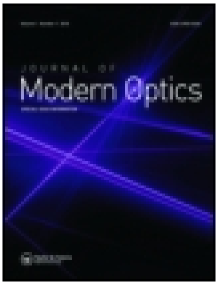

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Chao-Hsien Chen [a], Shin-Gwo Shiue [a] & Mao-Hong Lu [a]

[a] Institute of Electro-Optical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, 30050, Taiwan

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Method of solving triplets consisting of a singlet and air-spaced doublet with given primary aberrations

CHAO-HSIEN CHEN, SHIN-GWO SHIUE and
MAO-HONG LU

Institute of Electro-Optical Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 30050, Taiwan

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Abstract. An effective algebraic algorithm is proposed as a computational tool for solving the thin-lens structure of a triplet which consists of a singlet and an air-spaced doublet. The triplet is required to yield specified amounts of lens power and four primary aberrations: spherical aberration, coma, longitudinal chromatic aberration and secondary spectrum. In addition, the air spacing is used to control the zonal spherical aberration and spherochromatism. The problem is solved in the following manner. First, the equations for power and chromatic aberration are combined into a quartic polynomial equation if the object is at a finite distance, or combined into a quadratic polynomial equation if the object is at infinity. The roots give the element powers. Second, the lens shapes are obtained by solving the quartic polynomial equation which is obtained by combining the equations of spherical aberration and coma. Since quartic and quadratic equations can be solved using simple algebraic methods, the algorithm is rapid and guarantees that all the lens forms can be found.

1. Introduction

Optical decision problems in which the system will be designed from primary aberration theory normally begin with two stages. First, the optical thin-lens layout and the aberration targets of each optical component are decided. Thus the lens power and aberration targets of each component, the separation between adjacent components, and the paraxial marginal and chief ray heights at each component are determined. Second, there follows a thin-lens design stage for each component. During this stage, a suitable type of lens is chosen for each component and then the glass materials and curvatures are found to meet the power and aberration targets. According to the possibility of satisfying the targets, the lens type may be a single element, doublet, triplet or even a complex assembly of many elements. These thin-lens components are used as the starting points for future thickening and optimization. The primary aberration targets of thin lenses where the stop is at the lens are normally three in number: the Siedel coefficients S_1, S_{2C} and C_L, representing spherical aberration, central coma and longitudinal chromatic aberration respectively. Central coma is defined as the coma value for the case of the stop located at the lens. There is no need for the targets of the other primary aberrations: coma S_2, astigmatism S_3, field curvature S_4, distortion S_5 and lateral chromatic aberration C_T, since they can be expressed as combinations of K, S_1, S_{2C} and C_L by using the well known stop-shift formulae of aberration theory.

Hence, systematic thin lens design methods for various lens types are useful tools and have been the research subject of several workers. For example, the thin-lens designs of various doublet types have been provided by Dreyfus *et al.* [1], Kingslake [2], Khan and Macdonald [3], Smith [4] and Chen *et al.* [5], among many others.

Triplets are also elementary lenses which are used in many optical systems. Conrady [2, 6] has given a study of cemented triplets which have four surfaces as degrees of freedom to meet the specified lens power K, primary spherical aberration S_1, central coma S_{2C}, and longitudinal chromatic aberration C_L. The triplets are produced from dividing cemented doublets and thus have the same glass type for the first and third elements (figure 1). Recently, we [7] have derived an algorithm to solve cemented triplets with three different glass types (figure 2). In general, these triplets can have a lower secondary spectrum than the triplets with two glass types. However, they still cannot control the amount of secondary spectrum precisely because of the discontinuous distribution of glasses, and lenses with more than four degrees of freedom are required for the task. The triplet

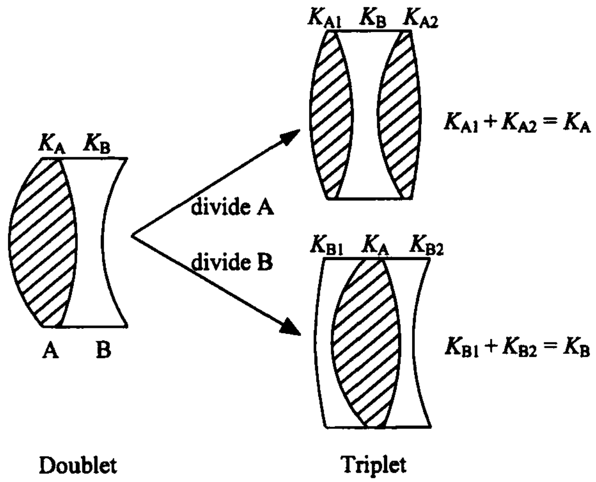


Figure 1. The cemented triplets with two glass types are obtained from dividing either the flint lens or the crown lens of the cemented doublets into two parts [2, 6]. The triplets have the same values of focal length and chromatic aberration with the doublet; they also produce the same amounts of secondary spectrum.

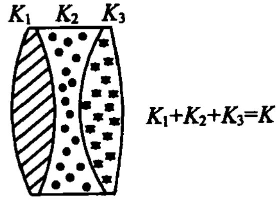


Figure 2. We have derived an algorithm to solve the cemented triplet with three different glass types [7]. This triplet is usually more effective to reduce the secondary spectrum than the triplet with two glass types.

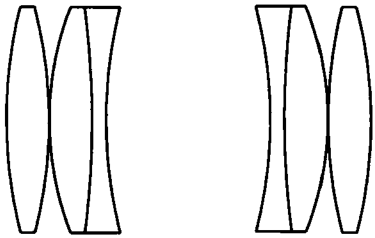


Figure 3. The schemes of the two thin-lens triplet types: singlet leading and doublet leading. These triplets have five surfaces as degrees of freedom to yield specified amounts of lens power, spherical aberration, central coma, chromatic aberration and secondary spectrum [8].

which consists of a singlet and a cemented doublet with the two lenses in contact has five surfaces as degrees of freedom to fit the purpose (figure 3). We have also derived an algorithm to solve this triplet type [8].

The chromatic variation of spherical aberration is often an important consideration in lens performance. Because of the residuals of zonal spherical aberration and spherochromatism, the triplets shown in figure 3 are usually limited to use at somewhat lower lens speeds. On the other hand, if the singlet and cemented doublet are separated by an air spacing, the extra variable may be used to reduce the zonal spherical aberration and spherochromatism, thus obtaining lenses with somewhat higher speeds. This has been discussed in detail elsewhere [2, 9].

In this paper, we propose an algorithm for solving the thin-lens solutions of a triplet which consists of a singlet and an air-spaced doublet. To control both longitudinal chromatic aberration and secondary spectrum, C_L is divided into C_{L1} and C_{L2} (defined in the following section). The triplet has five surfaces to meet five specified target values K, S_1, coma S_2, C_{L_1} and C_{L2}, and uses the air spacing d as an indirect parameter to control the spherical aberration curves which include not only primary but also higher-order aberrations. It is noteworthy that the comatic aberration is given by coma S_2 instead of central coma S_{2C}; this is required because S_{2C} cannot be properly defined for such an air-spaced lens unless d is zero.

From a mathematical point of view, the problem is to solve a set of five nonlinear simultaneous equations with five variables. However, the solving process can be simplified by two steps. First, the power of each element is obtained by solving the simultaneous equations of K, C_{L1} and C_{L2}. The three equations are combined into a quartic polynomial equation if the object is at a finite distance, or combined into a quadratic polynomial equation if the object is at infinity. The roots of the quartic or quadratic equation give the element powers. Second, the lens shapes are obtained by solving the quartic polynomial equation which is obtained by combining the equations of S_1 and S_2. Finally, the thin-lens solutions which meet the aperture requirement and has better spherical aberration curves are chosen as starting points for future thickening and optimization. Since the roots of quartic and quadratic equations can be solved by using simple algebraic methods [10], the algorithm is rapid and guarantees that all the lens structures can be found.

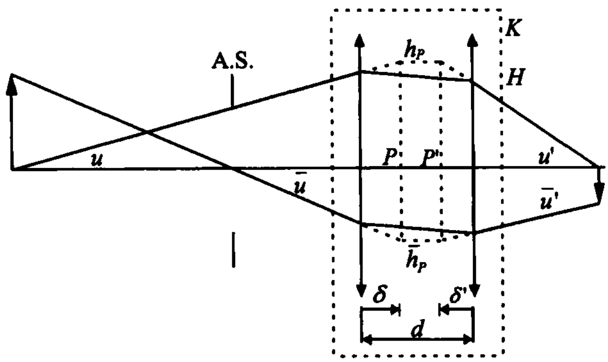


Figure 4. The notation for the Gaussian optics of the air-spaced thin-lens triplet.

2. Method

The first-order layout of the triplet is shown in figure 4. The notation is defined as follows.

(1) K is the power which is the inverse of the focal length.
(2) d is the spacing between the two lenses.
(3) P and P' are the first and second principal points respectively; they coincide if d is equal to zero.
(4) h_P and $\bar{h}_P$ are the marginal and principal ray heights at the principal planes, being positive if they are above the axis.
(5) u, $\bar{u}$, u' and $\bar{u}'$ are the paraxial angles of the marginal and princpal rays in the object and image spaces respectively. A ray angle is defined as positive if a clockwise rotation of the ray brings it parallel to the optical axis.
(6) $H = \bar{h}_P u - h_P \bar{u}$ is the optical invariant.
(7) δ (δ') is the distance from the front (rear) lens to the front (rear) principal surface; the sign is positive if the disance is to the right of the front (rear) lens.

Since the principal planes are at neither the singlet nor the doublet, it is important to keep h_P and $\bar{h}_P$ unchanged; otherwise the first-order layout of the whole system will be destroyed.

Two triplet types, singlet leading and doublet leading, are shown in figure 5. Because the algorithms of solving both types are similar, we only propose the one for the singlet-leading type. If c denotes the surface curvature and n is the refractive index, then $K_i = (n_i - 1)(c_i - c_{i+1})$ is the power of element i. In addition, we define the following.

(1) $E_i = \bar{h}_i/h_i$ is the eccentricity factor for surface i.
(2) $E_P = \bar{h}_P/h_P$ is the eccentricity factor for principal planes,
(3) $A_i = n_{i-1}(h_i c_i + u_{i-1}) = n_i(h_i c_i + u_i)$ is the refraction invariant for surface i.

In the case when $d = 0$, the triplet, which has the central coma value $S_{2C} = S_2 - (\bar{h}_P/h_P)S_1$, can be solved by using the method described in [8].

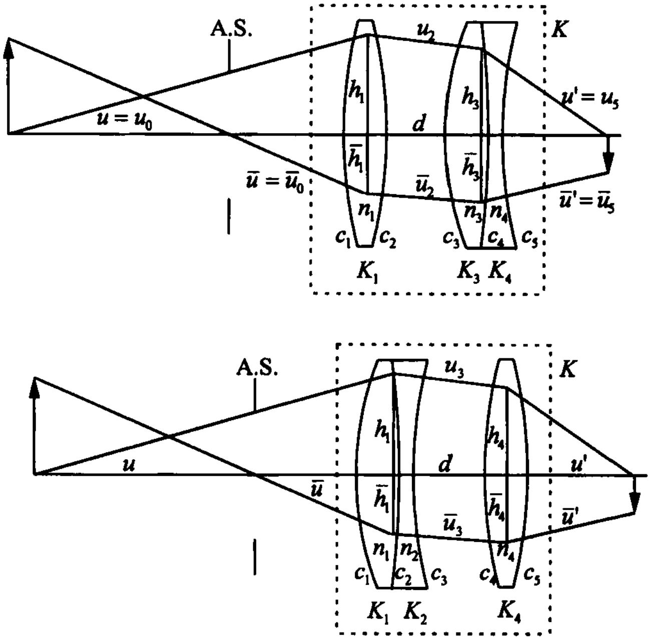


Figure 5. The schemes of the two types, singlet leading and doublet leading, of air-spaced thin-lens triplets. Since the algorithms of solving both types are similar, we only propose that for solving the singlet-leading type in this paper.

2.1. *Determining the element powers by solving the formulae for K, C_{L1} and C_{L2}*

Let λ_S, λ_M and λ_L be the short, middle and long wavelengths respectively over the band of interest. n_S, n_M and n_L are the associated refractive indices. The well known Seidel formula for C_L of a thin singlet in air is given by

$$C_L = \frac{h^2 K}{V}, \tag{1}$$

where

$$V = \frac{n_M - 1}{n_S - n_L} \tag{2}$$

denotes the Abbe number. If C_L is equal to zero, the foci of λ_S and λ_L are coincident, but they usually do not coincide with that of λ_M. To control all three foci, define C_{L1} and C_{L2} by analogy to C_L as follows:

$$C_{L1} = \sum \frac{h^2 K}{V_{SM}}, \qquad C_{L2} = \sum \frac{h^2 K}{V_{ML}}, \tag{3}$$

where V_{SM} and V_{ML} are defined analogously to V as

$$V_{SM} = \frac{n_M - 1}{n_S - n_M}, \qquad V_{ML} = \frac{n_M - 1}{n_M - n_L}. \tag{4}$$

C_{L1} and C_{L2} indicate the departures of the foci between λ_S and λ_M and between λ_M and λ_L respectively. It can be seen that C_L is equal to the sum of C_{L1} and C_{L2}. If C_{L1} and C_{L2} are both zero, the foci of all the three wavelengths are coincident and the triplet is apochromatic.

The formulae for K, C_{L1} and C_{L2} of the triplet are [11]

$$K_1 + (K_3 + K_4) - dK_1(K_3 + K_4) = K, \tag{5}$$

$$h_1^2 \frac{K_1}{(V_{SM})_1} + h_3^2\left(\frac{K_3}{(V_{SM})_3} + \frac{K_4}{(V_{SM})_4}\right) = C_{L1}, \tag{6}$$

$$h_1^2 \frac{K_1}{(V_{ML})_1} + h_3^2\left(\frac{K_3}{(V_{ML})_3} + \frac{K_4}{(V_{ML})_4}\right) = C_{L2}. \tag{7}$$

We wish to solve $K_{1,3,4}$ from these equations; therefore h_1 and h_3 have to be expressed as functions of $K_{1,3,4}$. It is usual to normalize the lens such that the corresponding normalized lens has unity power and unity h_P [3, 12]. To achieve this, we define the following dimensionless normalized parameters:

$$\begin{aligned}
\tilde{H} &= \frac{H}{h_P^2 K}, \\
\tilde{K}_{1,3,4} &= \frac{1}{K} K_{1,3,4}, \\
\tilde{d} &= Kd, \\
\tilde{c}_i &= \frac{c_i}{K}, i = 1, \ldots, 5, \\
(\tilde{u}_1, \tilde{\bar{u}}_i) &= \frac{1}{h_P K}(u_i, \bar{u}_i), i = 0, \ldots, 5, \\
(\tilde{h}_i, \tilde{\bar{h}}_i) &= \frac{1}{h_P}(h_i, \bar{h}_i), i = 1, \ldots, 5, \\
(\tilde{\delta}, \tilde{\delta}') &= (\delta, \delta')K, \\
\tilde{S}_1 &= \frac{1}{h_P^4 K^3} S_1, \\
\tilde{S}_2 &= \frac{1}{H h_P^2 K^2} S_2, \\
(\tilde{C}_{L1}, \tilde{C}_{L2}) &= \frac{1}{h_P^2 K}(C_{L1}, C_{L2}).
\end{aligned} \tag{8}$$

Consider

$$\begin{aligned}
\tilde{\delta} &= (\tilde{K}_3 + \tilde{K}_4)\tilde{d}, \\
\tilde{\delta}' &= -\tilde{K}_1 \tilde{d};
\end{aligned} \tag{9}$$

then $\tilde{h}_{1,3}$ and $\tilde{\bar{h}}_{1,3}$ can be expressed as functions of $\tilde{K}_{1,3,4}$ as follows:

$$\tilde{h}_1 = \frac{h_P - \delta u}{h_P} = 1 - (\tilde{K}_3 + \tilde{K}_4)\tilde{d}\tilde{u},$$

$$\tilde{h}_3 = \frac{h_P - \delta' u'}{h_P} = 1 + \tilde{K}_1 \tilde{d}\tilde{u}', \tag{10}$$

$$\tilde{\tilde{h}}_1 = E_P - (\tilde{K}_3 + \tilde{K}_4)\tilde{d}\tilde{\tilde{u}},$$

$$\tilde{\tilde{h}}_3 = E_P + \tilde{K}_1 \tilde{d}\tilde{\tilde{u}}'.$$

It is apparent that the eccentricity factors E_1 and E_3 are also equal to $\tilde{\tilde{h}}_1/\tilde{h}_1$ and $\tilde{\tilde{h}}_3/\tilde{h}_3$, respectively. Equations (1–3) can now be rewritten as

$$\tilde{K}_1 + (\tilde{K}_3 + \tilde{K}_4) - \tilde{d}\tilde{K}_1(\tilde{K}_3 + \tilde{K}_4) = 1, \tag{11}$$

$$[1 - \tilde{d}\tilde{u}(\tilde{K}_3 + \tilde{K}_4)]^2 \frac{\tilde{K}_1}{(V_{SM})_1} + (1 + \tilde{d}\tilde{u}'\tilde{K}_1)^2 \left(\frac{\tilde{K}_3}{(V_{SM})_3} + \frac{\tilde{K}_4}{(V_{SM})_4} \right) = \tilde{C}_{L1}, \tag{12}$$

$$[1 - \tilde{d}\tilde{u}(\tilde{K}_3 + \tilde{K}_4)]^2 \frac{\tilde{K}_1}{(V_{ML})_1} + (1 + \tilde{d}\tilde{u}'\tilde{K}_1)^2 \left(\frac{\tilde{K}_3}{(V_{ML})_3} + \frac{\tilde{K}_4}{(V_{ML})_4} \right) = \tilde{C}_{L2}. \tag{13}$$

These dimensionless normalized formulae are independent of K and h_P. By equation (11), we have

$$\tilde{K}_3 + \tilde{K}_4 = \frac{1 - \tilde{K}_1}{1 - \tilde{d}\tilde{K}_1}. \tag{14}$$

After substituting the above equation into equations (12) and (13), we get

$$\frac{1}{(V_{SM})_3}\tilde{K}_3 + \frac{1}{(V_{SM})_4}\tilde{K}_4 = Q_1, \tag{15}$$

$$\frac{1}{(V_{ML})_3}\tilde{K}_3 + \frac{1}{(V_{ML})_4}\tilde{K}_4 = Q_2, \tag{16}$$

where Q_1 and Q_2 are functions of $\tilde{K}_1$:

$$Q_1 = \frac{\tilde{C}_{L1} - [1 - \tilde{d}\tilde{u}(1 - \tilde{K}_1)/(1 - \tilde{d}\tilde{K}_1)]^2 \tilde{K}_1/(V_{SM})_1}{(1 + \tilde{d}\tilde{u}'\tilde{K}_1)^2},$$

$$Q_2 = \frac{\tilde{C}_{L2} - [1 - \tilde{d}\tilde{u}(1 - \tilde{K}_1)/(1 - \tilde{d}\tilde{K}_1)]^2 \tilde{K}_1/(V_{ML})_1}{(1 + \tilde{d}\tilde{u}'\tilde{K}_1)^2}. \tag{17}$$

Then, we can solve $\tilde{K}_3$ and $\tilde{K}_4$ as functions of $\tilde{K}_1$ as follows:

$$\tilde{K}_3 = \frac{-Q_2/(V_{SM})_4 + Q_1/(V_{ML})_4}{[1/(V_{SM})_3][1/(V_{ML})_4] - [1/(V_{ML})_3][1/(V_{SM})_4]},$$

$$\tilde{K}_4 = \frac{Q_2/(V_{SM})_3 + Q_1/(V_{ML})_3}{[1/(V_{SM})_3][1/(V_{ML})_4] - [1/(V_{ML})_3][1/(V_{SM})_4]}, \tag{18}$$

and hence

$$\tilde{K}_3 + \tilde{K}_4 = V_a Q_1 + V_b Q_2, \tag{19}$$

where

$$V_{\rm a} = \frac{1/(V_{\rm ML})_4 - 1/(V_{\rm ML})_3}{[1/(V_{\rm SM})_3][1/(V_{\rm ML})_4] - [1/(V_{\rm ML})_3][1/(V_{\rm SM})_4]},$$
$$V_{\rm b} = \frac{1/(V_{\rm SM})_3 - 1/(V_{\rm SM})_4}{[1/(V_{\rm SM})_3][1/(V_{\rm ML})_4] - [1/(V_{\rm ML})_3][1/(V_{\rm SM})_4]}. \tag{20}$$

Substituting equation (19) into equation (14) and after some mathematical manipulation, we can get the following quartic equation for $\tilde{K}_1$:

$$T_4\tilde{K}_1^4 + T_3\tilde{K}_1^3 + T_2\tilde{K}_1^2 + T_1\tilde{K}_1 + T_0 = 0, \tag{21}$$

where

$$\begin{aligned}
T_4 &= -\tilde{u}'^2\tilde{d}^3,\\
T_3 &= [\tilde{u}'(1+\tilde{d}) - 2]\tilde{u}'\tilde{d}^2 + \left(\frac{V_{\rm a}}{(V_{\rm SM})_1} + \frac{V_{\rm b}}{(V_{\rm ML})_1}\right)W_1,\\
T_2 &= (2\tilde{u}' - 1)\tilde{d} + \tilde{u}'\tilde{d}^2(2-\tilde{u}') + V_{\rm a}\left(W_3 + \frac{W_2}{(V_{\rm SM})_1}\right) + V_{\rm b}\left(W_6 + \frac{W_2}{(V_{\rm ML})_1}\right),\\
T_1 &= 1 + \tilde{d} - 2\tilde{d}\tilde{u}' + V_{\rm a}\left(W_5 + \frac{W_4}{(V_{\rm SM})_1}\right) + V_{\rm b}\left(W_7 + \frac{W_4}{(V_{\rm ML})_1}\right),\\
T_0 &= V_{\rm a}\tilde{C}_{\rm L1} + V_{\rm b}\tilde{C}_{\rm L2} - 1,
\end{aligned} \tag{22}$$

with

$$\begin{aligned}
W_1 &= -(\tilde{u}' - 1)^2\tilde{d}^2,\\
W_2 &= 2(1-\tilde{u})(1-\tilde{d}\tilde{u})\tilde{d},\\
W_3 &= \tilde{C}_{\rm L1}\tilde{d}^2,\\
W_4 &= -(1-\tilde{d}\tilde{u})^2,\\
W_5 &= -2\tilde{C}_{\rm L1}\tilde{d},\\
W_6 &= \tilde{C}_{\rm L2}\tilde{d}^2,\\
W_7 &= -2\tilde{C}_{\rm L2}\tilde{d}.
\end{aligned} \tag{23}$$

Once the values of coefficients T_0 to T_4 are calculated, the values of $\tilde{K}_1$ are obtained by solving the quartic equation (21). There may have at most four roots. For each root of $\tilde{K}_1$, the corresponding values of $\tilde{K}_{3,4}$, $\tilde{h}_{1,3}$, $\bar{\tilde{h}}_{1,3}$, $\tilde{\delta}$ and $\tilde{\delta}'$ are obscured from equations (18), (10) and (9), the eccentricity factors $E_{1,3}$ are obtained from $\bar{\tilde{h}}_{1,3}/\tilde{h}_{1,3}$, and the value of $\tilde{u}_2$ can also be obtained from

$$\tilde{u}_2 = \tilde{u} - \tilde{h}_1\tilde{K}_1. \tag{24}$$

Hence, the power distribution and the ray paths outside the lenses are obtained.

If the object is at infinity, $\tilde{u} = 0$ and $\tilde{u}' = -1$, Q_1 and Q_2 are reduced as follows:

$$Q_1 = \frac{\tilde{C}_{\rm L1} - \tilde{K}_1/(V_{\rm SM})_1}{(1-\tilde{d}\tilde{K}_1)^2}, \tag{25}$$

$$Q_2 = \frac{\tilde{C}_{\rm L2} - \tilde{K}_1/(V_{\rm ML})_1}{(1-\tilde{d}\tilde{K}_1)^2}. \tag{26}$$

From equations (14) and (19), it is easy to obtain the following quadratic equation in $\tilde{K}_1$:

$$\tilde{d}\tilde{K}_1^2 + \left(\frac{V_a}{(V_{SM})_1} + \frac{V_b}{(V_{ML})_1} - \tilde{d} - 1\right)\tilde{K}_1 + (1 - V_a\tilde{C}_{L1} - V_b\tilde{C}_{L2}) = 0, \quad (27)$$

for which there are at most two real roots. The next step is to find the lens shapes.

2.2. *Determining the curvatures by solving formulae for S_1 and S_2*

The spherical aberration and central coma of the front singlet are given by [11]

$$(S_1)_S = \frac{h_1^4 K_1^3}{4}[M_0(n_1)X_1^2 - M_1(n_1)X_1 Y_1 + M_2(n_1)Y_1^2 + M_3(n_1)], \quad (28)$$

$$(S_{2C})_S = \frac{-Hh_1^2 K_1^2}{2}[M_4(n_1)X_1 - M_5(n_1)Y_1] \quad (29)$$

respectively, where the shape factor X_1, the conjugate factor Y_1 and the functions $M_0(n)$ to $M_5(n)$ are defined as

$$\begin{aligned} X_1 &= \frac{c_1 + c_2}{c_1 - c_2} = \frac{\tilde{c}_1 + \tilde{c}_2}{\tilde{c}_1 - \tilde{c}_2}, \\ Y_1 &= \frac{u_2 + u}{u_2 - u} = \frac{\tilde{u}_2 + \tilde{u}}{\tilde{u}_2 - \tilde{u}}, \\ M_0(n) &= \frac{n+2}{n(n-1)^2}, \\ M_1(n) &= \frac{4(n+1)}{n(n-1)}, \\ M_2(n) &= \frac{3n+2}{n}, \\ M_3(n) &= \left(\frac{n}{n-1}\right)^2, \\ M_4(n) &= \frac{n+1}{n(n-1)}, \\ M_5(n) &= \frac{2n+1}{n}. \end{aligned} \quad (30)$$

The subscripts S in equations (28) and (29) denotes singlet. The normalized forms of $(S_1)_S$ and $(S_2)_S$ are

$$(\tilde{S}_1)_S = \frac{(S_1)_S}{h_P^4 K^3} = g_1 X_1^2 + g_2 X_1 + g_3, \quad (31)$$

$$\begin{aligned} (\tilde{S}_2)_S &= \frac{(S_{2C})_S + E_1(S_1)_S}{Hh_P^2 K^2} \\ &= \frac{E_1}{\tilde{H}}g_1 X_1^2 + \left(g_4 + \frac{E_1}{\tilde{H}}g_2\right)X_1 + g_5 + \frac{E_1}{\tilde{H}}g_3, \end{aligned} \quad (32)$$

where the coefficients g_1 to g_5 are as follows:

$$
\begin{aligned}
g_1 &= \frac{\tilde{h}_1^4 \tilde{K}_1^3}{4} M_0(n_1), \\
g_2 &= -\frac{\tilde{h}_1^4 \tilde{K}_1^3}{4} M_1(n_1) Y_1, \\
g_3 &= \frac{\tilde{h}_1^4 \tilde{K}_1^3}{4} [M_2(n_1) Y_1^2 + M_3(n_1)], \\
g_4 &= -\frac{\tilde{h}_1^2 \tilde{K}_1^2}{2} M_4(n_1), \\
g_5 &= \frac{\tilde{h}_1^2 \tilde{K}_1^2}{2} M_5(n_1) Y_1.
\end{aligned} \tag{33}
$$

The Hopkins–Venketeswara Rao [12] expressions for the spherical aberration and central coma of a cemented doublet are

$$
\begin{aligned}
\frac{(S_1)_\mathrm{D}}{h_3^4 K_\mathrm{D}^3} = {} & \left(N_0(n_3) \frac{1+\rho}{2} + N_0(n_4) \frac{1-\rho}{2} + 1 \right) \alpha^2 \\
& + \left[N_1(n_3) \left(\frac{1+\rho}{2} \right)^2 - N_1(n_4) \left(\frac{1-\rho}{2} \right)^2 + (\rho - Y_\mathrm{D}) \right] \alpha \\
& + \left[N_2(n_3) \left(\frac{1+\rho}{2} \right)^3 + N_2(n_4) \left(\frac{1-\rho}{2} \right)^3 \right. \\
& \left. - N_3(n_3) \left(\frac{1+\rho}{2} \right)^2 \frac{Y_\mathrm{D} - 1}{2} + N_3(n_4) \left(\frac{1-\rho}{2} \right)^2 \frac{Y_\mathrm{D} + 1}{2} \right],
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\frac{(S_{2\mathrm{C}})_\mathrm{D}}{H h_3^2 K_\mathrm{D}^2} = {} & \left(N_4(n_3) \frac{1+\rho}{2} + N_4(n_4) \frac{1-\rho}{2} + 1 \right) \alpha \\
& + \left[N_5(n_3) \left(\frac{1+\rho}{2} \right)^2 - N_5(n_4) \left(\frac{1-\rho}{2} \right)^2 + \frac{\rho - Y_\mathrm{D}}{2} \right]
\end{aligned} \tag{35}
$$

respectively, where the functions $N_0(n)$ to $N_5(n)$ are defined as

$$
\begin{aligned}
N_0(n) &= \frac{2}{n}, \\
N_1(n) &= \frac{3}{n-1}, \\
N_2(n) &= \frac{n}{(n-1)^2}, \\
N_3(n) &= \frac{n}{n-1}, \\
N_4(n) &= \frac{1}{n}, \\
N_5(n) &= \frac{1}{n-1}
\end{aligned} \tag{36}
$$

and

$$K_D = K_3 + K_4, \tilde{K}_D = \frac{K_D}{K} = \tilde{K}_3 + \tilde{K}_4$$
$$\rho = \frac{K_3 - K_4}{K_D} = \frac{\tilde{K}_3 - \tilde{K}_4}{\tilde{K}_D},$$
$$Y_D = \frac{u' + u_2}{u' - u_2} = \frac{\tilde{u}' + \tilde{u}_2}{\tilde{u}' - \tilde{u}_2}, \tag{37}$$
$$\alpha = \frac{1}{h_3 K_D} A_4 = \frac{1}{\tilde{h}_3 \tilde{K}_D} \tilde{A}_4.$$

The subscript D denotes doublet. K_D is the power of the doublet, ρ describes the power distribution and

$$\tilde{A}_4 = \frac{1}{h_P K} A_4 = n_3(\tilde{h}_3 \tilde{c}_4 + \tilde{u}_3). \tag{38}$$

The normalized forms of $(S_1)_D$ and $(S_2)_D$ are

$$(\tilde{S}_1)_D = \frac{(S_1)_D}{h_P^4 K^3} = g_6 \tilde{A}_4^2 + g_7 \tilde{A}_4 + g_8, \tag{39}$$

$$(\tilde{S}_2)_D = \frac{(S_{2C})_D + E_3 (S_1)_D}{H h_P^2 K^2}$$
$$= \frac{E_3}{\tilde{H}} g_6 \tilde{A}_4^2 + \left(g_9 + \frac{E_3}{\tilde{H}} g_7\right) \tilde{A}_4 + g_{10} + \frac{E_3}{\tilde{H}} g_8 \tag{40}$$

respectively, where the coeffficients g_6 to g_{10} are

$$g_6 = \tilde{h}_3^2 \tilde{K}_D \left(N_0(n_3) \frac{1+\rho}{2} + N_0(n_4) \frac{1-\rho}{2} + 1 \right),$$
$$g_7 = \tilde{h}_3^3 \tilde{K}_D^2 \left[N_1(n_3) \left(\frac{1+\rho}{2}\right)^2 - N_1(n_4) \left(\frac{1-\rho}{2}\right)^2 + (\rho - Y_D) \right],$$
$$g_8 = \tilde{h}_3^4 \tilde{K}_D^3 \left[N_2(n_3) \left(\frac{1+\rho}{2}\right)^3 + N_2(n_4) \left(\frac{1-\rho}{2}\right)^3 \right.$$
$$\left. - N_3(n_3) \left(\frac{1+\rho}{2}\right)^2 \frac{Y_D - 1}{2} + N_3(n_4) \left(\frac{1-\rho}{2}\right)^2 \frac{Y_D + 1}{2} \right], \tag{41}$$
$$g_9 = -\tilde{h}_3 \tilde{K}_D \left(N_4(n_3) \frac{1+\rho}{2} + N_4(n_4) \frac{1-\rho}{2} + 1 \right),$$
$$g_{10} = -\tilde{h}_3^2 \tilde{K}_D^2 \left[N_5(n_3) \left(\frac{1+\rho}{2}\right)^2 - N_5(n_3) \left(\frac{1-\rho}{2}\right)^2 + \frac{\rho - Y_D}{2} \right].$$

From

$$S_1 = (\tilde{S}_1)_S + (\tilde{S}_1)_D,$$
$$\tilde{S}_2 = (\tilde{S}_2)_S + (\tilde{S}_2)_D, \tag{42}$$

we have the following two second-order equations in X_1 and $\tilde{A}_4$:

$$g_1 X_1^2 + g_2 X_1 + g_6 \tilde{A}_4^2 + g_7 \tilde{A}_4 + (g_3 + g_8 - \tilde{S}_1) = 0, \tag{43}$$

$$\left(\frac{E_1}{\tilde{H}} g_1\right) X_1^2 + \left(g_4 + \frac{E_1}{\tilde{H}} g_2\right) X_1 + \left(\frac{E_3}{\tilde{H}} g_6\right) \tilde{A}_4^2 + \left(g_9 + \frac{E_3}{\tilde{H}} g_7\right) \tilde{A}_4 + \left(g_5 + \frac{E_1}{\tilde{H}} g_3 + g_{10} + \frac{E_3}{\tilde{H}} g_8 - \tilde{S}_2\right) = 0. \tag{44}$$

by reducing the above simultaneous equations into a quartic equation (see appendix A), the values of X_1 and $\tilde{A}_4$ can be solved. Consider

$$\tilde{A}_3 = \tilde{h}_3 \tilde{c}_3 + \tilde{u}_2; \tag{45}$$

then we can have the value of $\tilde{A}_3$ from $\tilde{A}_4$ using

$$\tilde{A}_3 = \tilde{A}_4 + \frac{n_3}{n_3 - 1} \tilde{h}_3 \tilde{K}_3. \tag{46}$$

Finally, it is straightforward to obtain $(\tilde{c}_1, \tilde{c}_2)$ from $(X_1, \tilde{K}_1)$ and $(\tilde{c}_3, \tilde{c}_4, \tilde{c}_5)$ from $(\tilde{A}_3, \tilde{K}_3, \tilde{K}_4)$ as follows:

$$\begin{aligned} \tilde{c}_1 &= \frac{(X_1 + 1)\tilde{K}_1}{2(n_1 - 1)}, \\ \tilde{c}_2 &= \frac{(X_1 - 1)\tilde{K}_1}{2(n_1 - 1)}, \\ \tilde{c}_3 &= \frac{\tilde{A}_3 - \tilde{u}_2}{\tilde{h}_3}, \\ \tilde{c}_4 &= \tilde{c}_3 - \frac{\tilde{K}_3}{n_3 - 1}, \\ \tilde{c}_5 &= \tilde{c}_4 - \frac{\tilde{K}_4}{n_4 - 1}. \end{aligned} \tag{47}$$

Thus, the surface curvatures are solved. Finally, the thin-lens solution which meets the aperture requirement and has better spherical aberration curves is chosen as the starting point for future thickening and optimization.

3. Example

As an example to demonstrate the calculation process of the method in detail, we solve the thin-lens structures of a telescope objective system which has a focal length of 300 mm, a relative aperture of $F/3$, a field of view of $\pm 3°$, and an image height of 15.72 mm (figure 6). The object is infinite. The optical invariant $H = -2.618$ mm. The wavelengths over the band of interest are the hydrogen F line (0.4861 μm), the helium d line (0.5876 μm), and the hydrogen C line (0.6563 μm). The glasses PK51A, LAK31 and FK54 are used for the triplet, and the glass BAK1 is used for the prism. These glasses are chosen from the Schott catalogue; the explicit values for all glass parameters are listed in table 1. The desired primary aberrations for the whole system are of zero spherical aberration and coma, and correction of both longitudinal chromatic aberration and secondary spectrum.

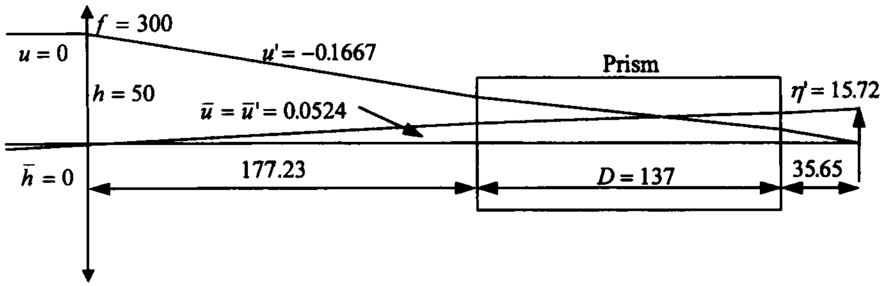


Figure 6. Gaussian design of the telescope objective system. The aberrations S_1, S_2, C_{L1} and C_{L2} produced by the prism are going to be balanced by the triplet to make the whole system both aplanatic and apochromatic.

Table 1. Data for the chosen glasses.

	Triplet			
Parameter	Glass 1 (PK51A)	Glass 3 (LAK31)	Glass 4 (FK54)	Prism (BAK1)
n_M	1.528 55	1.696 73	1.437	1.572 5
n_S	1.533 33	1.705 31	1.440 34	1.579 44
n_L	1.526 46	1.692 96	1.435 52	1.569 48
V	76.941 5	56.396 9	90.656 1	57.526
V_{SM}	110.611	81.157 2	130.832	82.499 1
V_{ML}	252.77	184.853	295.264	190.038
$M_0(n)$	8.263 02	—	—	—
$M_1(n)$	12.518 8	—	—	—
$M_2(n)$	4.308 43	—	—	—
$M_3(n)$	8.363 42	—	—	—
$M_4(n)$	3.129 7	—	—	—
$M_5(n)$	2.654 21	—	—	—
$N_0(n)$	—	1.178 74	1.391 79	—
$N_1(n)$	—	4.305 84	6.865 01	—
$N_2(n)$	—	3.495 31	7.524 83	—
$N_3(n)$	—	2.435 28	3.288 34	—
$N_4(n)$	—	0.589 37	0.695 895	—
$N_5(n)$	—	1.435 28	2.288 34	—
V_a		—5 318.43		—
V_b		122 98.8		—

The prism induces the following primary aberrations [11]:

$$S_1 = -\frac{n^2-1}{n^3}Du'^4 = -\frac{1.5725^2-1}{1.5725^3}137(-0.1667)^4 = -0.040\,038,$$

$$S_2 = \frac{\bar{u}'}{u'}S_1 = \frac{0.0524}{-0.1667}S_1 = 0.012\,59,$$

$$C_{L1} = -\frac{D}{V_{SM}}\frac{n-1}{n_2}u'^2 = -\frac{137}{82.497}+\frac{1.5725-1}{1.5725^2}(-0.1667)^2 = -0.010\,684\,1,$$

$$C_{L2} = -\frac{D}{V_{ML}}\frac{n-1}{n^2}u'^2 = -\frac{137}{190.038}\frac{1.5725-1}{1.5725^2}(-0.1667)^2 = -0.004\,638\,2, \tag{48}$$

Table 2. Abberation targets of the triplet.

Aberration	Seidel coefficient	Wave front[a] (in units of λ_d)	Normalized Seidel coefficient
Spherical	$S_1 = 0.040\,038\,1$	$\frac{S_1}{8\lambda_d} = 8.5173$	$\tilde{S}_1 = \frac{S_1}{h^4K^3} = 0.172\,965$
Coma	$S_2 = -0.012\,59$	$\frac{S_2}{2\lambda_d} = -10.7131$	$\tilde{S}_2 = \frac{S_2}{Hh^2K^2} = 0.1731$
Secondary chromatic	$C_{L1} = 0.010\,684\,1$	$\frac{C_{L1}}{2\lambda_d} = 9.0913$	$\tilde{C}_{L1} = \frac{C_{L1}}{h^2K} = 0.001\,283$
	$C_{L2} = 0.004\,638\,2$	$\frac{C_{L2}}{2\lambda_d} = 3.946\,73$	$\tilde{C}_{L2} = \frac{C_{L2}}{h^2K} = 5.565\,84 \times 10^{-4}$

[a] $\lambda_d = 0.000\,587\,6\,\text{mm}$.

where $D = 137\,\text{mm}$ is the thickness of the prism. To keep the whole system aplanatic and apochromatic, the aberrations induced by the prism must be balanced by the front triplet. The aberration targets for the triplet are expressed in three forms in table 2. To show the ability of reducing higher-order aberrations by using the air spacing, comparison between a triplet with $d = 0$ and a triplet with a finite air spacing is given.

In the case of $d = 0$, the triplet has the form shown in figure 3 which is solved by using the method proposed in [8]. As a general rule, weak lens surfaces are more preferred than strong surfaces because the former induces fewer higher-order aberrations. For this reason, we choose only the solution which has the weakest lens surfaces. Layout, longitudinal spherical and transverse aberration curves, and spot diagrams of the thin-lens solution are shown in figure 7. It can be seen that the three foci are consistent, but the tops of the longitudinal spherical and transverse aberration curves are far from the ideal positions.

As a comparison, a second triplet which has a suitable air spacing is solved. It is hard to give a suitable air spacing directly because our algorithm is based on the primary aberrations but the aberration curves and spot diagrams include not only primary but also high-order aberrations. Hence, we solve several triplets with different air spacings and find that the triplet with an air spacing $d = 2.5\,\text{mm}$ ($\tilde{d} = 8.333 \times 10^{-3}$) has the best aberration performance. To enable a user to validate several intermediate steps in the calculation as well as the final result, the values for derived parameters are also given. The normalized Gaussian parameters are shown in table 3. From equation (26), we have the following quadratic equation in $\tilde{K}_1$:

$$8.333 \times 10^{-3}\tilde{K}_1^2 - 0.434\,862\tilde{K}_1 + 0.973\,424 = 0. \tag{49}$$

The equation has two real roots which are 49.8397 and 2.343 73. The surface radii induced by the first root are too small to meet the aperture requirement. From the second root, we have the values shown in table 4. The locations of the front and rear principal planes are $\delta = -3.426\,24$ and $\delta' = -5.859\,32$ respectively. Using these values and the method in appendix A, we have the following quartic equation in X_1:

$$-1.009\,23X_1^4 + 79.8157X_1^3 - 823.332X_1^2 + 1223.4X_1 + 192.602 = 0, \tag{50}$$

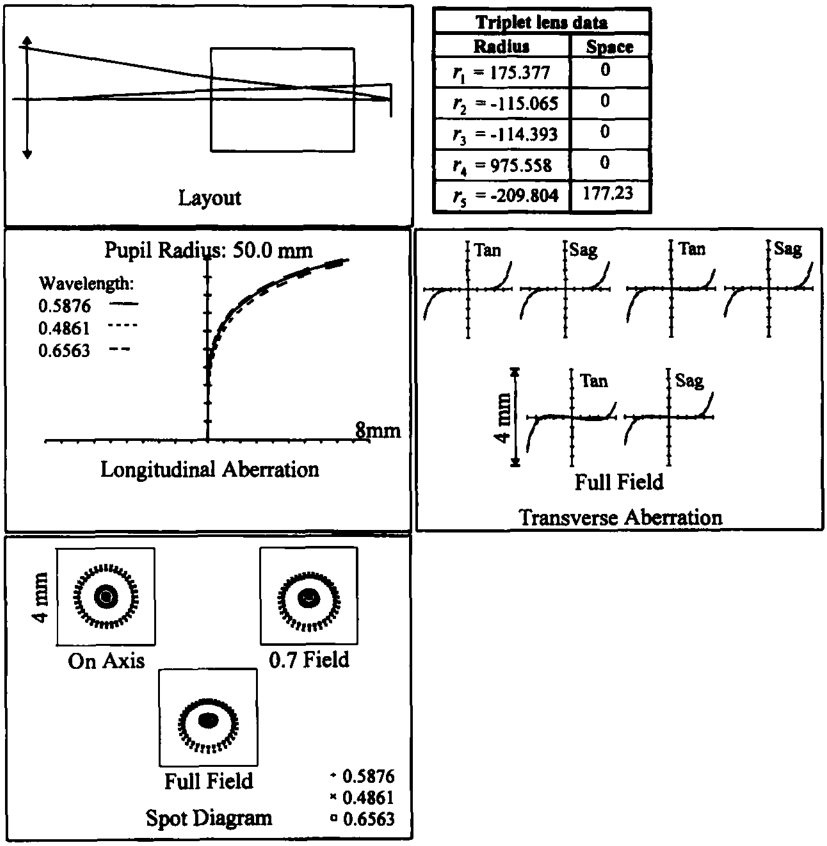


Figure 7. Layout, longitudinal spherical and transverse aberration curves, and spot diagram of the thin-lens telescope objective system in which the air spacing of the triplet is zero.

Table 3. Values of the normalized Gaussian parameters.

Normalized Gaussian parameters	Note
$\tilde{H} = -0.31416$ $\tilde{u} = 0, \tilde{u}' = -1$ $\bar{\tilde{u}} = 0.3142, \bar{\tilde{u}}' = 0.3142$	Equation (8)

which has four real roots. Table 5 shows the corresponding four solutions. Solutions 2, 3 and 4 are useless because their surface radii are too strong to meet the aperture requirement. Layout of solution 1 with its aberration is shown in figure 8. Compare figures 7 and 8; the latter has better aberration curves and smaller spots. This indicates that a suitable air spacing can be used to reduce aberrations.

Once the thin-lens systems are obtained, the next step is to find the corresponding thickened optical systems which have the same Gaussian designs and the primary aberrations. If the thin-lens system is not good enough, the aberrations of the corresponding thickened system may be changed by a significant factor. This

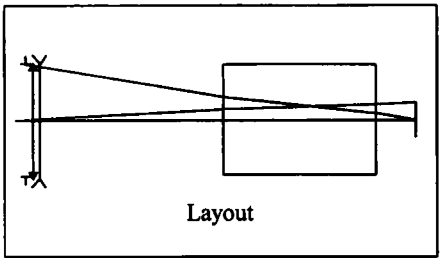
Table 4. Values obtained from $\tilde{K}_1 = 2.34373$.

Values	Note
$Q_1 = -0.020708, Q_2 = -0.009066$	Equations (25) and (26)
$\tilde{K} = -2.18731, \tilde{K}_4 = 0.816817$	Equation (18)
$\tilde{h}_1 = 1, \tilde{h}_3 = 0.980469,$	Equation (10)
$\tilde{\bar{h}}_1 = 0.00358796, \tilde{\bar{h}}_3 = 0.00613588,$	
$E_1 = 0.00358796, E_3 = 0.00625811$	
$\tilde{\delta} = -0.01142, \tilde{\delta}' = -0.01953$	Equation (9)
$\tilde{u} = -2.34373$	Equation (24)
$Y_1 = 1$	Equation (30)
$\tilde{K}_D = -1.3705, \rho = 2.192,$	Equation (37)
$Y_D = -2.48839$	
$g_1 = 26.5951, g_2 = -40.2926$	Equations (33) and (41)
$g_3 = 40.7852, g_4 = -8.59583$	
$g_5 = 7.28989, g_6 = -2.70316$	
$g_7 = 23.3858, g_8 = -53.6836$	
$g_9 = 2.05037, g_{10} = -9.35904$	

Table 5. Four thin-lens solutions.

Solution 1	Solution 2	Solution 3	Solution 4	Note
$X_1 = -0.1434$	$X_1 = 1.97218$	$X_1 = 10.039$	$X_1 = 67.218$	Roots of equation (50)
$\tilde{A}_4 = 0.29881$	$\tilde{A}_4 = 9.09487$	$\tilde{A}_4 = 33.5791$	$\tilde{A}_4 = -204.157$	Equation (A 3)
$\tilde{A}_3 = -4.92387$	$\tilde{A}_3 = 3.87218$	$\tilde{A}_3 = 28.3565$	$\tilde{A}_3 = -209.379$	Equation (46)
$\tilde{c}_1 = 1.89918$	$\tilde{c}_1 = 6.58969$	$\tilde{c}_1 = 24.4747$	$\tilde{c}_1 = 151.247$	Equation (47)
$\tilde{c}_2 = -2.53505$	$\tilde{c}_2 = 2.15545$	$\tilde{c}_2 = 20.0404$	$\tilde{c}_2 = 146.813$	
$\tilde{c}_3 = -2.63154$	$\tilde{c}_3 = 6.33974$	$\tilde{c}_3 = 31.3117$	$\tilde{c}_3 = -211.16$	
$\tilde{c}_4 = 0.507866$	$\tilde{c}_4 = 9.47914$	$\tilde{c}_4 = 34.4512$	$\tilde{c}_4 = -208.02$	
$\tilde{c}_5 = -1.36129$	$\tilde{c}_5 = 7.60999$	$\tilde{c}_5 = 32.582$	$\tilde{c}_5 = -209.89$	
$r_1 = 157.963$	$r_1 = 45.5257$	$r_1 = 12.2576$	$r_1 = 1.98351$	Surface radii
$r_2 = -118.341$	$r_2 = 139.182$	$r_2 = 14.9697$	$r_2 = 2.04341$	
$r_3 = -114.002$	$r_3 = 47.3206$	$r_3 = 9.58107$	$r_3 = -1.42073$	
$r_4 = 590.707$	$r_4 = 31.6484$	$r_4 = 8.70798$	$r_4 = -1.44217$	
$r_5 = -220.379$	$r_5 = 39.4219$	$r_5 = 9.20754$	$r_5 = -1.42932$	
Yes	No	No	No	Meet the aperture requirement

thickening procedure can be performed using the method of Hopkins and Venkateswara Rao [12] or using the optimization methods supported in commercial optical design software. Two suitable thickened systems corresponding to figures 7 and 8 are shown in figures 9 and 10 respectively. The aberration curves of the thickened systems shift only slightly from those of the original thin systems, and the aberrations in figure 10 are smaller than those in figure 9. The system in figure 10 is preferred to be used as a starting point for further full optimization.



Triplet lens data	
Radius	Space
A.S.	3.426
r_1 = 157.963	0
r_2 = -118.341	2.5
r_3 = -114.002	0
r_4 = 590.707	0
r_5 = -220.379	171.37

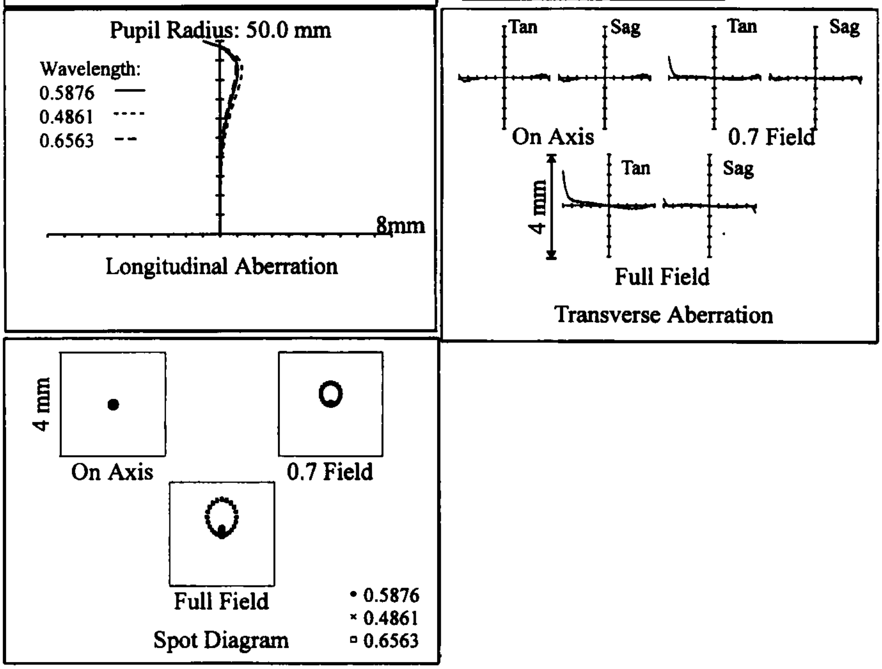
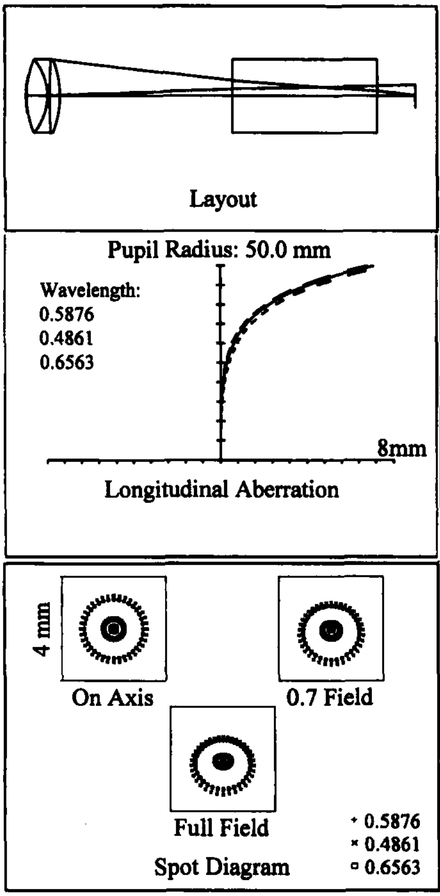


Figure 8. Layout, longitudinal and transverse aberration curves, and spot diagram of the thin-lens telescope objective system in which the air spacing of the triplet is 2.5 mm. The locations of the front and rear principal planes are $\delta = -3.426$ mm and $\delta' = -5.859$ mm respectively. To maintain the first order layout, the aperture stop is set on the first principal plane of the triplet, and the distance from the last surface of the triplet to the first plane of the prism is set as $177.23 + \delta' = 171.37$. It can be seen that the aberrations are smaller than in figure 7.

Air spacings cannot improve aberrations for every glass combination. For example, if we use the glasses Ohara BSL7, Schott PK51A and Ohara LAL59 for the triplet, then the triplet without an air spacing will induce the best aberrations. Figures 11 and 12 show the system with $d = 0$ and $d = 5$ mm respectively. As a conclusion, we can use several air spacings to solve the triplets and then find the best.

4. Discussion

The triplet consisting of a singlet and an air-spaced doublet is an important lens type which is widely used as a component in many optical systems, for



Triplet lens data	
Radius	Space
r_1 = 180.093	20
r_2 = -107.69	0
r_3 = -107.07	20
r_4 = 1067.85	10
r_5 = -190.592	162.871

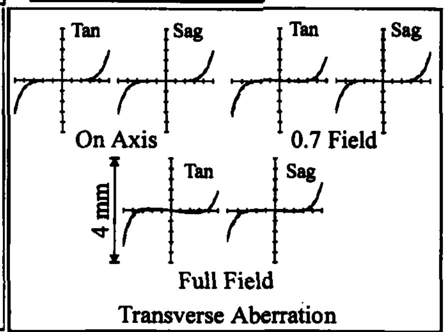
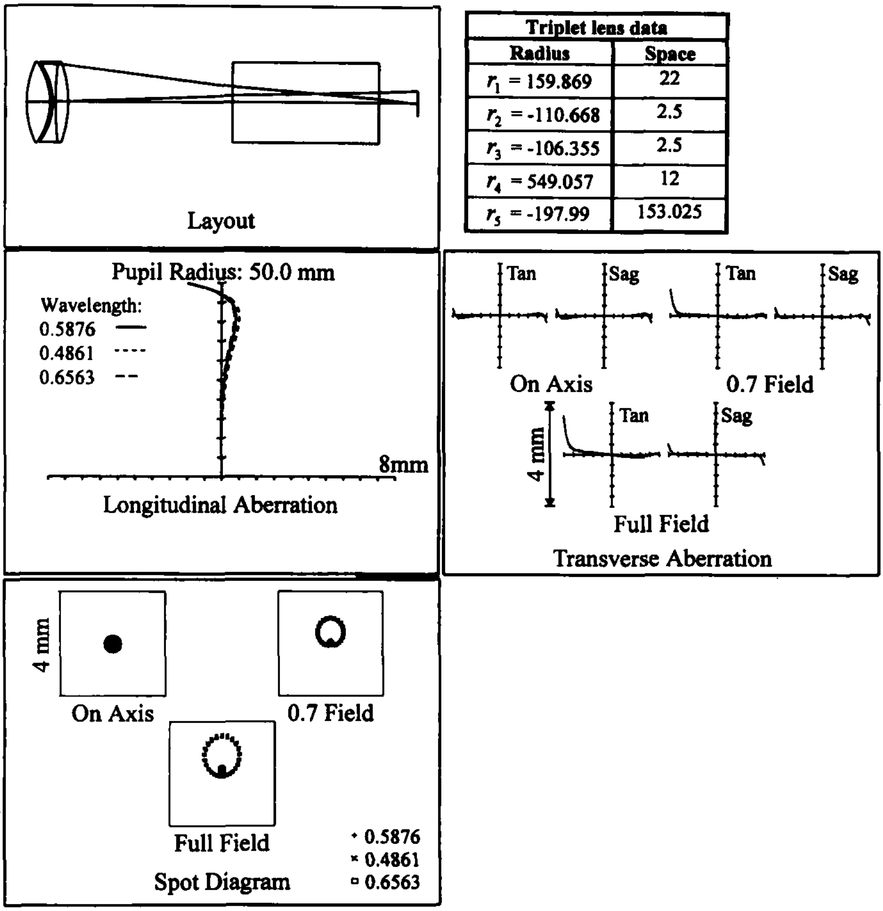


Figure 9. Layout, longitudinal and transverse aberration curves, and spot diagram of the thickened version corresponding to the system shown in figure 7. The aperture stop is located on the first surface. The aberrations do not change much.

example telescope objectives, telephoto lenses, zoom and Petzval lenses. We have proposed an algebraic algorithm to solve the lens with specified amounts of K, S_1, S_2, C_{L1} and C_{L2}. However, the complexity of derived coefficients prevents insights into the interactions of the parameters of the triplet; we therefore regard the algorithm as only a computational tool for the pre-design of such a lens.

Although we can directly solve the surface radii to meet the K, S_1, S_2, C_{L1} and C_{L2}, we cannot directly solve the air spacing d to meet the specified spherical aberration curves which include not only primary but also higher-order aberrations. However, the air spacing is not always a useful variable. For some glasses, triplets with air spacing may induce more aberrations than triplets without air spacing. Some iterations are needed to find a proper air spacing. The proposed algorithm is useful to investigate the aberration properties at the thin-lens design stage.

For a thin lens, the primary aberrations S_2, S_3, S_4 and S_5 can be expressed as combinations of K, S_1 and S_{2C} by using the well known stop-shift formulae of



Triplet lens data	
Radius	Space
r_1 = 159.869	22
r_2 = -110.668	2.5
r_3 = -106.355	2.5
r_4 = 549.057	12
r_5 = -197.99	153.025

Figure 10. Layout, longitudinal and transverse aberration curves, and spot diagram of the thickened version corresponding to the system shown in figure 8. The aperture stop is relocated on the first surface. The aberrations are better than those in figure 9.

aberration theory. Since the air-spaced triplet has an air spacing, we cannot define central coma S_{2C} for such a lens. Hence, it is noteworthy that, although each solution has the same value of K, S_1 and S_2, it may have different values of S_3, S_4 and S_5.

One might consider as an alternative method to give the pre-design of a triplet the use of intuitive guesswork. This may work well for an experienced designer but it is usually difficult for a beginner. The damped least-squares (DLS) method is a numerical method for solving nonlinear simultaneous equations. It can also be used to solve thin-lens design. We had used the DLS method for the task and usually suffered from the difficulty of giving a proper initial guess which is critical for the DLS method. As a comparison, the proposed algorithm is a simple algebraic process which is quick and guarantees to find all the solutions.

Triplet lens data	
Radius	Space
r_1 = 284.59	0
r_2 = 95.77	0
r_3 = 91.39	0
r_4 = -96.68	0
r_5 = -225.73	177.23

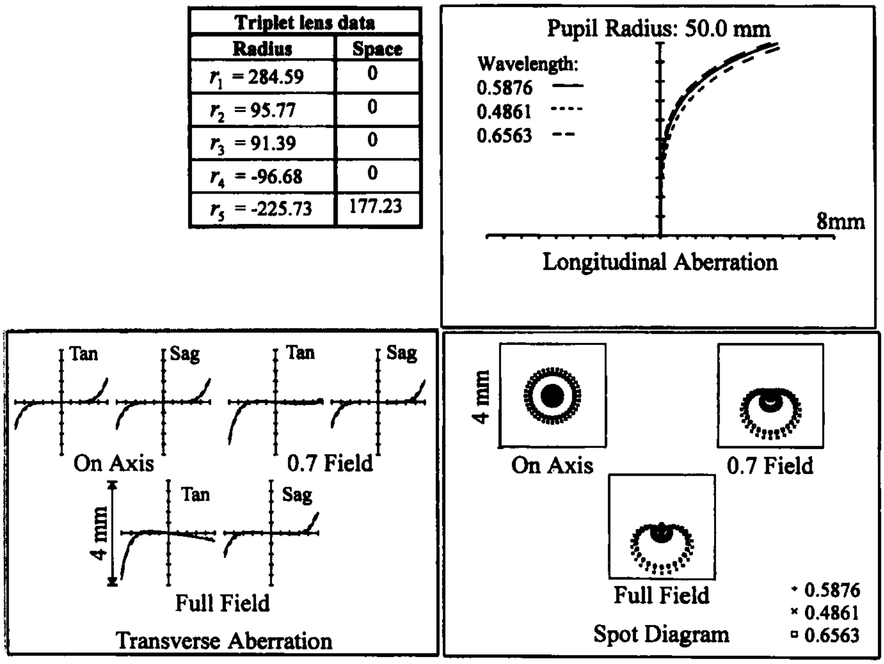


Figure 11. Longitudinal spherical and transverse aberration curves, and spot diagram of the thin-lens telescope objective system in which the glasses used for the triplet are Ohara BSL7, Schott PK15A and Ohara LAL59. The air spacing of the triplet is zero.

Triplet lens data	
Radius	Space
r_1 = 339.11	0
r_2 = 98.21	5
r_3 = 96.32	0
r_4 = -90.67	0
r_5 = -197.47	5.60
A.S.	177.23

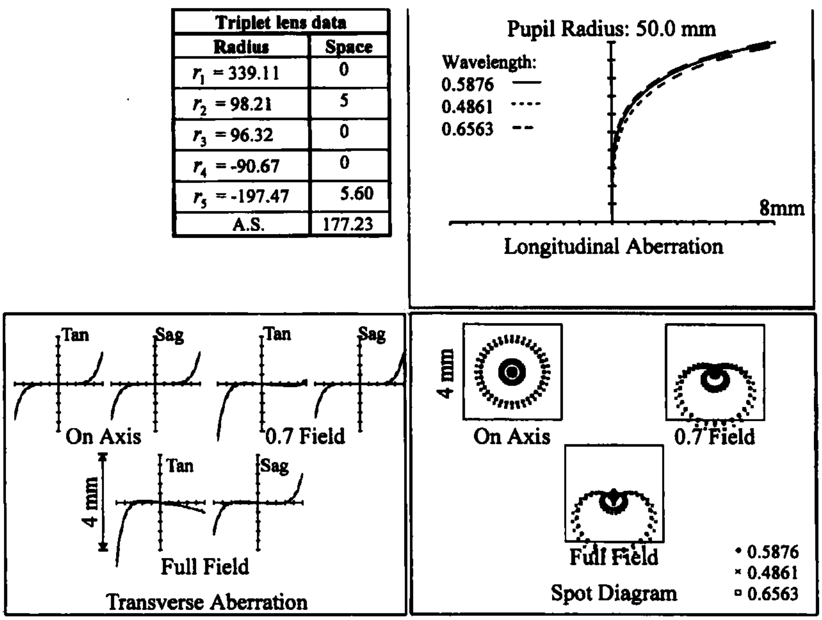


Figure 12. Longitudinal spherical and transverse aberration curves, and spot diagram of the thin-lens telescope objective system in which the air spacing of the triplet is 5 mm. In comparison with figure 11, this system induces more high-order aberrations.

Appendix A

Consider the following nonlinear simultaneous equations:

$$a_1X^2 + a_2X^2 + a_3Y^2 + a_4Y + a_5 = 0, \tag{A 1}$$

$$a_6X^2 + a_7X^2 + a_8Y^2 + a_9Y + a_{10} = 0, \tag{A 2}$$

where X and Y are the variables to be solved, and a_1 to a_{10} are constant coefficients. Using equation (A 1) multiplied by a_8 minus equation (A 2) multiplied by a_3 to eliminate the Y^2 term, we obtain

$$Y = \frac{p_2X^2 + p_1X + p_0}{p_3}, \tag{A 3}$$

where

$$\left.\begin{aligned} p_3 &= a_3a_9 - a_4a_8, \\ p_2 &= a_1a_8 - a_3a_6, \\ p_1 &= a_2a_8 - a_3a_7, \\ p_0 &= a_5a_8 - a_3a_{10}. \end{aligned}\right\} \tag{A 4}$$

After substituting equation (A 3) into equation (A 1), we obtain a quartic equation of X:

$$m_4X^4 + m_3X^3 + m_2X^2 + m_1X + m_0 = 0, \tag{A 5}$$

where the coefficients m_4 to m_0 are as follows:

$$\left.\begin{aligned} m_4 &= a_3p_2^2, \\ m_3 &= 2a_3p_1p_2, \\ m_2 &= a_3p_1^2 + 2a_3p_0p_2 + a_4p_2p_3 + a_1p_3^2, \\ m_1 &= 2a_3p_0p_1 + a_4p_1p_3 + a_2p_3^2, \\ m_0 &= a_3p_0^2 + a_4p_0p_3 + a_5p_3^2. \end{aligned}\right\} \tag{A 6}$$

The roots can be solved by the algebra described in [10]. Once the values of X are solved, the corresponding values of Y are then obtained from equation (A 3).

References

[1] Dreyfus, M. G., Bishop, R. E., and Moorhead, J. E., 1960, *J. opt. Soc. Am.*, **50,** 375.
[2] Kingslake, R., 1978, *Lens Design Fundamentals* (New York: Academic Press), p. 173.
[3] Iqbel Khan, M., and Macdonald, J., 1982, *Optica Acta*, **29,** 807.
[4] Smith, W. J., 1991, *Modern Optical Engineering* (New York: McGraw-Hill).
[5] Chen, C.-S., Shiue, S.-G., and Lu, M.-H., 1997, *Proc. Natln. Sci. Council ROC* A, **21,** 75.
[6] Conrady, A. E., 1960, *Applied Optics and Optical Design*, Part 2 (New York: Dover Publications), 554.
[7] Chen, C.-S., Shiue, S.-G., and Lu, M.-H., 1997, *J. mod. Optics*, **44,** 753.
[8] Chen, C.-S., Shiue, S.-G., and Lu, M.-H., 1997, *J. mod. Optics*, **44,** 1279.

[9] Smith, W. J., 1984, *Modern Lens Design* (New York: McGraw-Hill).
[10] Spiegel, M. R., 1968, *Mathematical Handbook* (New York: McGraw-Hill).
[11] Welford, W. T., 1974, *Aberrations of the Symnettrical Optical System* (London: Academic Press).
[12] Hopkins, H. H., and Venkateswara Rao, V., 1970, *Optica Acta*, **19**, 497.