and S_{pk}—have been proposed to handle processes with asymmetric tolerances. Unfortunately, these generalizations understate or overstate the process capability in many cases, particularly for cases where the preset production tolerances are asymmetric. Therefore, they reflect the process potential and performance inaccurately. In this paper, we first review the existing generalizations of C_{pk} and then propose a new index, which we refer to as C''_{pk}. The proposed new index C''_{pk} is compared with the existing generalizations of C_{pk} in terms of some process characteristics considered by Boyles (1994), Choi and Owen (1990) and Pearn *et al.* (1992). The results indicate that the proposed new index C''_{pk} is superior to the existing generalizations of C_{pk}. In addition, we investigate the statistical properties of the natural estimator of C''_{pk}, assuming that the process is normally distributed.

2 Existing generalizations of C_{pk}

The first generalization proposed for processes with asymmetric tolerances shifts one of the two specification limits, so that the new (shifted) specification limits are symmetric to the target value (see Chan *et al.*, 1988; Kane, 1986). In other words, the proposal replaces the original specification limits $(T - D_l,\ T + D_u)$ with the new symmetric limits (unjustified sometimes) $T \pm d^{\star}$, where $d^{\star} = \min\{D_l, D_u\}$, $D_u = \text{USL} - T$ and $D_l = T - \text{LSL}$, and then applies the standard definition of C_{pk}.

The generalization may be written as

$$C_{pk}^{\star} = \frac{d^{\star} - |\mu - T|}{3\sigma} \tag{2}$$

We note that this generalization can understate the process capability, by restricting the process to a proper subset of the actual specification range, as observed by Boyles (1994). For example, consider a process with $\mu = T - d/2 = M$ and $\sigma = d/3$, where the target value is $T = [3(\mathrm{USL}) + (\mathrm{LSL})]/4$. For this process, we have $C_{pk}^{\star} = 0$. However, the expected proportion non-conforming is approximately 0.27%. Therefore, the index $C_{pk}^{\star}$ understates the capability of the process in this case. Obviously, if $D_u = D_l$, then the specification tolerance becomes symmetric and the generalization defined in equation (2) reduces to C_{pk} defined in equation (1).

The second generalization proposed for processes with asymmetric tolerances shifts both specification limits to obtain one that is symmetric (Franklin & Wasserman, 1992; Kushler & Hurley, 1992). In other words, the proposal replaces the original specification limits $(T - D_l, T + D_u)$ with the new symmetric limits (unjustified sometimes) $T \pm (D_l + D_u)/2$, and then applies the standard definition of C_{pk}. With this generalization, the index defined in equation (1) can be rewritten as

$$C'_{pk} = \frac{d - |\mu - T|}{3\sigma} \tag{3}$$

This approach can either understate or overstate the process capability, depending on the position of μ relative to T, as noted by Boyles (1994). For example, consider the following two processes A and B with $\mu_A = T - d$, $\sigma_A = d/6$, $\mu_B = T + 3d/4$, $\sigma_B = d/12$ and $T = [3(\mathrm{USL}) + (\mathrm{LSL})]/4$. For process A, we have $C'_{pk} = 0$. However, the expected proportion non-conforming is approximately 0.135%. Thus, C'_{pk} understates the process capability in this case. In contrast, the index value given to process B is $C'_{pk} = 1$. However, the expected proportions non-conforming is approximately 99.865%. Obviously, C'_{pk} overstates the process capability in this case. We note that, if $D_u = D_l$, then the specification tolerance becomes symmetric and the generalization defined in equation (3) reduces to C_{pk} defined in equation (1).

Boyles (1994) defined a smooth function

$$S(x, y) = \frac{\Phi^{-1}[\Phi(x)/2 + \Phi(y)/2]}{3}$$

where $\Phi(x)$ is the cumulative function of the standard normal distribution. Based on this smooth function, Boyles (1994) considered a generalization of C_{pk} defined as

$$S_{pk} = S((\mathrm{USL} - \mu)/\sigma, (\mu - \mathrm{LSL})/\sigma)$$

which can be rewritten as

$$S_{pk} = \frac{1}{3}\Phi^{-1}\left\{\frac{\Phi[(\mathrm{USL} - \mu)/\sigma]}{2} + \frac{\Phi[(\mu - \mathrm{LSL})/\sigma]}{2}\right\} \tag{4}$$

We note that, given $S_{pk} = c$, we can calculate the process yield as

$$\Phi((\mathrm{USL} - \mu)/\sigma) - \Phi((\mathrm{LSL} - \mu)/\sigma) = 2\Phi(3c) - 1$$

for arbitrary values of c. Therefore, S_{pk} represents the actual process yield, unlike C_{pk}, which is only approximately related to the process yield (see Boyles, 1994).

However, we point out that, for a fixed standard deviation σ, S_{pk} obtains the maximal values not at $\mu = T$ but at $\mu = M = (\mathrm{USL} + \mathrm{LSL})/2$, which may reflect the process capability inaccurately in some cases.

For example, consider the following case with asymmetric tolerance $(\mathrm{LSL}, T, \mathrm{USL}) = (26, 50, 58)$. Assume that we have two processes A and B with $\mu_A = T = 50$ and $\mu_B = 34$, respectively, and standard deviation $\sigma_A = \sigma_B = 8$. It is easy to verify that the S_{pk} values for processes A and B are the same (0.468), so both processes have the same yield. While process A is on target, process B is severely off target.

3 New generalization C''_{pk}

In this section, we propose a new generalization of C_{pk} for processes with asymmetric tolerances, which we refer to as C''_{pk}. The design of the new index C''_{pk} is based on the following criteria by Boyles (1994), Choi and Owen (1990) and Pearn *et al.* (1992) in analyzing and comparing the existing capability indices: (a) process yield; (b) process centering; (c) other process characteristics.

The new index C''_{pk} may be defined as

$$C''_{pk} = \frac{d^\star - A^\star}{3\sigma} \tag{5}$$

where

$$A^\star = \max\{d^\star(\mu - T)/D_u, d^\star(T - \mu)/D_l\}$$

Obviously, if $T = M$ (symmetric tolerance), then $A^\star = |\mu - T|$ and C''_{pk} reduces to the basic index C_{pk} defined in equation (1).

We can show that $C''_{pk} < S_{pk}$ for any level c and all values of μ. Thus, given a process with capability $C''_{pk} = c$, the fraction non-conforming is guaranteed to be no greater than that of a process with $S_{pk} = c$; it is $2[1 - \Phi(3c)]$ if the process is normally distributed. Further, given $C''_{pk} > c$, the bounds on $|\mu - T|$ may be calculated as $T - D_l < \mu < T + D_u$. In developing the new index C''_{pk}, we replaced d and $|\mu - T|$ in equation (1) by $d^\star$ and $A^\star$ respectively. This replacement ensures that the new index C''_{pk} obtains the maximal values at $\mu = T$, regardless of whether the preset specification tolerances are symmetric or asymmetric. For processes with asymmetric tolerances, the corresponding loss function is also asymmetric to T. We take into account the asymmetry of the loss function by adding the factors $d^\star/D_u$ and $-d^\star/D_l$ to $\mu - T$, according to whether μ is greater or less than T. The factors $d^\star/D_u$ and $-d^\star/D_l$ ensure that, if there are two processes A and B with $\mu_A > T$ and $\mu_B < T$ satisfying $(\mu_A - T)/D_u = (T - \mu_B)/D_l$, then the index values given to processes A and B must be the same. It is easy to verify that, if the process is on the specification limits ($\mu = \mathrm{LSL}$ or $\mu = \mathrm{USL}$), then $C''_{pk} = 0$. In contrast, if $\mathrm{LSL} < \mu < \mathrm{USL}$, then we have $C''_{pk} > 0$.

To compare the new index with the existing indices, we consider the following example with specifications $(\mathrm{LSL}, T, \mathrm{USL}) = (10, 40, 50)$. Since $D_u = \mathrm{USL} - T = 10$ and $D_l = T - \mathrm{LSL} = 30$, the process has an asymmetric tolerance. Table 1 displays the values of the five indices, C_{pk}, $C^\star_{pk}$, C'_{pk}, S_{pk} and C''_{pk} for various values of μ with a fixed standard deviation $\sigma = 10/3$. In Table 1, we note that C_{pk} and S_{pk} are maximized by $\mu = M = 30$, and the two indices give the same index values to processes A and B, satisfying $\mu_A - M = M - \mu_B$. Thus, for $\mu_A = 40$ and $\mu_B = 20$, the

TABLE 1. Comparison between the five indices for various values of μ and fixed $\sigma = 10/3$, with (LSL, T, USL) = (10, 40, 50)

μ	C_{pk}	$C^{\star}_{pk}$	C'_{pk}	S_{pk}	C''_{pk}
10	0.000	0.000	0.000	0.225	0.000
11	0.100	0.000	0.000	0.291	0.033
12	0.200	0.000	0.000	0.364	0.067
13	0.300	0.000	0.000	0.443	0.100
14	0.400	0.000	0.000	0.525	0.133
15	0.500	0.000	0.000	0.611	0.167
16	0.600	0.000	0.000	0.699	0.200
17	0.700	0.000	0.000	0.789	0.233
18	0.800	0.000	0.000	0.881	0.267
19	0.900	0.000	0.000	0.974	0.300
20	1.000	0.000	0.000	1.068	0.333
21	1.100	0.000	0.100	0.163	0.367
22	1.200	0.000	0.200	1.259	0.400
23	1.300	0.000	0.300	1.355	0.433
24	1.400	0.000	0.400	1.451	0.467
25	1.500	0.000	0.500	1.548	0.500
26	1.600	0.000	0.600	1.646	0.533
27	1.700	0.000	0.700	1.743	0.567
28	1.800	0.000	0.800	1.841	0.600
29	1.900	0.000	0.900	1.938	0.633
30	<u>2.000</u>	0.000	1.000	<u>2.000</u>	0.667
31	1.900	0.100	1.100	1.938	0.700
32	1.800	0.200	1.200	1.841	0.733
33	1.700	0.300	1.300	1.743	0.767
34	1.600	0.400	1.400	1.646	0.800
35	1.500	0.500	1.500	1.548	0.833
36	1.400	0.600	1.600	1.451	0.867
37	1.300	0.700	1.700	1.355	0.900
38	1.200	0.800	1.800	1.259	0.933
39	1.100	0.900	1.900	1.163	0.967
40	1.000	<u>1.000</u>	<u>2.000</u>	1.068	<u>1.000</u>
41	0.900	0.900	1.900	0.974	0.900
42	0.800	0.800	1.800	0.881	0.800
43	0.700	0.700	1.700	0.789	0.700
44	0.600	0.600	1.600	0.699	0.600
45	0.500	0.500	1.500	0.611	0.500
46	0.400	0.400	1.400	0.525	0.400
47	0.300	0.300	1.300	0.443	0.300
48	0.200	0.200	1.200	0.364	0.200
49	0.100	0.100	1.100	0.291	0.100
50	0.000	0.000	1.000	0.225	0.000

process yields (approximately 99.865%) and the index values are all the same for both processes. While process A is on target, process B is severely off target. For $C^{\star}_{pk}$, the index values given to processes with $\mu < T$ are too low. In fact, we have $C^{\star}_{pk} = 0$ for all $\mu \leqslant 30$. We note that, for $\mu = 30$, the process yield is approximately 100%. Clearly, $C^{\star}_{pk}$ understates the process capability in this case. Similarly, the index C'_{pk} understates the process capability for $\mu \leqslant 30$ and overstates it for $\mu > 30$. In fact, for $\mu = 50$, the process yield is approximately 50%, but $C'_{pk} = 1$.

Further, the new index C''_{pk} has taken into account the asymmetry of the loss function. Thus, given two processes A and B with $\mu_A > T$ and $\mu_B < T$, satisfying $(\mu_A - T)/D_u = (T - \mu_B)/D_l$, the (new) index values given to processes A and B are

TABLE 2. Corresponding index values for processes that satisfy $(\mu_A - T)/D_u = (T - \mu_B)/D_l$

μ	C_{pk}	$C^{\star}_{pk}$	C'_{pk}	S_{pk}	C''_{pk}
37	1.300	0.700	1.700	1.355	0.900
41	0.900	0.900	1.900	0.974	0.900
34	1.600	0.400	1.400	1.646	0.800
42	0.800	0.800	1.800	0.881	0.800
31	1.900	0.100	1.100	1.938	0.700
43	0.700	0.700	1.700	0.789	0.700
28	1.800	0.000	0.800	1.841	0.600
44	0.600	0.600	1.600	0.699	0.600
25	1.500	0.000	0.500	1.548	0.500
45	0.500	0.500	1.500	0.611	0.500
22	1.200	0.000	0.200	1.259	0.400
46	0.400	0.400	1.400	0.525	0.400
19	0.900	0.000	0.000	0.974	0.300
47	0.300	0.300	1.300	0.443	0.300
16	0.600	0.000	0.000	0.699	0.200
48	0.200	0.200	1.200	0.364	0.200
13	0.300	0.000	0.000	0.443	0.100
49	0.100	0.100	1.100	0.291	0.100
10	0.000	0.000	0.000	0.225	0.000
50	0.000	0.000	1.000	0.225	0.000

the same. Table 2 is a summary of the processes (taken from Table 1) that satisfy $(\mu_A - T)/D_u = (T - \mu_B)/D_l$. For example, consider processes A and B with $\mu_A = 41 > T$ and $\mu_B = 37 < T$. Clearly, we have $(\mu_A - T)/D_u = 1/10$ and $(T - \mu_B)/D_l = 3/30 = 1/10$. Checking Table 2 for the index values that correspond to $\mu_A = 41$ and $\mu_B = 37$, we have $C''_{pk} = 0.900$ for both processes A and B. However, the values of C_{pk} and S_{pk} given to process B are considerably higher than those given to process A, and the values of $C^{\star}_{pk}$ and C'_{pk} given to process B are lower than those given to process A.

4 Estimation of C''_{pk}

To estimate the new index C''_{pk}, we consider the natural estimator which can be defined as

$$\hat{C}''_{pk} = \frac{d^{\star} - \hat{A}^{\star}}{3S}$$

where

$$\hat{A}^{\star} = \max\{d^{\star}(\bar{X} - T)/D_u, d^{\star}(T - \bar{X})/D_l\}$$

and

$$\bar{X} = \left(\sum_{i=1}^{n} x_i\right) \bigg/ n$$

$$S = \left[(n-1)^{-1} \sum_{i=1}^{n} (x_i - \bar{X})^2\right]^{1/2}$$

are conventional estimators of μ and σ which may be obtained from a process that is demonstrably stable (well in control). In the case where the production tolerance is symmetric (i.e. $T = M$), $\hat{A}^\star$ may be simplified as $|\bar{X} - T|$ and the estimator $\hat{C}''_{pk}$ is reduced to

$$\hat{C}_{pk} = \min\{(\mathrm{USL} - \bar{X})/3S, (\bar{X} - \mathrm{LSL})/3S\}$$

i.e. the natural estimator of C_{pk} discussed by Kotz *et al.* (1993). Therefore, we may view the estimator $\hat{C}''_{pk}$ as a direct extension of $\hat{C}_{pk}$. Assume that the process is normally distributed. Then, the estimator $\hat{C}''_{pk}$ can be rewritten as

$$\hat{C}''_{pk} = \frac{f^{1/2}}{3} K^{-1/2} \left[\frac{d^\star}{\sigma} - \frac{d^\star}{n^{1/2}} \max\left\{\frac{Z}{D_u}, \frac{-Z}{D_l}\right\}\right]$$

where $K = fS^2/\sigma^2$ is distributed as χ^2_f, and $Z = n^{1/2} (\bar{X} - T)/\sigma$ is distributed as $N(\delta, 1)$, with $f = n - 1$ and $\delta = n^{1/2} (\mu - \mathrm{T})/\sigma$. Further, since $\bar{X}$ and S^2 are mutually independent, Z and K are also mutually independent. To obtain the expected value and variance of $\hat{C}''_{pk}$, we first calculate the following:

$$E\left(\max\left\{\frac{Z}{D_u}, \frac{-Z}{D_l}\right\}\right) = \left(\frac{1}{D_u} + \frac{1}{D_l}\right)\frac{1}{(2\pi)^{1/2}}\exp\left(-\frac{\delta^2}{2}\right) + \max\left\{\frac{\delta}{D_u}, \frac{-\delta}{D_l}\right\}[1 - 2\Phi(-|\delta|)] + \left(\frac{\delta}{D_u} - \frac{\delta}{D_l}\right)\Phi(-|\delta|)$$

$$E\left(\max^2\left\{\frac{Z}{D_u}, \frac{-Z}{D_l}\right\}\right) = \frac{1}{2}\left(\frac{1}{D_u^2} + \frac{1}{D_l^2}\right) + \left(\frac{\delta^2}{D_u^2} + \frac{\delta^2}{D_l^2}\right)\Phi(-|\delta|) + \left(\frac{1}{D_u^2} - \frac{1}{D_l^2}\right)\left\{\frac{\delta}{(2\pi)^{1/2}}\exp\left(-\frac{\delta^2}{2}\right) + \frac{\delta}{2|\delta|}[1 - 2\Phi(-|\delta|)]\right\} + \max^2\left\{\frac{\delta}{D_u}, \frac{-\delta}{D_l}\right\}[1 - 2\Phi(-|\delta|)]$$

Therefore, the rth moment (about zero) of $\hat{C}''_{pk}$ may be obtained as

$$E(\hat{C}''_{pk})^r = \frac{f^{r/2}}{3^r} E(K^{-r/2}) \sum_{j=0}^{r} \binom{r}{j} \left(\frac{d^\star}{\sigma}\right)^j \left(\frac{-d^\star}{n^{1/2}}\right)^{r-j} E\left(\max\left\{\frac{Z}{D_u}, \frac{-Z}{D_l}\right\}\right)^{r-j}$$

Hence, we have

$$E(\hat{C}''_{pk}) = \left\{ C''_{pk} - \frac{1}{6}\left(\frac{d^\star}{D_u} + \frac{d^\star}{D_l}\right)\left(\frac{2}{n\pi}\right)^{1/2} \exp\left(-\frac{\delta^2}{2}\right) - \frac{1}{3}\left(\frac{d^\star}{D_u} - \frac{d^\star}{D_l}\right)\left(\frac{\mu - T}{\sigma}\right)\Phi(-|\delta|) + \frac{2}{3}\max\left\{\left(\frac{d^\star}{D_u}\right)\left(\frac{\mu - T}{\sigma}\right), \left(\frac{d^\star}{D_l}\right)\left(\frac{T-\mu}{\sigma}\right)\right\}\Phi(-|\delta|)\right\} b_f^{-1}$$

$$\begin{aligned}\mathrm{Var}(\hat{C}''_{pk}) = \frac{f-2}{f}\Bigg(& (C''_{pk})^2 + \frac{4}{9}\left(\frac{d^\star}{\sigma}\right)\max\left\{\left(\frac{d^\star}{D_u}\right)\left(\frac{\mu - \mathrm{T}}{\sigma}\right), \left(\frac{d^\star}{D_l}\right)\left(\frac{T-\mu}{\sigma}\right)\right\}\Phi(-|\delta|) \\ & - \frac{1}{9}\left(\frac{d^\star}{\sigma}\right)\left(\frac{d^\star}{D_u} + \frac{d^\star}{D_l}\right)\left(\frac{2}{n\pi}\right)^{1/2}\exp\left(-\frac{\delta^2}{2}\right) + \frac{1}{18n}\left[\left(\frac{d^\star}{D_u}\right)^2 + \left(\frac{d^\star}{D_l}\right)^2\right] \\ & + \frac{1}{9}\left[\left(\frac{d^\star}{D_u}\right)^2 + \left(\frac{d^\star}{D_l}\right)^2\right]\left(\frac{\mu - T}{\sigma}\right)^2\Phi(-|\delta|) \\ & - \frac{2}{9}\left(\frac{d^\star}{D_u} - \frac{d^\star}{D_l}\right)\left(\frac{d^\star}{\sigma}\right)\left(\frac{\mu - T}{\sigma}\right)\Phi(-|\delta|) \\ & - \frac{2}{9}\max^2\left\{\left(\frac{d^\star}{D_u}\right)\left(\frac{\mu - T}{\sigma}\right), \left(\frac{d^\star}{D_l}\right)\left(\frac{T-\mu}{\sigma}\right)\right\}\Phi(-|\delta|) \\ & + \frac{1}{9n}\left[\left(\frac{d^\star}{D_u}\right)^2 - \left(\frac{d^\star}{D_l}\right)^2\right]\left\{\frac{\delta}{(2\pi)^{1/2}}\exp\left(-\frac{\delta^2}{2}\right) \right. \\ & \left. + \frac{\delta}{2|\delta|}[1 - 2\Phi(-|\delta|)]\right\}\Bigg) - [E(\hat{C}''_{pk})]^2\end{aligned}$$

In the case where the production tolerance is symmetric ($T = M$), we have

$$\begin{aligned}E(\hat{C}''_{pk}) &= \left[C_{pk} - \frac{1}{3}\left(\frac{2}{n\pi}\right)^{1/2}\exp\left(-\frac{\delta^2}{2}\right) + \frac{2}{3}\frac{|\mu - T|}{\sigma}\Phi(-|\delta|)\right] b_f^{-1} \\ &= E(\hat{C}_{pk})\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(\hat{C}''_{pk}) = \frac{f-2}{f}\Bigg[& (C_{pk})^2 + \frac{4}{9}\left(\frac{d^\star}{\sigma}\right)\frac{|\mu - T|}{\sigma}\Phi(-|\delta|) + \frac{1}{9n} \\ & - \frac{2}{9}\left(\frac{d^\star}{\sigma}\right)\left(\frac{2}{n\pi}\right)^{1/2}\exp\left(-\frac{\delta^2}{2}\right)\Bigg] - [E(\hat{C}_{pk})]^2 = \mathrm{Var}(\hat{C}_{pk})\end{aligned}$$

The results are the same as those calculated by Kotz *et al.* (1993).

TABLE 3. Moments of $\hat{C}''_{pk}$

	Results for the following values of $(\mu - T)/\sigma$									
	− 3.0		− 1.5		0.0		0.5		1.0	
$d^\star/\sigma$	EV	Var	EV	Var	EV	Var	EV	Var	EV	Var
$n = 10$										
2	0.365	0.011	0.547	0.024	0.668	0.037	0.543	0.034	0.365	0.024
3	0.729	0.041	0.912	0.063	1.033	0.083	0.908	0.073	0.729	0.054
4	1.094	0.090	1.277	0.122	1.398	0.148	1.273	0.132	1.094	0.103
5	1.459	0.159	1.641	0.200	1.762	0.233	1.638	0.210	1.459	0.171
$n = 20$										
2	0.347	0.004	0.521	0.009	0.653	0.014	0.520	0.014	0.347	0.010
3	0.695	0.015	0.868	0.023	1.000	0.032	0.868	0.028	0.695	0.021
4	1.042	0.033	1.215	0.045	1.348	0.056	1.215	0.050	1.042	0.039
5	1.389	0.058	1.563	0.074	1.695	0.087	1.562	0.079	1.389	0.064
$n = 30$										
2	0.342	0.003	0.513	0.005	0.651	0.009	0.513	0.009	0.342	0.006
3	0.685	0.009	0.856	0.014	0.994	0.020	0.856	0.018	0.685	0.013
4	1.027	0.020	1.198	0.027	1.336	0.034	1.198	0.031	1.027	0.024
5	1.369	0.035	1.540	0.045	1.678	0.054	1.540	0.048	1.369	0.039
$n = 40$										
2	0.340	0.002	0.510	0.004	0.651	0.007	0.510	0.006	0.340	0.004
3	0.680	0.007	0.850	0.010	0.991	0.014	0.850	0.013	0.680	0.009
4	1.020	0.014	1.190	0.020	1.331	0.025	1.190	0.022	1.020	0.017
5	1.360	0.025	1.530	0.032	1.671	0.039	1.530	0.035	1.360	0.028
$n = 50$										
2	0.339	0.001	0.508	0.003	0.652	0.005	0.508	0.005	0.339	0.004
3	0.677	0.005	0.846	0.008	0.990	0.011	0.846	0.010	0.677	0.007
4	1.016	0.011	1.185	0.015	1.329	0.020	1.185	0.017	1.016	0.013
5	1.354	0.020	1.523	0.025	1.667	0.030	1.523	0.027	1.354	0.022
$n = 60$										
2	0.338	0.001	0.506	0.002	0.652	0.004	0.506	0.004	0.338	0.003
3	0.675	0.004	0.844	0.006	0.990	0.009	0.844	0.008	0.675	0.006
4	1.013	0.009	1.182	0.013	1.327	0.016	1.182	0.014	1.013	0.011
5	1.351	0.016	1.519	0.021	1.665	0.025	1.519	0.022	1.351	0.018

Note: EV, expected value.

Some numerical values of $E(\hat{C}''_{pk})$ and $\mathrm{Var}(\hat{C}''_{pk})$ are presented in Table 3. The readers are encouraged to examine the column that corresponds to $\mu = T$ most carefully. Corresponding values of C''_{pk} are presented in Table 4. We note that $\hat{C}''_{pk}$ is a biased estimator of C''_{pk}. The resultant bias is positive for all cases shown in Table 3 for which $\mu \neq T$. When $\mu = T$, the bias is positive for $n = 10$ but becomes negative for larger values of n. (For $d^\star/\sigma = 2.0$, it is negative for all $n \geqslant 20$; for $d^\star/\sigma = 3.0$, it is negative for all $n \geqslant 30$; for $d^\star/\sigma = 4.0$, it is negative for all $n \geqslant 40$; for $d^\star/\sigma = 5.0$, it is negative for all $n \geqslant 60$.) As n becomes very large, the bias becomes 0. This is explored in more detail in Table 5, which presents the values of $E(\hat{C}''_{pk})$ for $(\mu - T)/\sigma = 0$ and $d^\star/\sigma = 3$. We note that, in this case, the 'theoretical' value of C''_{pk} is 1.

TABLE 4. Values of C''_{pk}

	Results for the following values of $(\mu - T)/\sigma$				
d^*/σ	-3.0	-1.5	0.0	0.5	1.0
2.0	0.333	0.500	0.667	0.500	0.333
3.0	0.667	0.833	1.000	0.833	0.667
4.0	1.000	1.167	1.333	1.167	1.000
5.0	1.333	1.500	1.667	1.500	1.333

TABLE 5. Values of $E(\hat{C}''_{pk})$ for $\mu = T$ and $d^*/\sigma = 3$, corresponding to $C''_{pk} = 1$ for a series of increasing values of n

Sample size n	$E(\hat{C}''_{pk})$	Sample size n	$E(\hat{C}''_{pk})$
10	1.033	750	0.995
20	1.000	1 200	0.996
30	0.994	2 120	0.997
40	0.991	4 420	0.998
50	0.990	12 960	0.999
150	0.992	122 740	1.000
490	0.994		

REFERENCES

BOYLES, R. A. (1991) The Taguchi capability index, *Journal of Quality Technology*, 23, pp. 17–26.

BOYLES, R. A. (1994) Process capability with asymmetric tolerances, *Communications in Statistics: Computation & Simulation*, 23, pp. 615–643.

CHAN, L. K., CHENG, S. W. & SPIRING, F. A. (1988) A new measure of process capability C_{pm}, *Journal of Quality Technology*, 20, pp. 162–175.

CHOI, B. C. & OWEN, D. B. (1990) A study of a new capability index, *Communications in Statistics: Theory and Methods*, 19, pp. 1231–1245.

FRANKLIN, L. A. & WASSERMAN, G. (1992) Bootstrap lower confidence limits for capability indices, *Journal of Quality Technology*, 24, pp. 196–210.

JOHNSON, N. L., KOTZ, S. & PEARN, W. L. (1994) Flexible process capability indices, *Pakistan Journal of Statistics*, 10A, 23–31.

KANE, V. E. (1986) Process capability indices, *Journal of Quality Technology*, 18, pp. 41–52.

KOTZ, S., PEARN, W. L. & JOHNSON, N. L. (1993) Some process capability indices are more reliable than one might think, *Applied Statistics*, 42, pp. 55–62.

KUSHLER, R. H. & HURLEY, P. (1992) Confidence bounds for capability indices, *Journal of Quality Technology*, 24, pp. 188–195.

PEARN, W. L., KOTZ, S. & JOHNSON, N. L. (1992) Distributional and inferential properties of process capability indices, *Journal of Quality Technology*, 24, pp. 216–231.

PEARN, W. L. & CHEN, K. S. (1996) Bayesian-like Estimators of C_{pk}, *Communications in Statistics: Computation & Simulation*, 25, pp. 321–329.