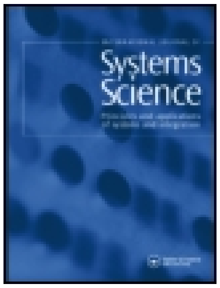

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Global stabilization of axial compressors using nonlinear cancellation and backstepping designs

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Global stabilization of axial compressors using nonlinear cancellation and backstepping designs

DER-CHERNG LIAW† and JENG TZE HUANG†

Issues concerning the global stabilization of axial compressors are presented. Instead of locally stabilizing the bifurcated rotating stall equilibria to avoid suddern performance drop, control schemes are proposed to globally stabilize the unstalled nominal operating points. These are achieved by employing nonlinear cancellation and backstepping designs via either throttle or direct mass flow rate control if the compressor characteristic satisfies the 'left-tilt' property. The designs not only guarantee a safe operation close to the maximum achievable pressure rise, but the bifurcated system equilibria are also found to be stabilized due to the elimination of the jump phenomenon. Though both throttle and direct mass flow rate control schemes can be implemented for global stabilization, numerical simulations demonstrate that direct mass flow rate control achieves better post-stall system behaviour.

Nomenclature

- A amplitude of the first angular mode of rotating wave.
- ΔP non-dimensional pressure rise within the plenum.
- $\dot{m}_C$ non-dimensional compressor mass flow rate.
- B Greitzer B-parameter, proportional to rotor speed.
- α a geometry-related constant.
- W scaling parameter for normalized velocities.
- $\dot{m}_T$ non-dimensional throttle mass flow rate.
- C_{ss} non-dimensional axisymmetric compressor characteristic.
- C'_{ss} derivative of C_{ss} function with its own argument.
- F non-dimensional throttle function.
- γ control parameter of throttle function.

1. Introduction

Axial flow compressors are widely utilized in both aerospace and industrial applications because of their potential of high efficient operations. The efficiency of an engine is heavily dependent on compressing the air prior to combustion to a high pressure condition (Kerrebrock 1992). However, when a compressor operates close to the peak of its achievable pressure rise, two aerodynamic instabilities are likely to occur which reduce its performance drastically. One is the so-called 'rotating stall', which refers to a dynamic instability that occurs when a non-axisymmetric flow pattern develops in the blade passages of a compressor stage. The other is surge behaviour, which denotes a large amplitude, axisymmetric oscillation in the overall pumping system.

Conventionally, a stall (or surge) line is drawn to provide a boundary of safe operation for compressors. It is obvious that such a restriction of the feasible operating region unduly restricts the capabilities of jet engines. Therefore, recently various control schemes have been proposed to enable compressors to operate safely beyond the stall line thereby increasing its efficiency (Paduano *et al.* 1993, Liaw and Abed 1996, Krstić *et al.* 1995b). Among these, bifurcation theory is successfully applied to stabilize the stalled branch, which results in a smooth behaviour rather than an abrupt drop of performance. Hence, the feasible operating region can be effectively enlarged (Liaw and Abed 1996). An experimental study of that design was also carried out to demonstrate the applicability to the real compressors (Badmus *et al.* 1993).

A Lyapunov function based backstepping design has been proposed to achieve global stabilization for com-

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pressor dynamics (Krstić *et al.* 1995b). However, the scheme was designed only for cubic compressor characteristics. Inspired by the works of Krstić *et al.* (1995b), in this paper we try to extend the study to the more general compression systems. There are three major differences between the works of Krstić *et al.* (1995b) and the study of this paper. Firstly, Krstić *et al.* tried to stabilize globally *a priori* known stalled equilibria, while we focus on the global stabilization of the unstalled operating points. Secondly, Krstić *et al.* only dealt with the compression systems with cubic compressor characteristics. In this paper, we only require the compressor characteristics to possess the so-called 'left-tilt' property. Thirdly, instead of using the throttle as the sole actuator, in this paper two types of actuators are considered for the implementation of the design. One is the modulation of the mass flow directly and the other is the setting of the throttle. The implementation of the former actuator can be carried out in several ways (Hendricks and Gysling 1994).

The paper is organized as follows. In section 2, the Moore–Greitzer model (Moore and Greitzer 1986) for a compression system is recalled. A brief description of compressor dynamics is also given to highlight the motivation of the paper. It is followed by the stabilization design of the unstalled equilibria via nonlinear cancellation and the backstepping control schemes. In the proposed designs, nonlinear dynamics are suitably cancelled to attain global stability for the unstalled system equilibria. The cubic compressor model is adopted in section 4 to demonstrate the validity of the designs. Finally, a conclusion is given in section 5.

2. Dynamical equations for axial flow compression systems

Conceptually, a compression system can be represented by a series of components comprising inlet duct, compressor, exit duct, plenum, and throttle as depicted in figure 1. In the practical application, the plenum represents the combustion chamber, while the throttle represents the first-stage turbine nozzles. The air flow is normally designed to be axisymmetric in an axial compressor. Though the flow within a compression system is distributed in nature and can only be described fully by partial differential equations, however, a lumped-parameter third-order ordinary differential equation (ODE) model introduced by Moore and Greitzer (1986) captures the most essential features of compressor dynamics. To adopt the notation of Liaw and Abed (1996), the model can be represented as

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(\dot{m}_\mathrm{C} + WA\sin\theta)\sin\theta\,\mathrm{d}\theta, \qquad (1)$$

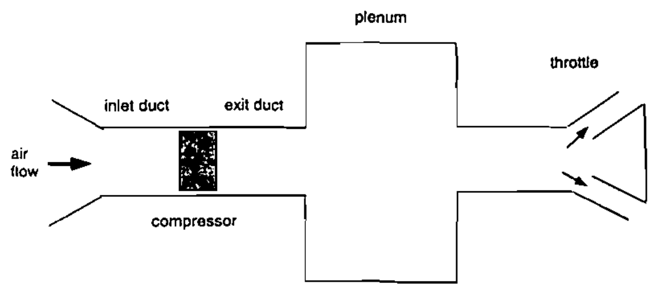

Figure 1. Schematic diagram of axial flow compression system.

$$\frac{\mathrm{d}\dot{m}_\mathrm{C}}{\mathrm{d}t} = -\Delta P + \frac{1}{2\pi}\int_0^{2\pi} C_{ss}(\dot{m}_\mathrm{C} + WA\sin\theta)\,\mathrm{d}\theta, \quad (2)$$

$$\frac{\mathrm{d}\Delta P}{\mathrm{d}t} = \frac{1}{4B^2}\{\dot{m}_\mathrm{C} - \dot{m}_\mathrm{T}\}, \qquad (3)$$

where the definitions of the quantities are given in the Nomenclature.

In the dynamical equations above, equation (2) is obtained from a momentum balance and implies that the acceleration of the fluid in the inlet and outlet ducts is proportional to the difference between the pressure rise across the compressor and the pressure rise in the plenum. The variable of integration θ represents the angular displacement from a reference stationary with the first harmonic mode of the stall wave (Moore and Greitzer 1986). Moreover, equation (1) determines the rate of the amplitude $A(t)$, while equation (3) governs the change rate of the plenum pressure. In this paper, the non-dimensional throttle mass flow, $\dot{m}_\mathrm{T}$, is taken as

$$\dot{m}_\mathrm{T} \triangleq F^{-1}(\gamma, \Delta P) = \gamma(\Delta P)^{1/2}, \qquad (4)$$

which is the same as the one in McCaughan (1989). The axisymmetric compressor characteristic $C_{ss}(\cdot)$ is often a S-shaped function and is modelled as a cubic polynomial (Moore and Greitzer 1986). Based on the cubic compressor characteristic of Moore and Greitzer (1986), a bifurcation analysis of the compression system (1)–(3) was studied by McCaughan (1989). A more general study was later investigated by Liaw and Abed (1992) without the assumption of cubic compressor characteristics.

Normally, the compression system is required to operate at the so-called 'unstalled equilibrium', i.e. $A = 0$. Denote $x^0(\gamma) = (0, \dot{m}_\mathrm{C}(\gamma), \Delta P(\gamma))^\mathrm{T}$ as such an equilibrium point for given throttle parameter γ. The values of $\dot{m}_\mathrm{C}^0(\gamma)$ and $\Delta P^0(\gamma)$ can then be obtained from setting the right-hand side of equations (1)–(3) to zero. This leads to the relationships: $\dot{m}_\mathrm{C}^0 = \gamma(\Delta P^0)^{1/2}$ and $\Delta P^0 = C_{ss}(\dot{m}_\mathrm{C}^0)$. It means that the unstalled equilibrium occurs at the intersection of the $C_{ss}(\cdot)$ and the $F(\cdot)$ function graphed in the ΔP–$\dot{m}_\mathrm{C}$ phase plane. Thus, as the throttle control parameter γ continuously varies, the nominal operating points will trace out a branch of unstalled equilibria in the phase plane. Moreover, there usually exist other equilibrium branches, the so-called 'stalled equilibria', i.e. the equilibrium solutions of system (1)–(3) with $A \neq 0$.

From Liaw and Abed's work (1996), the linearization of system (1)–(3), with $\dot{m}_T$ as in equation (4), shows that the nominal point is locally asymptotically stable (resp. unstable) for $C'_{ss}(\dot{m}_C^0) < 0$ (resp. $C'_{ss}(\dot{m}_C^0) > 0$). Moreover, the nominal unstalled points are found to lose linear stability at the point with $C'_{ss}(\dot{m}_C^0) = 0$. The operating point with $C'_{ss}(\dot{m}_C^0) = 0$ is the so-called 'stall inception point' at which the unstalled and the stalled equilibria joined as a consequence of the occurrence of *stationary bifurcation*. When the system operates at a point close to the stall inception point, a perturbation of the throttle control value might lead the compression system to exhibit a jump from a stable operation to a further stable stalled equilibrium. This results in a suddern change of the pressure rise and the occurrence of rotating stall in the real operation. Moreover, the system might also run into rotating stall for large enough disturbance due to the coexistence of multiple equilibria.

For illustrations, typical bifurcation diagrams and timing responses for a cubic compressor characteristic defined as in equation (36) of section 4 are depicted in figures 2 and 3. In figure 2 (a), the solid lines represent the stable equilibria while the dotted lines denote the unstable equilibria as the throttle setting varies. Denote γ_s and γ_c the values of γ at which the compression system exhibits saddle-node bifurcation and stationary bifurcation, respectively (McCaughan 1989). The throttle control functions for $\gamma = \gamma_s$ and $\gamma = \gamma_c$ are depicted in figure 2 (a) as two dash-dotted lines. Multiple system equilibria are observed to coexist for $\gamma_c \leq \gamma \leq \gamma_s$. For instance, the two transient trajectories denoted as 1 and 2 in figure 2 (a) go to different final states for the same value of the throttle control value. The jump behaviour associated with trajectory 2 is obviously caused by a large disturbance of the initial condition. When such a jumping phenomenon occurs, the system state eventually settles down to an equilibirum with a finite amplitude of stall wave and a much lower pressure rise instead of returning back to the original unstalled equilibrium as trajectory 1 does, as depicted in figure 2 (b) and 2 (d). The jumping behaviour, as discussed above, might also attribute to a perturbation of the throttle control value as depicted in figure 3. In figure 3 (a), the system jumps to a stable stalled equilibrium along trajectory 3 for the perturbed control value $\gamma < \gamma_c$, as depicted in figures 3 (b)–3 (d). The bifurcation diagram as depicted in figure 4 demonstrates the jumping behaviour of stable system equilibria near the stall inception point where the solid lines (resp. dotted lines) denote the stable (resp. unstable) system equilibria.

The main goal of this study is to design a suitable control law for eliminating the coexistence of multiple system equilibria and guaranteeing the stability of the unique system equilibrium for all $\gamma > \gamma_c$. By continuity, such design might also prevent the jumping behaviour for γ being close to γ_c with $\gamma < \gamma_c$. Details are given in the next section.

3. Global stabilization

In this section, we consider employing nonlinear cancellation and backstepping control schemes to globally stabilize the unstalled compressor dynamics so that the coexistence of multiple equilibria can be annihilated. Two types of actuators are considered for the study. One is the direct mass flow rate control (Hendricks and Gysling 1994), and the other is the dynamic setting of throttle. Detailed designs are given as follows.

3.1. *Direct mass flow rate control*

First, we consider applying the technique of Proposition 6.3 of Byrnes and Isidori (1991) for global stabilization of compression systems with control input appearing at the mass flow dynamics only. Implementation of such a control algorithm is found to be carried out in a variety of ways (Hendricks and Gysling 1994).

Denote $(0, \dot{m}_C^0, \Delta P^0)^T$ the unstalled operating point. Let the system state variation be given by $x_1 = A$, $x_2 = \dot{m}_C - \dot{m}_C^0, x_3 = \Delta P - \Delta P^0$. System (1)–(3) becomes

$$\frac{dx_1}{dt} = \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(\dot{m}_C^0 + x_2 + W x_1 \sin\theta)\sin\theta \, d\theta, \quad (5)$$

$$\frac{dx_2}{dt} = -x_3 - \Delta P^0 + \frac{1}{2\pi}\int_0^{2\pi} C_{ss}(\dot{m}_C^0 + x_2 + W x_1 \sin\theta)\, d\theta + u, \quad (6)$$

$$\frac{dx_3}{dt} = \frac{1}{4B^2}\{x_2 + \dot{m}_C^0 - \gamma(x_3 + \Delta P^0)^{1/2}\}, \quad (7)$$

where $\dot{m}_T$ is replaced by equation (4) and u denotes the applied control input. The linearization of system (5)–(7) at $(0, \dot{m}_C^0, \Delta P^0)^T$ gives

$$\frac{dX}{dt} = AX + Bu, \quad (8)$$

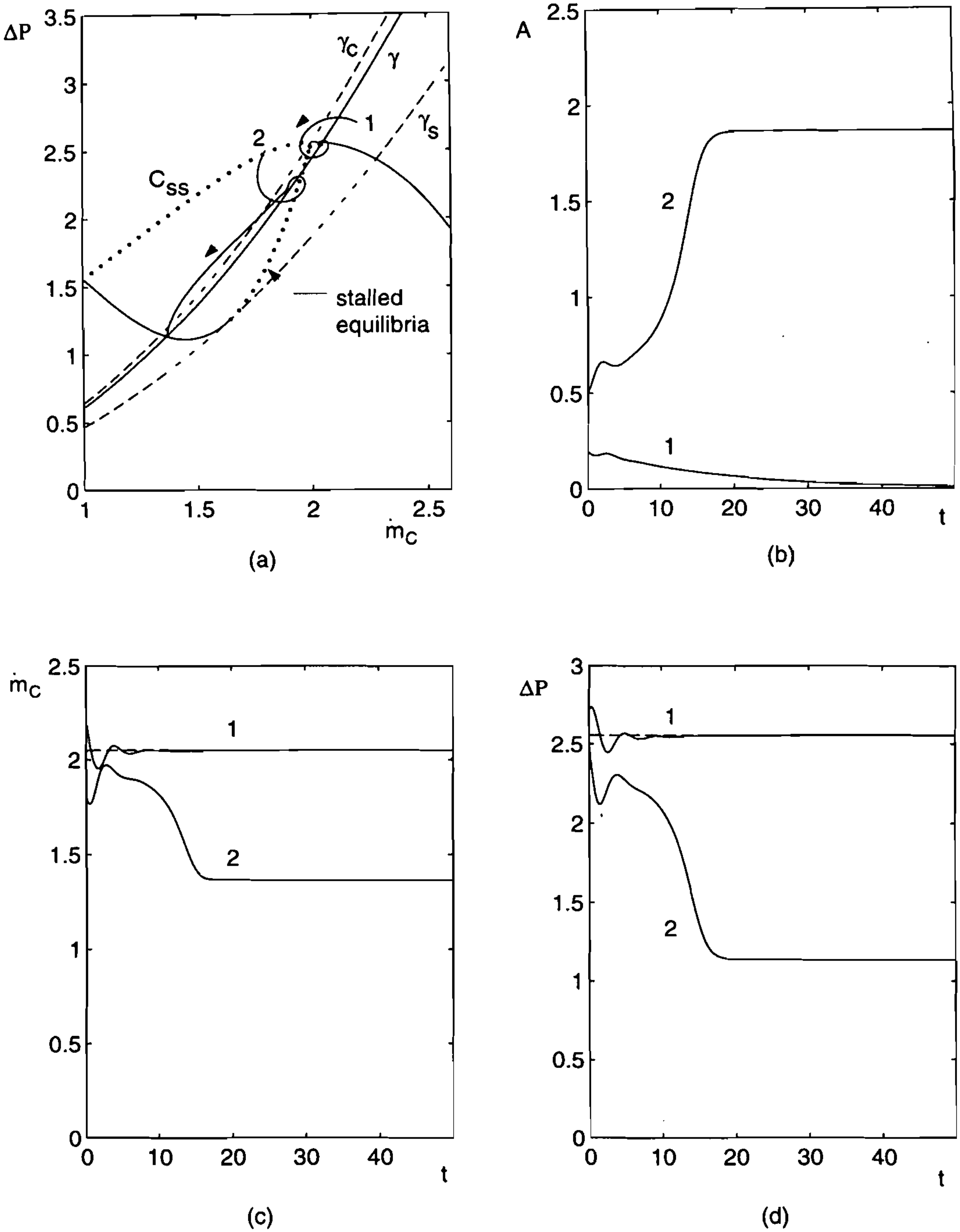

Figure 2. Time response of uncontrolled pre-stall behaviour.

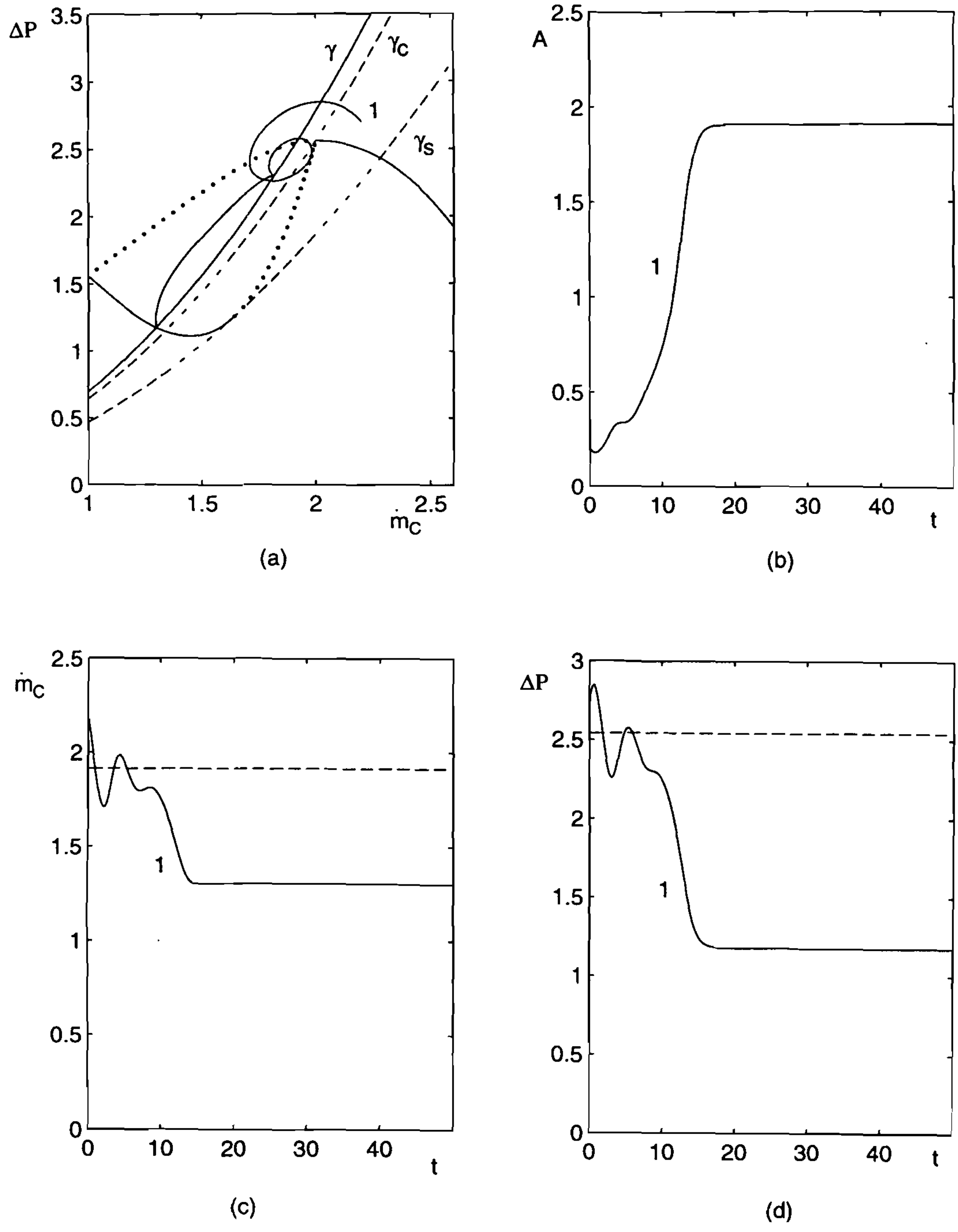

Figure 3. Time response of uncontrolled post-stall behaviour.

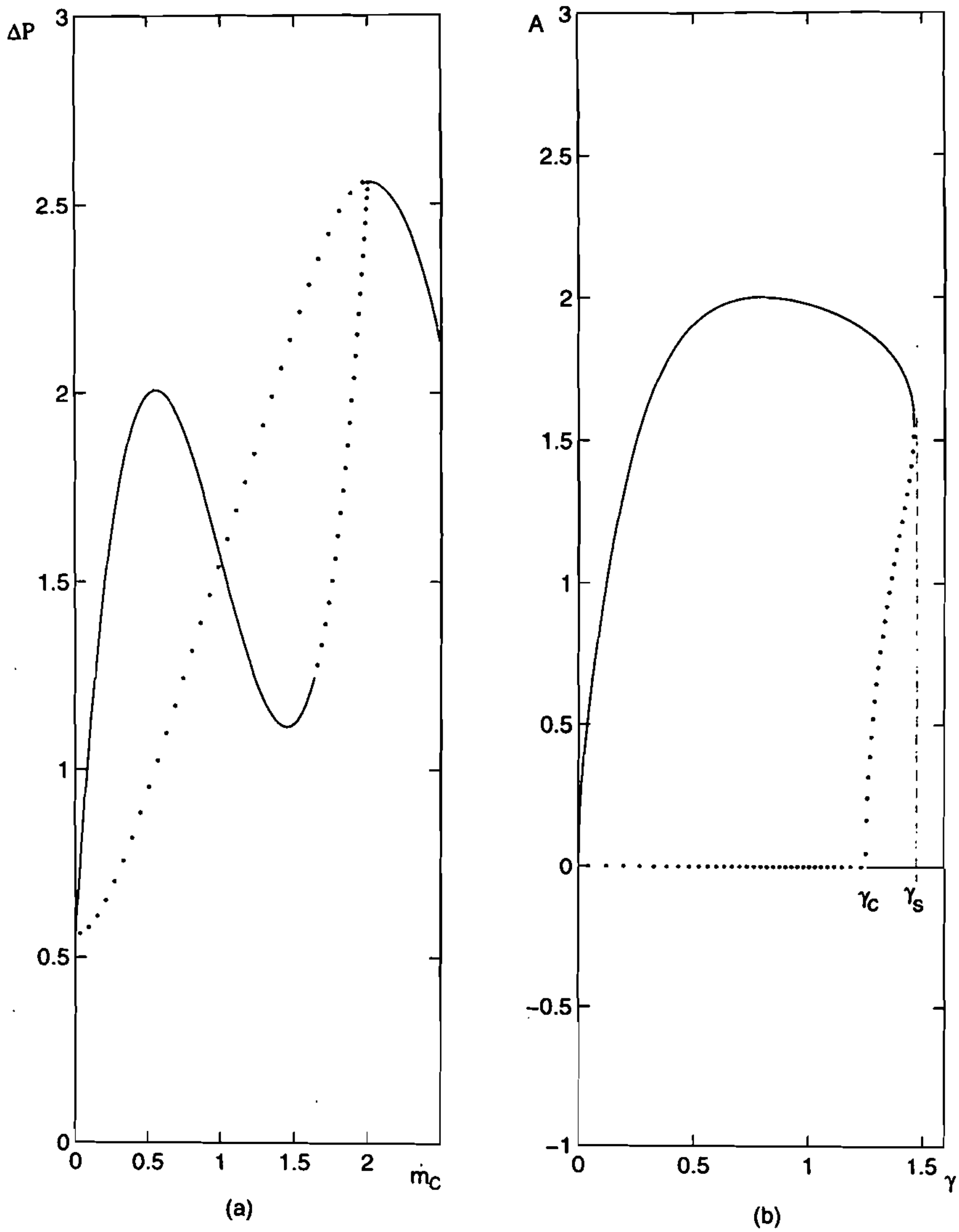

Figure 4. Bifurcation diagram of uncontrolled compressor dynamics.

where

$$A = \begin{pmatrix} \alpha C'_{ss}(\dot{m}_C^0) & 0 & 0 \\ 0 & C'_{ss}(\dot{m}_C^0) & -1 \\ 0 & \dfrac{1}{4B^2} & -\dfrac{\gamma}{8B^2\,(\Delta P^0)^{1/2}} \end{pmatrix} \tag{9}$$

and

$$B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} . \tag{10}$$

It is easy to check that system (5)–(7) is linearly uncontrollable by using the PBH test (e.g. Kailath 1980). Moreover, $x_1 = 0$ always makes the right-hand side of equation (5) zero. That means the system (5)–(7) has an invariant manifold with $x_1 = 0$ disregarding the value of u. However, it is known that a system is nonlinearly stabilizable if it belongs to the family of the so-called minimum phase nonlinear systems (Isidori 1989). Interestingly, by some reasonable assumptions on the $C_{ss}(\cdot)$ functions made in the following, the compression systems are found to belong to such a family.

It is known that the axisymmetric compressor characteristic C_{ss} is a S-shaped function of the mass flow rate (Moore and Greitzer 1986). This implies that the characteristic function $C_{ss}(\cdot)$ has a local maximum. Denote $\dot{m}_C^P$ the value of mass flux at which C_{ss} has its local maximum value. We have the following result.

Lemma 1: *Suppose $\dot{m}_C^0 \geq \dot{m}_C^P$. Then we have $x_1\dot{x}_1 \leq 0$ for equation (5) with $x_2 = 0$ if the following two conditions hold:*

(i) *$C_{ss}(\xi)$ is a monotonically decreasing function for $\xi \geq \dot{m}_C^P$.*

(ii) *C_{ss} satisfies the 'left-tilt' property:*

$$C_{ss}(\dot{m}_C^P - \eta) > C_{ss}(\dot{m}_C^P + \eta) \qquad \text{for all } \eta > 0.$$

Moreover, $x_1\dot{x}_1 = 0$ only occurs at $x_1 = 0$.

Proof: It is clear that equation (5) with $x_2 = 0$ can be written as

$$\begin{aligned}\dot{x}_1 &= \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(\dot{m}_C^0 + Wx_1 \sin\theta)\sin\theta \, d\theta \\ &= \frac{\alpha}{\pi W}\int_0^{\pi}\{C_{ss}(\dot{m}_C^0 + Wx_1\sin\theta) \\ &\quad - C_{ss}(\dot{m}_C^0 - Wx_1\sin\theta)\}\sin\theta\, d\theta. \end{aligned} \tag{11}$$

First, we consider the case of which $\dot{m}_C^0 - |Wx_1| \geq \dot{m}_C^P$. Then we have

$$\begin{aligned} x_1\dot{x}_1 &= \frac{\alpha}{\pi W}\int_0^{\pi}\{C_{ss}(\dot{m}_C^0 + Wx_1\sin\theta) \\ &\quad - C_{ss}(\dot{m}_C^0 - Wx_1\sin\theta)\}x_1\sin\theta\, d\theta. \end{aligned} \tag{12}$$

Since $\dot{m}_C^0 - |Wx_1| \geq \dot{m}_C^P$, it is obvious that we have $x_1\dot{x}_1 \leq 0$ and $x_1\dot{x}_1 = 0$ only occurs at $x_1 = 0$ if condition (i) holds.

Next, we consider the case of which $\dot{m}_C^0 - |Wx_1| < \dot{m}_C^P$. To facilitate the proof, in the following we only concern ourselves with the case of $x_1 \geq 0$. It is not difficult to construct a similar proof for the case of $x_1 < 0$.

Let $\eta = \dot{m}_C^P - \dot{m}_C^0 + Wx_1\sin\theta$ and

$$\theta_1 = \sin^{-1}\left((\dot{m}_C^0 - \dot{m}_C^P)/(Wx_1)\right)$$

for $x_1 > 0$. Since $\dot{m}_C^0 - Wx_1 < \dot{m}_C^P$, θ_1 is always solvable. Moreover, it is obvious that $\eta > 0$ for $\theta_1 \leq \theta \leq \pi - \theta_1$. We then have

$$\begin{aligned} x_1\dot{x}_1 = \frac{\alpha}{\pi W}\Bigg\{ &\int_0^{\theta_1}[C_{ss}(\dot{m}_C^0 + Wx_1\sin\theta) \\ &- C_{ss}(\dot{m}_C^0 - Wx_1\sin\theta)]x_1\sin\theta\, d\theta \\ &+ \int_{\theta_1}^{\pi-\theta_1}[C_{ss}(\dot{m}_C^0 + Wx_1\sin\theta) \\ &- C_{ss}(\dot{m}_C^P + \eta)]x_1\sin\theta\, d\theta \\ &+ \int_{\theta_1}^{\pi-\theta_1}[C_{ss}(\dot{m}_C^P + \eta) - C_{ss}(\dot{m}_C^P - \eta)]x_1\sin\theta\, d\theta \\ &+ \int_{\pi-\theta_1}^{\pi}[C_{ss}(\dot{m}_C^0 + Wx_1\sin\theta) \\ &- C_{ss}(\dot{m}_C^0 - Wx_1\sin\theta)]x_1\sin\theta\, d\theta\Bigg\}. \end{aligned} \tag{13}$$

From the definition of η, we have $\dot{m}_C^0 + Wx_1\sin\theta - (\dot{m}_C^P + \eta) = 2(\dot{m}_C^0 - \dot{m}_C^P) > 0$. This implies that

$$\int_{\theta_1}^{\pi-\theta_1}[C_{ss}(\dot{m}_C^0 + Wx_1\sin\theta) - C_{ss}(\dot{m}_C^P + \eta)]x_1\sin\theta\, d\theta \leq 0 \tag{14}$$

and the equality holds only for $x_1 = 0$ if the characteristic function C_{ss} satisfies condition (i) of Lemma 1. Now, it is not difficult to check from equation (13) that $x_1\dot{x}_1 \leq 0$ and $x_1\dot{x}_1 = 0$ only occurs at $x_1 = 0$ if both conditions (i) and (ii) hold. □

Suppose x_2 is selected as the output of system (5)–(7). Since x_2 and x_3 are controllable state variables by the control input u, we then have just shown that the zero dynamics of system (5)–(7), i.e. equation (5) with $x_2 = 0$, is globally stable. Motivated by the stabilization results of Isidori (1989), we can construct a global stabilization control law for system (5)–(7) via nonlinear cancellation. Details are given as follows.

Suppose the compressor characteristic function C_{ss} is a C^1 function. By Mean Value Theorem (e.g. Courant and John 1989), for given x_2 we then have

$$\begin{aligned} &C_{ss}(\dot{m}_C^0 + x_2 + Wx_1\sin\theta) \\ &\quad = C_{ss}(\dot{m}_C^0 + Wx_1\sin\theta) + C'_{ss}(\dot{m}_C^0 + Wx_1\sin\theta + \zeta x_2)\cdot x_2 \end{aligned} \tag{15}$$

for some $\zeta \in [0, 1]$. Now, we construct an energy-like Lyapunov function for system (5)–(7) as

$$V(x) = \tfrac{1}{2}(x_1^2 + x_2^2 + 4B^2x_3^2), \tag{16}$$

where $x = (x_1, x_2, x_3)^{\mathrm{T}}$. Taking the derivative of $V(x)$ along trajectories of system (5)–(7), we have

$$\begin{aligned}\dot V(x) &= x_1\dot x_1 + x_2\dot x_2 + 4B^2\cdot x_3\dot x_3\\ &= \frac{\alpha}{\pi W}\left\{\int_0^{2\pi} C_{ss}(\dot m_C^0 + Wx_1\sin\theta)x_1\sin\theta\,d\theta\right.\\ &\quad\left.+\int_0^{2\pi} x_2C'_{ss}(\dot m_C^0 + Wx_1\sin\theta+\zeta x_2)x_1\sin\theta\,d\theta\right\}\\ &\quad+\frac{x_2}{2\pi}\int_0^{2\pi}[C_{ss}(\dot m_C^0 + x_2 + Wx_1\sin\theta)\\ &\quad - C_{ss}(\dot m_C^0)]\,d\theta + x_2u\\ &\quad -\gamma x_3((x_3+\Delta P^0)^{1/2} - (\Delta P^0)^{1/2}), \end{aligned}\tag{17}$$

since $\Delta P^0 = C_{ss}(\dot m_C^0)$ and $\dot m_C^0 = \gamma(\Delta P^0)^{1/2}$. From Lemma 1, we have

$$\int_0^{2\pi} C_{ss}(\dot m_C^0 + Wx_1\sin\theta)x_1\sin\theta\,d\theta \le 0$$

for all $x_1\in R$ and the equality holds only at $x_1 = 0$ if conditions of Lemma 1 hold. Moreover, it is not difficult to check that $-\gamma x_3((\Delta P^0 + x_3)^{1/2} - \Delta P^0) \le 0$ for all $x_3 \in R$ and the equality holds only at $x_3 = 0$. Thus, $\dot V(x)$ in (17) will be a negative definite function if the applied control input u is chosen so that V_1 defined below is a function of x_2 only as well as a negative definite function of x_2.

$$\begin{aligned}V_1 &:= \frac{\alpha}{\pi W}\int_0^{2\pi} x_2C'_{ss}(\dot m_C^0 + Wx_1\sin\theta+\zeta x_2)x_1\sin\theta\,d\theta\\ &\quad+\frac{x_2}{2\pi}\int_0^{2\pi}[C_{ss}(\dot m_C^0 + x_2 + Wx_1\sin\theta) - C_{ss}(\dot m_C^0)]\,d\theta + x_2u.\end{aligned}\tag{18}$$

Obviously, one of such choices for u can be selected as

$$\begin{aligned}u &= -\frac{\alpha}{\pi W}\int_0^{2\pi} C'_{ss}(\dot m_C^0 + Wx_1\sin\theta+\zeta x_2)x_1\sin\theta\,d\theta\\ &\quad-\frac{1}{2\pi}\int_0^{2\pi}[C_{ss}(\dot m_C^0 + x_2 + Wx_1\sin\theta) - C_{ss}(\dot m_C^0)]\,d\theta - x_2.\end{aligned}\tag{19}$$

Let the control input u be a function of x_1 and x_2 to make V_1 to be negative definite in x_2 for all $x_1 \in R$. We then have

$$\begin{aligned}\dot V(x) &= \frac{\alpha}{\pi W}\int_0^{2\pi} x_1C_{ss}(\dot m_C^0 + Wx_1\sin\theta)\\ &\quad\times\sin\theta\,d\theta + V_1(x_2) - \gamma x_3(x_3+\Delta P^0)^{1/2}\\ &\le 0\end{aligned}\tag{20}$$

for all $(x_1, x_2, x_3) \in R^3$ and $\dot V(x) = 0$ only occurs at $x = 0$.

Employing Lyapunov stability criteria (e.g. Vidyasaga 1992), we have the next global stabilization result following directly from the discussions above.

Theorem 1: *Suppose $\dot m_C^0 \ge \dot m_C^P$. Then the system equilibrium $(0, \dot m_C^0, \Delta P^0)$ of the uncontrolled version of system (5)–(7) can be globally stabilized by the direct control of mass flow rate if the compressor characteristic function C_{ss} satisfies the two conditions of Lemma 1.*

Remark 1: The control input u to fulfil the nonlinear cancellation as required by the global stabilization design might seem to be very complicated and hard to obtain. However, it is only for the proof of the existence of a global stabilizer for general compressor dynamics. In fact, especially for polynomial-type C_{ss} functions, it is not difficult to calculate. For instance, to adopt the cubic compressor characteristic function from Liaw and Abeds' result (1996), the applied control input u can be easily calculated as presented in section 4.

3.2. *Throttle control*

In this section, we extend the global design discussed in section 3.1 to the case in which the throttle setting is the only applied control input. The motivation of the study is that the throttle control will be easier for practical implementations. However, the nonlinear dynamics of a compression system is found not to be directly cancelled by throttle control only. Motivated by Krstić *et al.* (1995b), the backstepping control scheme is considered to fulfil the global stabilization design.

Similarly, denote $x_1 = A$, $x_2 = \dot m_C - \dot m_C^0$, $x_3 = \Delta P - \Delta P^0$ and let $\gamma = \gamma^0 + u$. Here, u denotes the throttle control input. The throttle controlled version of system (1)–(3) becomes:

$$\frac{dx_1}{dt} = \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(\dot m_C^0 + x_2 + Wx_1\sin\theta)\sin\theta\,d\theta,\tag{21}$$

$$\frac{dx_2}{dt} = -x_3 - \Delta P^0 + \frac{1}{2\pi}\int_0^{2\pi} C_{ss}(\dot m_C^0 + x_2 + Wx_1\sin\theta)\,d\theta,\tag{22}$$

$$\begin{aligned}\frac{dx_3}{dt} &= \frac{1}{4B^2}\{x_2 + \dot m_C^0 - \gamma^0(x_3+\Delta P^0)^{1/2}\}\\ &\quad-\frac{1}{4B^2}(x_3+\Delta P^0)^{1/2}u.\end{aligned}\tag{23}$$

Following the backstepping design procedure (Krstić *et al.* 1995a), we first try to globally stabilize the subsystem (21)–(22) by treating x_3 as a virtual control input. According to the discussions in section 3.1, x_3 can be chosen such that for any $x_1 \in R$ the value of V_2 defined below is a function of x_2 only as well as a negative definite function of x_2.

$$V_2 := \frac{\alpha}{\pi W}\int_0^{2\pi} x_2 C'_{ss}(\dot{m}_C^0 + Wx_1 \sin\theta + \zeta x_2)x_1 \sin\theta \, d\theta + \frac{x_2}{2\pi}\int_0^{2\pi}[C_{ss}(\dot{m}_C^0 + x_2 + Wx_1\sin\theta) - C_{ss}(\dot{m}_C^0)]\, d\theta - x_2 x_3. \quad (24)$$

Denote x_{3d} the designed virtual control x_3 which makes V_2 to be a negative definite function of x_2. It is obvious that x_{3d} is a function of x_1 and x_2. Similarly, one such choice of x_{3d} can be obtained as

$$x_{3d} = \frac{\alpha}{\pi W}\int_0^{2\pi} x_1 C'_{ss}(\dot{m}_C^0 + \zeta x_2 + Wx_1 \sin\theta)\sin\theta \, d\theta + \frac{1}{2\pi}\int_0^{2\pi}\{C_{ss}(x_2 + \dot{m}_C^0 + Wx_1\sin\theta) - C_{ss}(\dot{m}_C^0)\}\, d\theta + x_2 \quad (25)$$

for some $\zeta \in [0, 1]$ and ζ is known to be a function of x_2 (e.g. Courant and John 1989).

Next, we define the error term ξ of x_3 from the designed function x_{3d}, which makes V_2 defined in equation (24) to be a negative definite function of x_2 only, by

$$\xi := x_3 - x_{3d}. \quad (26)$$

The throttle control system (21)–(23) can then be rewritten in the new coordinates (x_1, x_2, ξ) as

$$\dot{x}_1 = \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(\dot{m}_C^0 + x_2 + Wx_1\sin\theta)\sin\theta\, d\theta, \quad (27)$$

$$\dot{x}_2 = -(\xi + x_{3d}) - \Delta P^0 + \frac{1}{2\pi}\int_0^{2\pi} C_{ss}(\dot{m}_C^0 + x_2 + Wx_1\sin\theta)\, d\theta, \quad (28)$$

$$\dot{\xi} = \frac{1}{4B^2}\{\dot{m}_C^0 + x_2 - \gamma^0(x_{3d} + \xi + \Delta P^0)^{1/2}\} - \dot{x}_{3d} - \frac{1}{4B^2}(x_{3d} + \xi + \Delta P^0)^{1/2}u. \quad (29)$$

Choose an energy-like Lyapunov function $V(x_1, x_2, \xi)$ as

$$V(x_1, x_2, \xi) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}(4B^2\xi^2). \quad (30)$$

The time derivative of V along trajectories of system (27)–(29) is then calculated as

$$\begin{aligned}\dot{V}(x_1, x_2, \xi) &= x_1\dot{x}_1 + x_2\dot{x}_2 + 4B^2\cdot\xi\dot{\xi} \\ &= \frac{\alpha}{\pi W}\left\{\int_0^{2\pi} x_1 C_{ss}(\dot{m}_C^0 + Wx_1\sin\theta)\sin\theta\, d\theta + \int_0^{2\pi} x_1 x_2 C'_{ss}(\dot{m}_C^0 + \zeta x_2 + Wx_1\sin\theta)\sin\theta\, d\theta\right\} \\ &\quad - x_2 x_{3d} \\ &\quad + x_2\cdot\frac{1}{2\pi}\int_0^{2\pi}[C_{ss}(\dot{m}_C^0 + x_2 + Wx_1\sin\theta) - C_{ss}(\dot{m}_C^0)]\, d\theta - 4B^2\dot{x}_{3d}\xi \\ &\quad - \gamma^0((x_{3d} + \xi + \Delta P^0)^{1/2} - (\Delta P^0)^{1/2})\xi - \xi(x_{3d} + \xi + \Delta P^0)^{1/2}u. \end{aligned} \quad (31)$$

By using the definition of V_2 as in equation (24), we have

$$\begin{aligned}\dot{V}(x_1, x_2, \xi) &= \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(\dot{m}_C^0 + Wx_1\sin\theta)x_1 \times \sin\theta\, d\theta + V_2(x_2) - 4B^2\dot{x}_{3d}\xi \\ &\quad - \gamma^0((x_{3d} + \xi + \Delta P^0)^{1/2} - (\Delta P^0)^{1/2})\xi \\ &\quad - \xi(x_{3d} + \xi + \Delta P^0)^{1/2}u. \end{aligned} \quad (32)$$

From the discussions above and those in section 3.1, the function $V_3(x_1, x_2)$ defined below will be a negative definite function in both x_1 and x_2 if x_{3d} is selected to make $V_2(x_2)$ as in equation (24) be negative definite.

$$V_3(x_1, x_2) := \frac{\alpha}{\pi W}\int_0^{2\pi} C_{ss}(\dot{m}_C^0 + Wx_1\sin\theta)x_1 \times \sin\theta\, d\theta + V_2(x_2). \quad (33)$$

First, it is observed from equation (31) that $\dot{V}(x_1, x_2, \xi)$ will be a negative definite function in x_1, x_2 and ξ if the applied throttle control u is chosen so that

$$V_4 := -\gamma^0((x_{3d} + \xi + \Delta P^0)^{1/2} - (\Delta P^0)^{1/2})\xi - 4B^2\dot{x}_{3d}\xi - \xi(x_{3d} + \xi + \Delta P^0)^{1/2}u \quad (34)$$

is a function of ξ only as well as a negative definition of ξ. From equation (34), one choice of such u can be selected as

$$u = \frac{1}{(x_{3d} + \xi + \Delta P^0)^{1/2}}\{-\gamma^0((x_{3d} + \xi + \Delta P^0)^{1/2} - (\Delta P^0)^{1/2}) - 4B^2\dot{x}_{3d} + \xi\} \quad (35)$$

if $x_{3d} + \xi + \Delta P^0 \neq 0$. Similarly, by employing Lyapunov stability criteria and the discussions above, we have the next theorem.

Theorem 2: *Suppose $\dot{m}_C^0 \geq \dot{m}_C^P$. Then the unstalled equilibrium point $(0, \dot{m}_C^0, \Delta P^0)$ of the uncontrolled version of* *system* (21)–(23) *is near-globally stabilized by throttle if the axisymmetric compressor characteristic C_{ss} satisfies the two conditions of Lemma* 1.

Note that the total pressure rise ΔP is known to be $\Delta P = \Delta P^0 + x_{3d} + \xi$, which is a positive value in practical application. Thus, in the practical application the throttle control input u as in equation (35) is always solvable.

4. Illustrative example

In the following, numerical simulations are presented to demonstrate the application of the proposed control algorithms to a given compressor dynamics. Here, we adopt the cubic axisymmetric compressor characteristic from (Liaw and Abed 1996) as given by

$$C_{ss}(\dot{m}_C) = 1.56 + 1.5(\dot{m}_C - 1) - 0.5(\dot{m}_C - 1)^3. \quad (36)$$

We have the specific compression system model as given by

$$\frac{dx_1}{dt} = \alpha x_1\{1.5 - 1.5(x_2 + \dot{m}_C^0 - 1)^2 - \tfrac{3}{8}W^2x_1^2\}, \quad (37)$$

$$\frac{dx_2}{dt} = -x_3 - 0.75W^2(\dot{m}_C^0 - 1)x_1^2 + (1.5 - 0.75W^2x_1^2 - 1.5(\dot{m}_C^0 - 1)^2)x_2 - 1.5(\dot{m}_C^0 - 1)x_2^2 - 0.5x_2^3 + u_1, \quad (38)$$

$$\frac{dx_3}{dt} = \frac{1}{4B^2}\{x_2 + \dot{m}_C^0 - \gamma^0(x_3 + \Delta P^0\})^{1/2} - \frac{1}{4B^2}(\Delta P^0 + x_3)^{1/2}u_2, \quad (39)$$

where u_1 and u_2 denote the direct mass flow rate and throttle control, respectively. In the following numerical study, we assume only one of them will be in effect for the control application.

The system equilibrium for unstalled operation (i.e. $A = 0$) and stalled operation (i.e. $A \neq 0$), as depicted in figure 2, are obtained by using the numerical continuation and bifurcation analysis code AUTO (Doedel 1981). Here, the values of γ corresponding to the occurrence of saddle-node bifurcation and stationary bifurcation are determined as $\gamma_s = 1.4638$ and $\gamma_c = 1.254$. Typical responses of the uncontrolled version of system (37)–(39) are shown in figures 2 and 3, where the system parameters are selected as $\alpha = 0.4114$, $W = 1.0$, and $B = 0.35$. Detailed descriptions have been presented in section 2 for illustrating the goal of this study.

It is observed from figures 2 and 3 that compressors will exhibit the jumping behaviour. In the following, we will try to apply Theorems 1 and 2 as in section 3 to globally stabilize unstalled system equilibria. First, we check whether the conditions of Lemma 1 will hold for the compressor characteristic C_{ss} as given in (36). By taking the first derivative of $C_{ss}(\dot{m}_C)$ with respect to $\dot{m}_C$, we have

$$C'_{ss}(\dot{m}_C) = \frac{dC_{ss}(\dot{m}_C)}{d\dot{m}_C} = 1.5[1 - (\dot{m}_C - 1)^2]. \quad (40)$$

It is not difficult to check that C_{ss} has a local maximum at $\dot{m}_C = 2$ by letting the right-hand side of equation (40) be zero and checking the second derivative of C_{ss} with respective to $\dot{m}_C$. That is, we have $\dot{m}_C^P = 2$. Moreover, we have negative values of $C'_{ss}(\dot{m}_C)$ for all $\dot{m}_C > \dot{m}_C^P$. Furthermore, for $\eta > 0$ we have

$$C_{ss}(\dot{m}_C^P - \eta) - C_{ss}(\dot{m}_C^P + \eta) = \eta + \eta^3 > 0. \quad (41)$$

Thus, the cubic compressor characteristic C_{ss} as in equation (36) satisfies the two conditions of Lemma 1. According to Theorems 1 and 2, there exist both direct mass flow and throttle control laws to globally or near-globally stabilize all the unstalled equilibrium $(0, \dot{m}_C^0, \Delta P^0)$ of compression system with $\dot{m}_C^0 \geq \dot{m}_C^P$ and $\dot{m}_C^0 = \gamma(\Delta P^0)^{1/2}$ while $\Delta P^0 = C_{ss}(\dot{m}_C^0)$ for given $\gamma = \gamma^0$. Now, we can construct the stabilizing control laws by following the derivations as in section 3. Details are given as follows.

Case 1. *Direct mass flow rate control*

First, consider the case of which $u_1 \neq 0$ and $u_2 = 0$. Rewriting equation (37), we have

$$\frac{dx_1}{dt} = \alpha x_1\{1.5 - 1.5(\dot{m}_C^0 - 1)^2 - \tfrac{3}{8}W^2x_1^2\} - 1.5\alpha x_1x_2[x_2 + 2(\dot{m}_C^0 - 1)]. \quad (42)$$

Thus, the value of V_1 defined in equation (18) is obtained as

$$\begin{aligned} V_1 = {} & -1.5\alpha x_1^2x_2[x_2 + 2(\dot{m}_C^0 - 1)] \\ & + x_2 \cdot \{-0.75W^2(\dot{m}_C^0 - 1)x_1^2 \\ & + [1.5 - 0.75W^2x_1^2 - 1.5(\dot{m}_C^0 - 1)^2]x_2 \\ & - 1.5(\dot{m}_C^0 - 1)x_2^2 - 0.5x_2^3 + u_1\}. \end{aligned} \quad (43)$$

Choosing

$$u_1 = 1.5\alpha x_1^2[x_2 + 2(\dot{m}_C^0 - 1)] + 0.75W^2x_1^2(\dot{m}_C^0 + x_2 - 1) + 1.5(\dot{m}_C^0 - 1)x_2^2, \quad (44)$$

we have

$$V_1(x_2) = [1.5 - 1.5(\dot{m}_C^0 - 1)^2]x_2^2 - 0.5x_2^4. \quad (45)$$

It is obvious that V_1 is a negative definite function of x_2 for all $\dot{m}_C^0 \geq \dot{m}_C^P = 2$. Numerical simulations with $\gamma = 1.2$ and $\gamma = 1.281$ are shown in figures 5 and 6, respectively, to demonstrate the stabilization results of

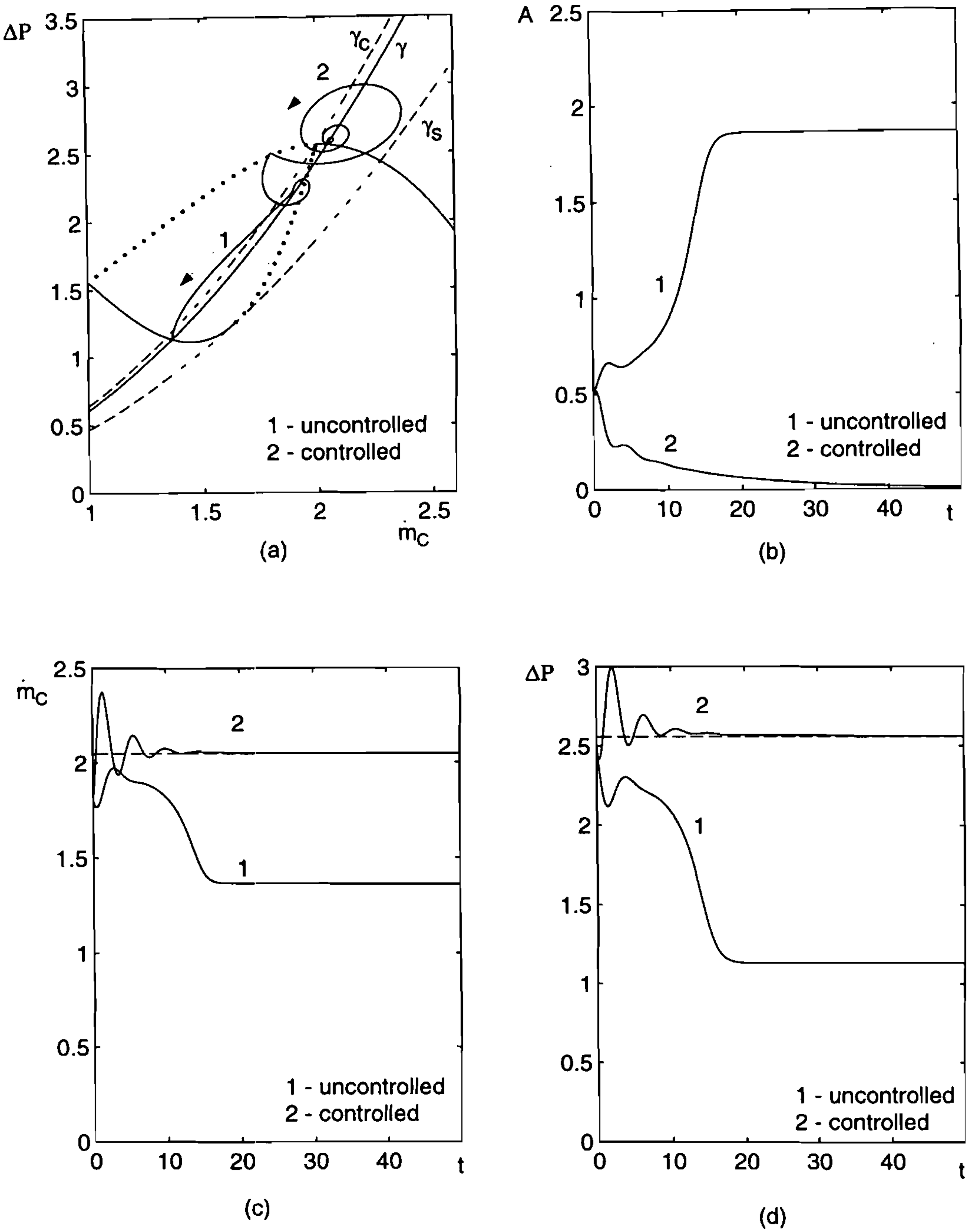

Figure 5. Time response of pre-stall behaviour with and without direct mass flow rate control.

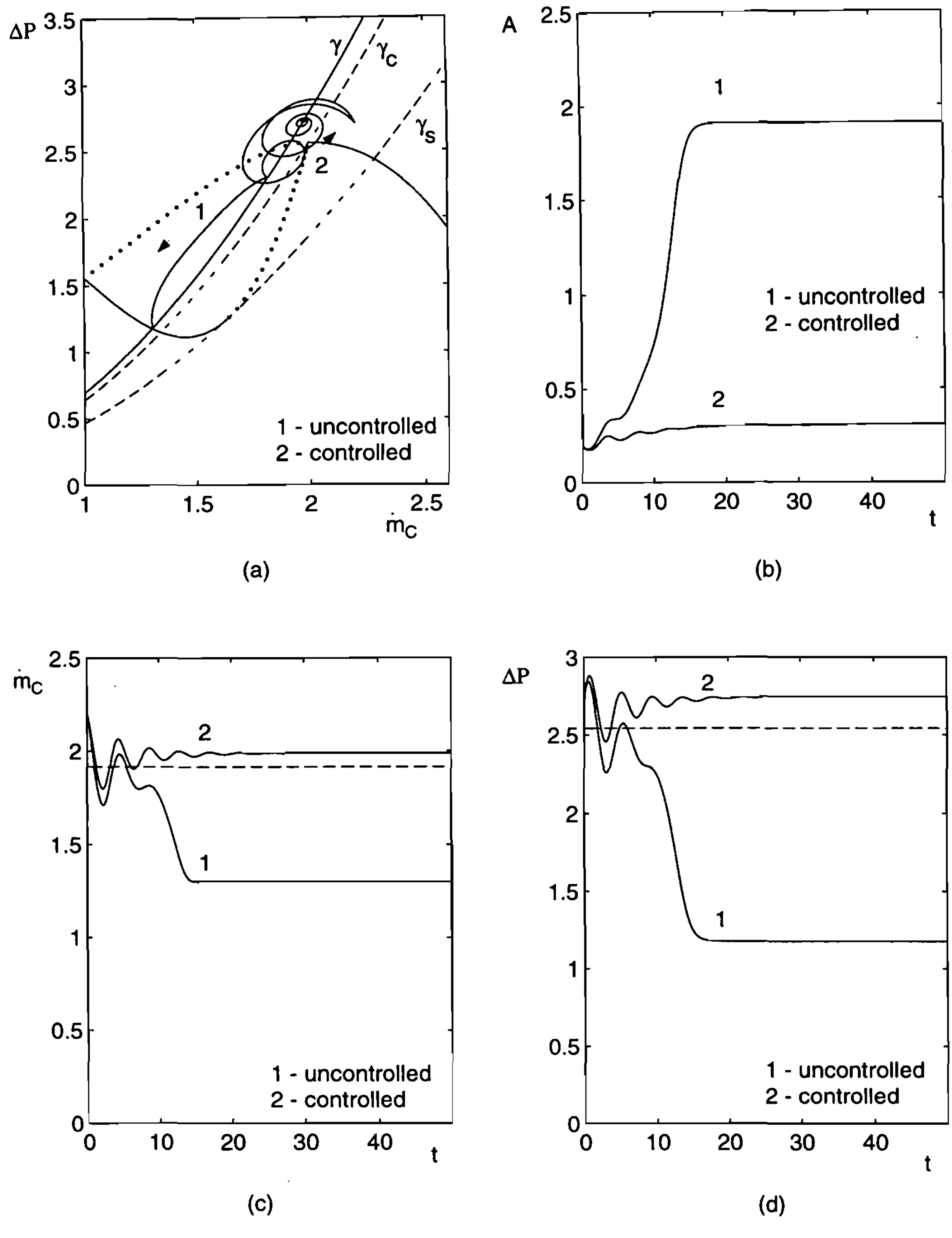

Figure 6. Time response of post-stall behaviour with and without direct mass flow rate control.

both pre-stall and post-stall behaviours via direct mass flow rate control. In both figures 5 and 6, trajectories 1 and 2 denote the timing responses of compression system without and with control, respectively. As discussed in Liaw and Abed's (1996) work, the unstalled equilibrium points (i.e. $A = 0$) after the stall inception point, will become unstable since $C'_{ss}(\dot{m}^0_C) > 0$. Moreover, the system is found to be uncontrollable via direct mass flow rate control. Thus, the timing response of post-stall behaviour will eventually go to some close-by stalled operating point (i.e. $A \neq 0$) even after adding control as depicted in figure 6. However, there is a big difference between the uncontrolled behaviour and controlled behaviour. That is, as depicted in figure 7, there will be no sudden pressure rise drop for the controlled system as those found in the bifurcation diagram of the

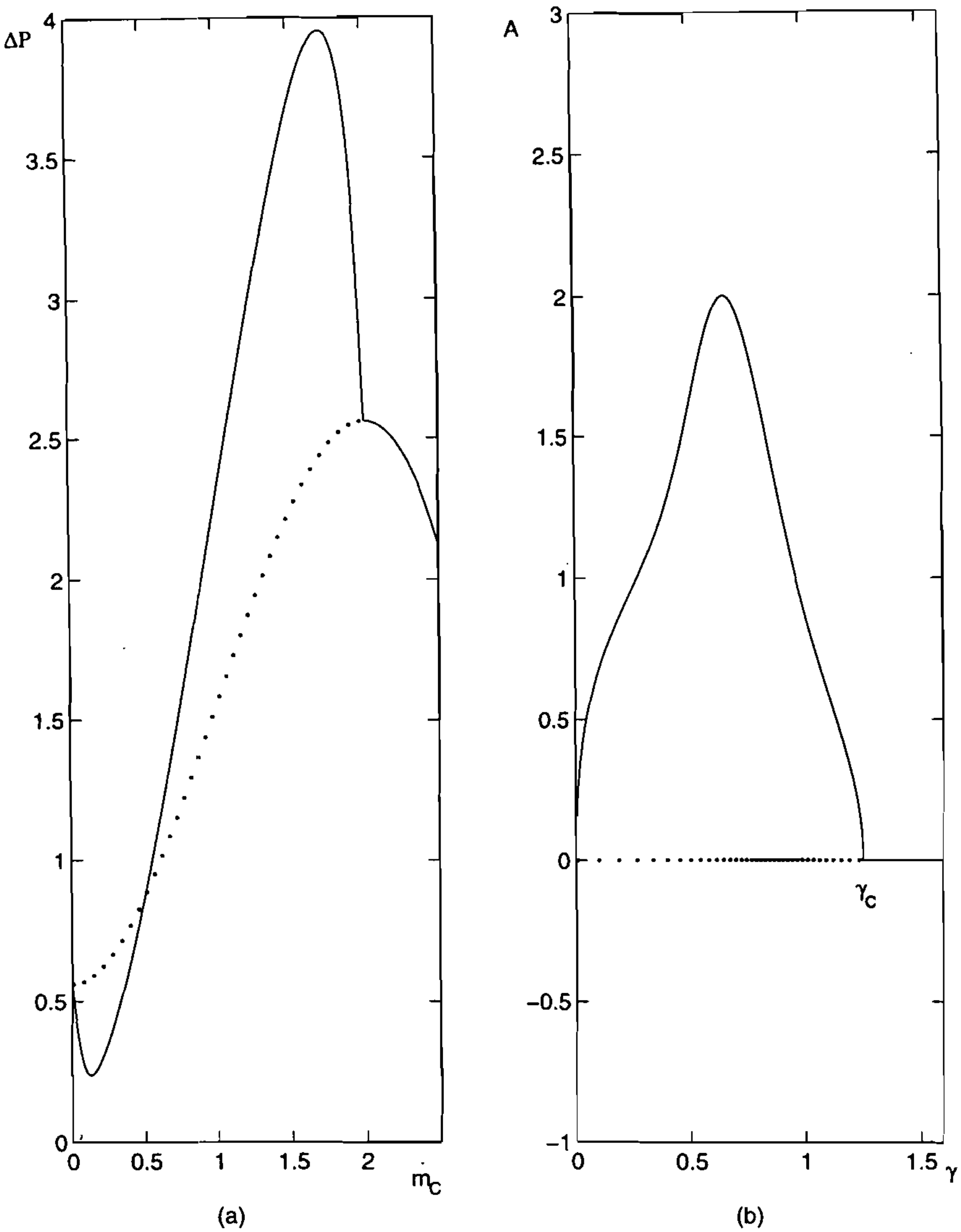

Figure 7. Bifurcation diagram of compressor dynamics with direct mass flow rate control.

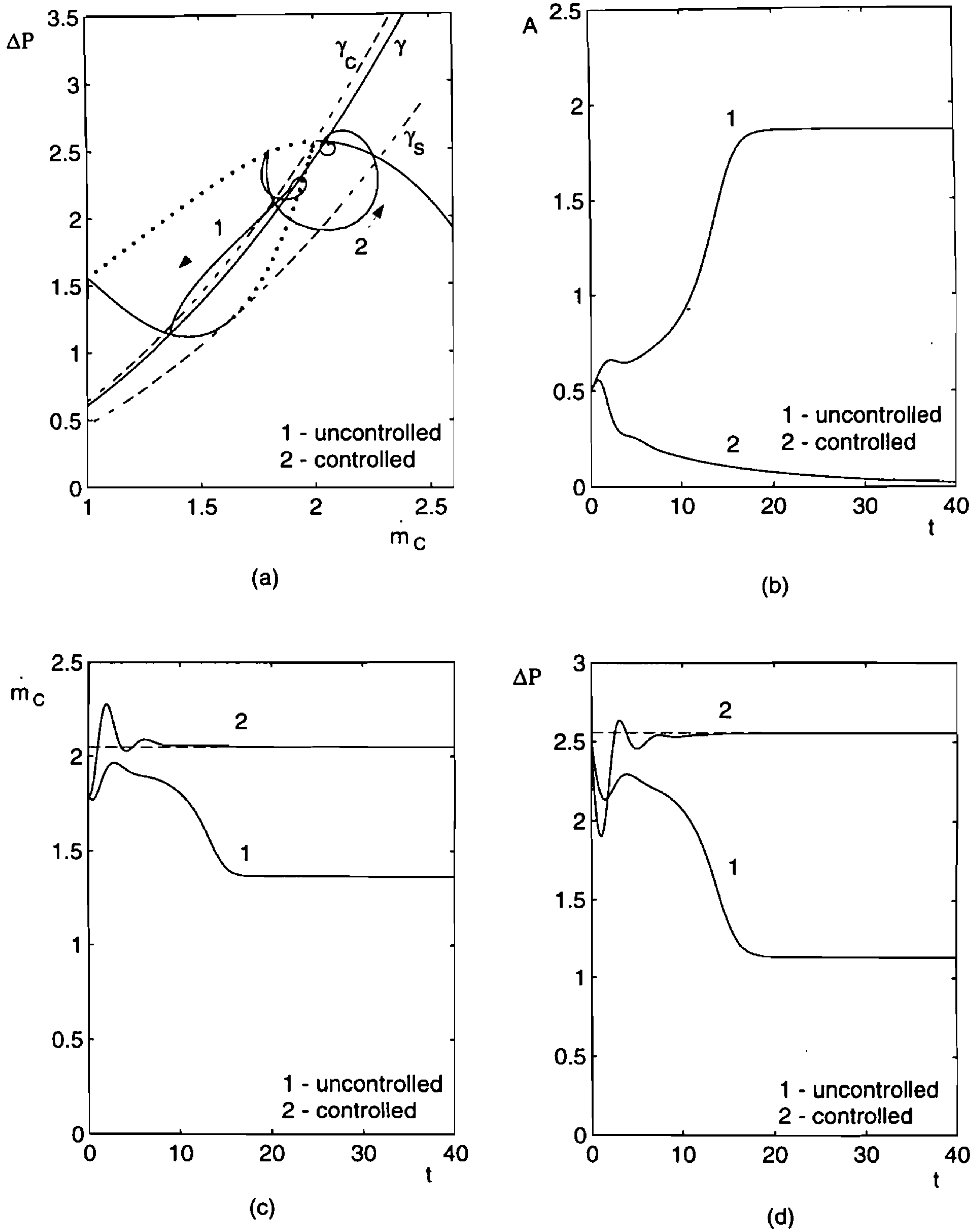

Figure 8. Time response of pre-stall behaviour with and without throttle control.

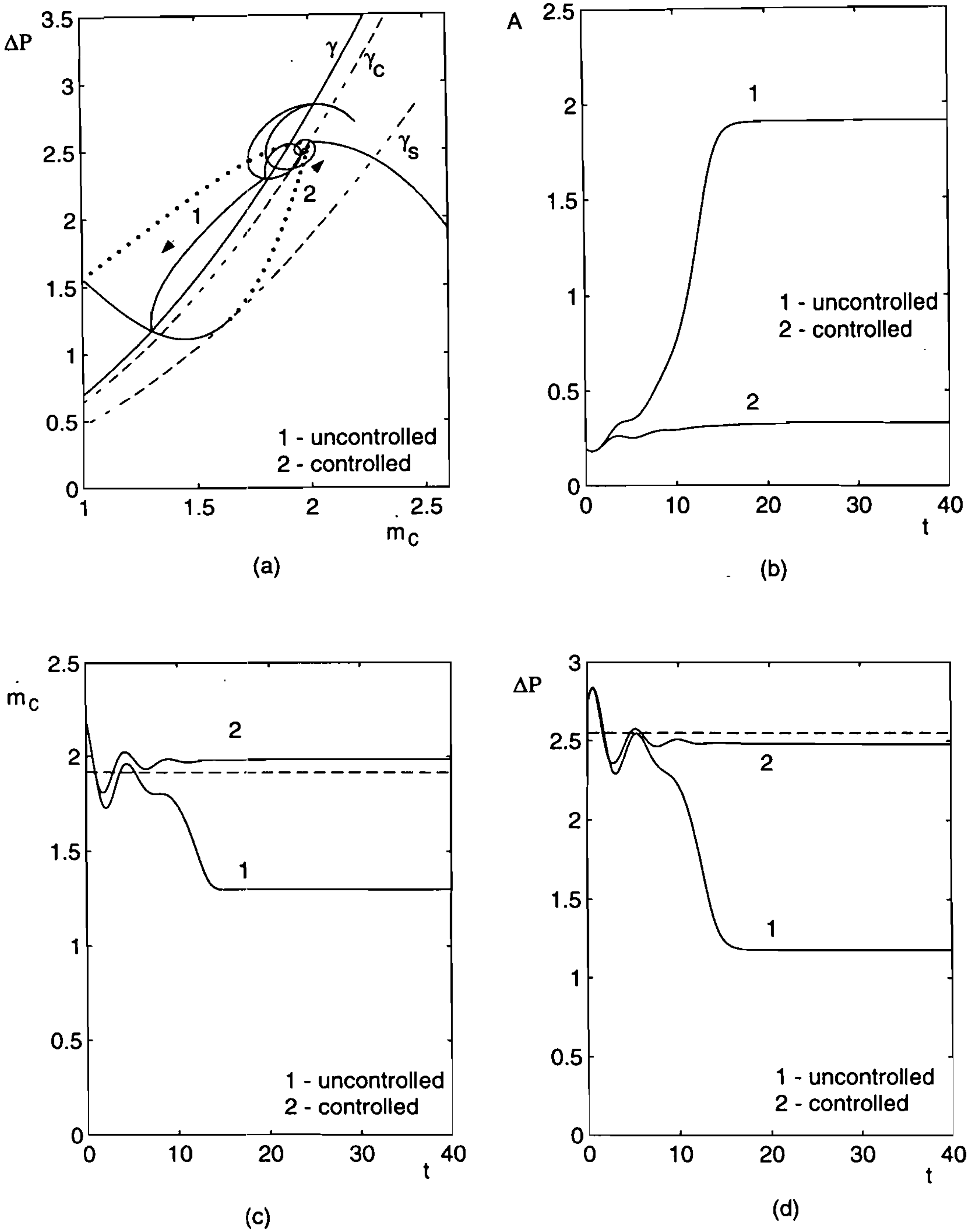

Figure 9. Time response of post-stall behaviour with and without throttle control.

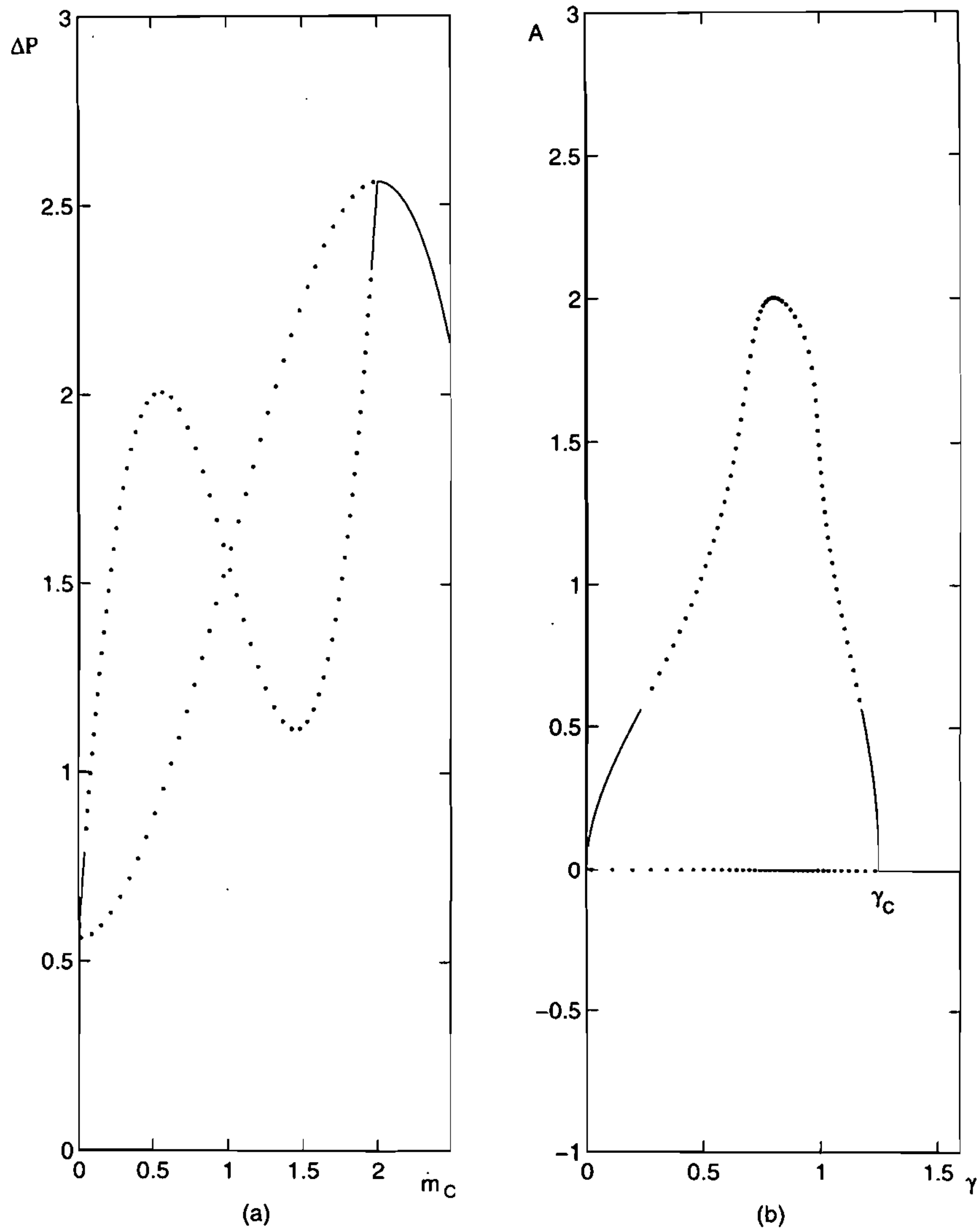

Figure 10. Bifurcation diagram of compressor dynamics with throttle control.

uncontrolled system as depicted in figure 4. In addition, it is observed from figure 7 that all the stalled equilibria are found to be asymptotically stable, which becomes a by-product of the design. As shown in figure 7 (*a*), the value of the pressure rise of the controlled operating point after stall inception point is much higher than that of the pre-stall equilibria. That means lots of control efforts might be consumed through direct mass flow rate control input. This is impractical for real implementation. As discussed in section 2, the goals of this study are only to eliminate the jump behaviour from the unstalled operating point and to provide the global stability of the unstalled equilibria before the stall inception point. The dynamical behaviour of post-stall operation, which is not close to the stall inception point, is not of interest in this paper. Thus, the proposed direct mass flow rate control scheme is recommended for the global stabilization of unstalled equilibria but not stalled ones.

Case 2. *Throttle control*

Next, we consider employing a throttle control scheme to globally stabilize the unstalled operating points of system (37)–(39) with $\dot{m}_C^0 \geq 2$. Choose

$$x_{3d} = -1.5\alpha x_1^2[x_2 + 2(\dot{m}_C^0 - 1)] - 0.75W^2x_1^2(\dot{m}_C^0 + x_2 - 1) - 1.5(\dot{m}_C^0 - 1)x_2^2. \quad (46)$$

We then have $V_2(x) = V_1(x_2)$, where $V_1(x_2)$ is determined in (43) above. Thus, we have the tracking error

$$\begin{aligned}\xi &= x_3 - x_{3d}\\ &= x_3 + 1.5\alpha x_1^2[x_2 + 2(\dot{m}_C^0 - 1)] + 0.75W^2x_1^2(\dot{m}_C^0 + x_2 - 1)\\ &\quad + 1.5(\dot{m}_C^0 - 1)x_2^2 \end{aligned} \quad (47)$$

and the applied throttle control input u_2 can be selected as defined in equation (35) by substituting x_{3d} and ξ with those given in (46) and (47), respectively.

Numerical results for pre-stall and post-stall behaviour of the throttle controlled system are shown in figures 8 and 9 to demonstrate the effectiveness of the stabilization design. The static settings of throttle control parameter γ^0 are chosen as 1.2 and 1.281 for figures 8 and 9, respectively, which are the same as those discussed above. A bifurcation diagram of compressor dynamics with throttle control is obtained in figure 10. It shows that there does not exist multiple system equilibria for all $\gamma \geq \gamma_c$. In addition, there is no jumping behaviour for the stable system equilibria which are close to the stall inception point.

5. Conclusion

In this paper, we have proposed control schemes for the global stabilization of unstalled compressor dynamics. It is demonstrated by the numerical example that not only the global stabilization of unstalled compressor dynamics has been successfully achieved, but also the phenomena of abrupt change of pressure rise of the stalled behaviour at the operating point near the stall inception point has been eliminated. Though our design of using throttle control is the same as that proposed by Krstić *et al.* (1995b), however, there are two main differences between two studies. Firstly, we focus on the stabilization design of the unstalled equilibrium branch but not stalled ones as discussed in (Krstić *et al.* 1995b). Secondly, we study the general stabilization problem for compressor dynamics satisfying 'left-tilt' property but not for a specific compression system only.

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