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Magnetically Charged Black Holes In Induced Gravity

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It is shown that the regular SO_3 non-abelian spherically symmetric monopole solution does not exist in an induced Einstein-Yang-Mills-Higgs model. It is also shown that Higgs scalar hair cannot exist in the presence of a black hole event horizon in this model. The corresponding global behavior of the gauge field is analyzed in a black hole background. In particular, it is shown that a non-trivial non-abelian monopole solution exists only if the radius of the event horizon is smaller than the characteristic radius of the classical monopole.

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I. Introduction

There has been much research activity on the classical behavior of a non-abelian monopole in a static and spherically symmetric curved space [1,2,3,4]. Note that it was first shown by 't Hooft and Polyakov [5], motivated by the singular solution with a Dirac string in the abelian U_1 gauge theory [6], that a regular spherically symmetric monopole solution does exist in a non-abelian SO_3 gauge theory with a Higgs potential that exhibits spontaneously symmetry breaking (SSB).

It is also known that the monopole configuration exists only when the associated second homotopy group is nontrivial, i.e. $\Pi_2(\mathcal{M}) \neq 0$. Here $\mathcal{M} \equiv \mathcal{G}/\mathcal{H}$ denotes the Higgs vacuum configuration associated with the symmetry breaking pattern $\mathcal{G} \to \mathcal{H}$ [7]. There is, however, no general rule that can guarantee the regularity and stability of the classical monopole solution in a model independent version. As a result, there has been considerable interest in the study of the non-abelian monopole in various models of interest.

Note that the monopole solution has also been found for similar models in curved space-time [1,2,3]. Moreover, the physical behavior of a monopole solution in the presence of a black hole background has also been a focus of related studies [1,2,3,4]. It is known that the no-hair theorem is a widely-believed conjecture in black hole physics [8]. Evidence also indicates that only three kinds of physical quantities: the electric charge Q, the gravitational mass M and the angular momentum J, can be detected outside the event horizon of a black hole [9]. Hence a consistency check of the no-hair theorem for any model of interest is thus an important and routine program.

Induced gravity model has also been a focus of research interests following the introductory work by Zee [10]. It was shown that induced gravity may have important implications in inflationary universe [11]. Note that gravitational constant and cosmological *constant* are in fact dynamical variables in the induced gravity model [12]. Hence it was raised as a research problem to study the behavior of the Higgs scalar field in the presence of the classical black hole [2,4,13]. Existence and stability of monopole solution has also been studied in the induced Einstein-Yang-Mills-Higgs (EYMH) models [4,13].

In this paper we will present a more complete analysis of the existence and properties of non-abelian monopole in an SO_3 induced EYMH model. We will show that regular spherically symmetric monopole solution does not exist in this model. It will also be shown that no-hair theorem also applies to the Higgs scalar field in this model. There exists, however, black hole solution with non-trivial gauge configuration that differs from the well-known Reissner-Nordstrom solution under certain constraint, in particular $r_H > 1/ev$, on the coupling constants. Here r_H denotes the radius of the event horizon while $1/(ev)$ is the characteristic radius of the classical monopole configuration. Note that e will denote the gauge coupling constant. We will discuss these results in this paper.

This paper will be organized as follows: (i) in section 2, we will present a brief review on the derivation of the field equations; (ii) In section 3, we will show that regular spherically symmetric monopole solution does not exist in this model; (iii) a no-hair theorem and the global behavior of the magnetically charged black hole solution will be studied in section 4; (iv) some concluding remarks will be drawn in section V; (v) finally, in the appendix, we will verify the conservation of the generalized energy momentum tensor.

II. The action and field equations

The Einstein-Yang-Mills-Higgs model with real SO_3 triplet scalar field is given by the following action:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\epsilon\phi^2 R - \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - V(\phi^2)\right]. \tag{1}$$

Here R is the scalar curvature and ϵ denotes some dimensionless coupling constant of order one. Moreover, the Higgs scalar field ϕ^a, with $a,b,\cdots=1,2,3$ denoting the SO_3 gauge indices, is the real triplet scalar field. Note that the gauge covariant derivative $D_\mu\phi^a$ of ϕ^a and the curvature tensor $F^a_{\mu\nu}$ of the gauge field A^a_μ are

$$D_\mu\phi^a = \partial_\mu\phi^a + e\epsilon_{abc}A^b_\mu\phi^c, \tag{2}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon_{abc}A^b_\mu A^c_\nu, \tag{3}$$

respectively with e denoting the dimensionless gauge coupling constant. Furthermore, we will also assume the well-known Higgs potential $V(\phi^2) = \frac{\lambda}{8}(\phi^2 - v^2)^2$. Here $\phi^2 \equiv \phi^a\phi^a$ denotes the squared-norm of the scalar field. Note also that the action (1) exhibits global scale invariance in the limit $v = 0$ since all coupling constants are dimensionless (except v) by construction.

The Euler-Lagrange equations of motion can be obtained from varying action (1) with respect to $g_{\mu\nu}$, A^a_μ and ϕ^a. After some algebra, one has:

$$G_{\mu\nu} = \frac{1}{\phi^2}(D_\mu \partial_\nu \phi^2 - g_{\mu\nu} D^2 \phi^2) + \frac{1}{\epsilon\phi^2} T_{\mu\nu} \equiv \mathcal{T}_{\mu\nu}, \tag{4}$$

$$D^\mu F^a_{\mu\nu} = e\epsilon_{abc}\phi^b D_\nu \phi^c, \tag{5}$$

and

$$D_\mu D^\mu \phi^a - \epsilon R \phi^a = \frac{1}{\phi}\frac{\partial V(\phi)}{\partial \phi}\phi^a. \tag{6}$$

Here $G_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R - R_{\mu\nu}$ is the Einstein tensor while $T_{\mu\nu}$ is the energy momentum tensor associated with the scalar field ϕ and the gauge field A^a_μ given by the following expression:

$$T_{\mu\nu} = D_\mu \phi^a D_\nu \phi^a + F^a_{\mu\rho} - g_{\mu\nu}\left[\frac{1}{4} F^a_{\rho\sigma} F^{a\rho\sigma} + \frac{1}{2} D_\rho \phi^a D^\rho \phi^a + V(\phi)\right]. \tag{7}$$

Moreover, $\mathcal{T}_{\mu\nu}$ can be thought of as the 'generalized energy momentum tensor' which has to be divergentless, $D^\mu \mathcal{T}_{\mu\nu} = 0$, due to the Bianchi Identity $D^\mu G_{\mu\nu} = 0$. Proof for the conservation of the generalized energy momentum tensor will be given in the Appendix. For simplicity in notation, we have written both gauge covariant derivatives and tensorial covariant derivatives uniquely with the same notation as $D_\nu T^a_\mu$. Here T^a_μ denotes physical fields with gauge index a and some space-time tensor indices μ. To be more specific, $D_\alpha F^a_{\mu\nu} \equiv \partial_\alpha F^a_{\mu\nu} - \Gamma^\lambda_{\alpha\mu} F^a_{\lambda\nu} - \Gamma^\lambda_{\alpha\nu} F^a_{\mu\lambda} + \epsilon_{abc} A^b_\alpha F^c_{\mu\nu}$ is the correct covariant derivative of $F^a_{\mu\nu}$ which is covariant with the cooperation of the gauge connection A^a_μ and the affine connection $\Gamma^\lambda_{\alpha\nu}$.

Indeed, the correct details of the covariant derivatives can be read off directly from any field equations since all field equations are covariant by construction.

For latter convenience, we will write $\phi^a = \varphi\tilde{\phi}^a$ with $\tilde{\phi}^a$ denoting the unitnorm scalar field such that $\tilde{\phi}^a\tilde{\phi}^a = 1$. Note that $\tilde{\phi}^a$ is in fact the gauge-phase part of the Higgs field. Therefore, one has

$$\tilde{\phi}^a D_\mu \tilde{\phi}^a = 0, \tag{8}$$

$$D_\mu \tilde{\phi}^a D_\nu \tilde{\phi}^a + \tilde{\phi}^a D_\mu D_\nu \tilde{\phi}^a = 0 \tag{9}$$

from successively differentiating the identity $\tilde{\phi}^a\tilde{\phi}^a = 1$.

Hence one can write Eq. (6) as

$$D_\mu D^\mu \tilde{\phi}^a + \epsilon R \tilde{\phi}^a + \frac{1}{\varphi}\left[2\partial_\mu \varphi D^\mu \tilde{\phi}^a + D_\mu \partial^\mu \varphi \tilde{\phi}^a - \frac{\partial V(\varphi)}{\partial \varphi}\tilde{\phi}^a\right] = 0. \tag{10}$$

Moreover, one has

$$R = -\frac{1}{\epsilon}\left[D_\mu \tilde{\phi}^a D^\mu \tilde{\phi}^a - \frac{1}{\varphi}\left(D_\mu \partial^\mu \varphi + \frac{\partial V(\varphi)}{\partial \varphi}\right)\right] \tag{11}$$

from multiply Eq.(10) with $\tilde{\phi}^a$. Note that one has used the identity (9).

Note that the trace of Equation (4) gives:

$$-R = \left(6 + \frac{1}{\epsilon}\right) \frac{1}{\varphi^2}\partial_\mu\varphi\partial^\mu\varphi + \frac{6}{\varphi}D_\mu\partial^\mu\varphi + \frac{1}{\epsilon}\left[D_\mu\tilde{\phi}^a D^\mu\tilde{\phi}^a + 4\frac{V(\varphi)}{\varphi}\right]. \tag{12}$$

Therefore one has

$$D^\mu\partial_\mu\varphi + \frac{1}{\varphi}\partial_\mu\varphi\partial^\mu\varphi = \frac{1}{1+6\epsilon}\left[\frac{\partial V(\varphi)}{\partial\varphi} - 4\frac{V(\varphi)}{\varphi}\right]. \tag{13}$$

Finally, equations (10), (12) and (13) give

$$D_\mu D^\mu\tilde{\phi}^a + \tilde{\phi}^a D_\mu\tilde{\phi}^b D^\mu\tilde{\phi}^b + \frac{2}{\varphi}\partial_\mu\varphi D^\mu\tilde{\phi}^a = 0. \tag{14}$$

Therefore, the field equations can be written, in terms of the new variables φ and $\tilde{\phi}^a$ as

$$\begin{aligned} G_{\mu\nu} = \Big\{ & 2\left[\frac{1}{\varphi^2}(\partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi) + \frac{1}{\varphi}(D_\mu\partial_\nu\varphi - g_{\mu\nu}D_\rho\partial^\rho\varphi)\right] \\ & + T^{(F)}_{\mu\nu} + T^{(\varphi)}_{\mu\nu} + T^{(\tilde{\phi})}_{\mu\nu}\Big\}, \end{aligned} \tag{15}$$

$$D^\mu F^a_{\mu\nu} = e\varphi^2\epsilon_{abc}D_\nu\tilde{\phi}^b\tilde{\phi}^c, \tag{16}$$

$$D_\mu\partial^\mu\varphi + \frac{1}{\varphi}\partial_\mu\varphi\partial^\mu\varphi = \frac{1}{1+6\epsilon}\left[\frac{\partial V(\varphi)}{\partial\varphi} - 4\frac{V(\varphi)}{\varphi}\right], \tag{17}$$

$$D_\mu D^\mu\tilde{\phi}^a + \tilde{\phi}^a D_\mu\tilde{\phi}^b D^\mu\tilde{\phi}^b + \frac{2}{\varphi}\partial_\mu\varphi D^\mu\tilde{\phi}^a = 0. \tag{18}$$

Here

$$T^{(F)}_{\mu\nu} = \frac{1}{\epsilon\varphi^2}\left(F^a{}_{\mu\rho}F^a_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F^a_{\rho\sigma}F^{a\rho\sigma}\right), \tag{19}$$

$$T^{(\varphi)}_{\mu\nu} = \frac{1}{\epsilon\varphi^2}\left[\partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi)\right)\right], \tag{20}$$

$$T^{(\tilde{\phi})}_{\mu\nu} = \frac{1}{\epsilon}\left(D_\mu\tilde{\phi}^a D_\nu\tilde{\phi}^a - \frac{1}{2}g_{\mu\nu}D_\rho\tilde{\phi}^a D^\rho\tilde{\phi}^a\right) \tag{21}$$

represent the energy momentum tensor associated with the gauge field A^a_μ, norm of the scalar field φ and the phase content of the scalar field $\tilde{\phi}^a$ respectively.

In this paper, we will focus on the implications of the induced gravity model in a static and spherically symmetric pseudo-Riemannian space with the metric given by:

$$ds^2 = -B^2(r)C(r)dt^2 + \frac{dr^2}{C(T)} + r^2(d\theta^2 + \sin^2\theta\varphi^2). \tag{22}$$

Note that non-vanishing Einstein tensor can be shown to be

$$G_{\hat{t}\hat{t}} = \frac{1}{r^2} - \frac{C}{r^2} - \frac{C'}{r}, \tag{23}$$

$$G_{\hat{r}\hat{r}} = -\frac{1}{r^2} + \frac{C}{r^2} + \frac{C'}{r} + \frac{2CB'}{rB}, \tag{24}$$

$$G_{\hat{\vartheta}\hat{\vartheta}} = G_{\hat{\varphi}\hat{\varphi}} = \frac{CB'}{rB} + \frac{C'}{r} + \frac{3B'C'}{2B} + \frac{CB''}{B} + \frac{C''}{2}, \tag{25}$$

in an orthonormal frame. Here the prime denotes differentiation with respect to the argument r. We will also adopt the spherically symmetric 't Hooft-Polyakov ansatz for gauge field and Higgs field. It is given by

$$\phi^a = \varphi(r)\hat{r}^a, \tag{26}$$

$$A_i^a = \epsilon_{aij}\frac{1 - w(r)}{er}\hat{r}^j, \tag{27}$$

$$A_0^a = 0, \tag{28}$$

in Cartesian coordinates. After some algebra, one can show that equations (15-18) become

$$C' = \frac{1}{r}(1 - C) - r\rho, \tag{29}$$

$$B' = \frac{rB}{2C}(\rho - \tau), \tag{30}$$

$$(BCw')' + Bw\left[\frac{1 - w^2}{r^2} - e^2\varphi^2\right] = 0, \tag{31}$$

$$(BCr^2\varphi')' + \frac{BCr^2\varphi'^2}{\varphi} = \frac{\lambda v^2}{2(1 + 6\epsilon)}\frac{Br^2(\varphi^2 - v^2)}{\varphi}, \tag{32}$$

once the gauge ansatz and Higgs ansatz are substituted. Here

$$\rho = C\left\{\frac{2}{\varphi}\left[\varphi'' + \left(\frac{C'}{2C} + \frac{2}{r}\right)\varphi'\right] + \frac{1}{\varphi^2}\left[\left(2 + \frac{1}{2\epsilon}\right)\varphi'^2 + \frac{1}{\epsilon}\frac{w'^2}{e^2r^2}\right]\right\} + \frac{1}{\epsilon}\left[\frac{w^2}{r^2} + \frac{1}{\varphi^2}\left(\frac{(1 - w^2)^2}{2e^2r^4} + V(\varphi)\right)\right], \tag{33}$$

$$\tau = C\left[\frac{2\varphi'}{\varphi}\left(\frac{B'}{B} + \frac{C'}{2C} + \frac{2}{r}\right) - \frac{1}{\varphi^2}\left(\frac{\varphi'^2}{2} + \frac{w'^2}{e^2r^2}\right)\frac{1}{\epsilon}\varphi'^2\right] + \frac{1}{\epsilon}\left[\frac{w^2}{r^2} + \frac{1}{\varphi^2}\left(\frac{(1 - w^2)^2}{2e^2r^4} + V(\varphi)\right)\right], \tag{34}$$

are the generalized energy density and the generalized radial pressure measured with respect to the orthonormal frame. Note that the Higgs equation (32) is independent of the gauge field. This is a very unique result due to the induced coupling $\phi^2 R$. It will be used to show that (i) regular monopole solution does not exist, (ii) no hair theorem does apply to the norm of the Higgs scalar φ.

III. Non-existence of regular monopoles

In this section, we will show that regular monopole solutions to the system of Eqs. (29-32) does not exist due to special coupling in the induced gravity model. Note that the most general boundary conditions of various field variables at the origin (r = 0) can be argued to be:

$$C(0) = 1, \ \varphi(0) = 0, \ w(0) = 1, \ B(0) \text{ finite.} \tag{35}$$

Note the argument leading to the non-existence of regular solution to be shown in a moment remains valid as one relaxes the boundary condition on $\varphi(0)$ to be finite at $r = 0$. Therefore, one can expand these fields as

$$\varphi(r) = k_\varphi r + \ldots \tag{36}$$

$$\mathrm{W(T)} = 1 - k_w r^2 + \cdots \tag{37}$$

$$C(r) = 1 - k_C r^m + \ldots \tag{38}$$

$$\mathrm{B(T)} = B_0 - k_B r^n + \cdots \tag{39}$$

near the origin. Here k_φ, k_w, kc, kg and $B_0 \equiv$ B(0) are constants to be fit with the field equations. Moreover, m, n are both positive integers which can be shown to satisfy the inequality $m, n \geq$ **2.**

On the other hand, asymptotic flatness and regularity of various physical quantities will also impose on these fields the following boundary conditions:

$$C(\infty) = B(\infty) = 1, \quad w(\infty) = 0, \ \varphi(\infty) = v \tag{40}$$

at spatial infinity.

In order to show the non-existence theorem, one needs to show a few Lemmas. First of all, one will show that φ is a monotonically increasing function if $\varphi < v$ or monotonically decreasing function if $\varphi > v$.

This can be proved by showing that φ has no maxima for $\varphi > v$ and no minima for $\varphi < v$. Indeed, Eq. (32) shows that

$$\varphi'' = \frac{\lambda v^2 \quad \varphi^2 - v^2}{2(1 \ \mathbf{t} \ \mathbf{64} \ C\varphi} \tag{41}$$

at local extremum of φ. Let r_0 be the point such that $\varphi'(r_0) = 0$. It follows that $\varphi'' > 0$ at r_0 if $\varphi > v$. This means that φ is a local minima if $\varphi > v$ at r_0. Similar argument leads to the statement that φ is a maxima if $\varphi < v$ at r_0. This proves the statement stated above. As a result, φ has to be a monotonic function because $\varphi \to v$ at spatial infinity. Hence one proves that φ will monotonically approach v as r increases no matter what its initial value is.

Note that one can derive

$$\frac{\lambda v^2}{2(1+6\epsilon)} \int_0^\infty dr B r^2 (\varphi^2 - v^2) = BCr^2 \varphi' \varphi|_{r=\infty} \tag{42}$$

from multiplying both sides of Eq. (32) by φ and integrating over r. Here one has dropped the vanishing boundary term $BCr^2\varphi'\varphi|_{r=0}$ as the boundary condition C(0) = 0 demands. Equation (42) indicates that the left hand side of equation (42) is positive definite if $\varphi >$ v. On the other hand the right hand side of equation (42) is negative definite since $\varphi' < 0$ everywhere due to the Lemma just proved. Hence there is a contradiction here. Similar argument show that contradiction also appears if $\varphi < v$. Hence there does not exist non-trivial solution to φ that satisfies different boundary conditions at the origin and the spatial infinity. Hence the only consistent solution of is the constant v. Note that $\varphi = v$ will make ϕ singular at $r = 0$. Therefore, one reaches the conclusion that regular 't Hooft-Polyakov monopole solution does not exist in this induced EYMH model. Note that the φ = v monopole solution has been discussed in great details in reference [13]. Our analysis provides a complete proof that trivial Higgs solution is the only possible regular spherically symmetric monopole field configuration this induced system can admit. We will also show that Higgs hair can not exist if a black hole event horizon exists shortly in the following chapter.

IV. Magnetically charged black holes

A black hole is characterized by the event horizon. In the presence of the event horizon, we consider solutions of Eqs. (29)–(32) in the region $r \in [r_H, \infty)$, where r_H is the radius of the (outermost) horizon. The boundary conditions at $r = r_H$ are

$$C(r_H) = 0, \; C'(r_H) \geq 0, \tag{43}$$

and the functions φ, φ', w, w' and B are assumed to be finite [14]. At infinity the asymptotic flatness conditions (40) must be satisfied.

We will show that no-hair theorem is true for the φ field such that Higgs field has to be frozen at its global minima $\phi^2 = v^2$. Indeed, one has

$$\int_{r_H}^\infty dr B r^2 \left[\frac{vC\varphi'^2}{\varphi} + \frac{\lambda v^2}{2(1 t 6\epsilon)} (1 t \frac{v}{\varphi})(\varphi - v)^2 \right] = BCr^2 \varphi'(\varphi - v)|_{r_H}^\infty \tag{44}$$

from multiplying Eq. (32) by $(\varphi - v)$ and integrating with respect to r from the event horizon r_H (where $C(r_H) = 0$) to spatial infinity, Note that properly dropping boundary terms after appropriate integration by parts are required to derive equation (44).

It is known that B and φ' are both finite as required by the regularity of these field variables. Hence the right hand side of above equation vanishes because (i) $C(r_H) = 0$ at the lower bound r_H and (ii) $\varphi \to v$ at spatial infinity. Therefore equation (44) shows that $\varphi(r)=v$ for all $r > r_H$ because the integrand on the left hand side of the equation (44) are all positive. This proves the no-hair theorem for the Higgs field in this model.

Similarly, one can derive, from multiplying Eq. (31) by w and integrating from the r_H to infinity,

$$\int_{r_H}^{\infty} drB\left[Cw'^2 + \frac{w^2}{r^2}(e^2v^2r^2 + w^2 - 1)\right] = BCw'w|_{r_H}^{\infty} \tag{45}$$

after proper integration by part. Note that the right-hand side of above equation vanishes by similar argument due to the boundary conditions. If $(e^2v^2r^2 - 1) \geq 0$ for all $r \geq r_H$, or equivalently $r_H \geq 1/(ev)$, one has $w(r) = 0$ for all $r > r_H$. Note that $1/(ev)$ is the characteristic radius of the classical monopole. Hence the constraint $r_H \geq 1/(ev)$ indicates that the event horizon of a non-trivial w monopole black hole solution does fall inside the characteristic size of a monopole configuration. Equivalently, it is a black hole in a monopole [2].

Hence a black hole with a horizon $r_H \geq 1/(ev)$ will demand that w = 0 which can be shown to give the Reissner-Nordstrom black hole solution. Indeed, when $\varphi = v$ and w = 0, the field equations can be solved to give:

$$C = 1 - \frac{m}{4\pi\epsilon v^2 r} + \frac{1}{2\epsilon v^2 e^2 r^2}, \tag{46}$$

$$B = 1, \tag{47}$$

which is exactly the Reissner-Nordstrom solution. Here m is the ADM mass of the black hole and the constraint m $\geq \frac{4\pi v\sqrt{2\epsilon}}{e}$ is required to prevent the exposure of the naked singularity.

Note that the field equation can be reduced to

$$C' = \frac{1}{x}(1 - C) - \frac{1}{\epsilon x}\left[Cw'^2 + \frac{(1-w^2)^2}{2x^2} + w^2\right], \tag{48}$$

$$B' = \frac{Bw'^2}{\epsilon x}, \tag{49}$$

$$(BCw')' = \frac{Bw}{x^2}(w^2 + x^2 - 1), \tag{50}$$

Here we have written $x = evr$, $C = C(x)$, $B = B(x)$ and $w = w(x)$ for sim-plicity. Hence the condition for a non-trivial w black hole solution becomes $x_H < 1$. We will hence work on the case where $x_H < 1$. Note also that the prime denotes differentiation with respect to the argument x. One notes that equations (48850) are similar to that of the gravity minimally coupled to the SU_2 gauge field and the Higgs field system in the limit of infinite Higgs self-coupling constant studied by Aichelburg and Bizon [4], and our numerical results are in agreement with theirs.

We will show two lemmas [4] that control the behavior of w under the influence of the field equation in certain domain. Lemma one states that w is a monotonically decreasing (increasing) function of x if $w \geq 1 (w \leq -1)$. Lemma two states that w monotonically approaches 0 in the range $x \geq 1$.

Since the field equations are invariant under $w \to -w$, one can assume $w(x_H) > 0$ without any loss of generality. Note that, at local extrema such that $w'(x_0) = 0$, equation (50) becomes

$$Cw'' = \frac{w}{x_0^2}(w^2 + x_0^2 - 1). \tag{51}$$

If $w \geq 1$ one has $w'' > 0$ at the local extrema for all $x_0 > 0$. This means that $w(x_0)$ is a local minima. Similarly, if $w \leq -1$, one can show that $w(x_0)$ is a local maxima. Therefore the boundary condition of the variable w at spatial infinity, $w(\infty) \to 0$, will rule out the existence of local extrerna for w in the region $|w| \geq 1$. Hence any physical solution of w has to be a monotonically decreasing (increasing) function in x as long as the field value of w remains in the domain $w \geq 1 (w \leq -1)$. This proves Lemma one.

On the other hand, sign (w'') = sign (w) at these extremum (where $w'(x_0) = 0$ throughout the whole region $x_0 \geq 1$. This means that local extrema is a local minima (maxima) if $w > 0 (w < 0)$. Hence one proves Lemma two that states: w monotonically approaches 0 in the range $x \geq 1$.

Note that, at the event horizon of a black hole where $x = x_H$, equation (50) reads

$$C'w' = \frac{w}{x_H^2}(w^2 + x_H^2 - 1) \tag{52}$$

since $C(x_H) = 0$. Hence $w(x_H) \geq 1$ will imply $w'(x_H) > 0$. Note that we have used the fact the $C'(x_H) \geq 0$ since C is assumed to be positive definite for all $x > x_H$. Recall that $w(x_H) \geq 1$ implies that w is a monotonically decreasing function such that $w'(x) < 0$ for all x according to Lemma one. Therefore we have a contradiction here. Hence one has the conclusion that $w(x_H) < 1$. Note that similar argument also holds if $w < 0$.

Note that one can also show that $w'(x_H) < 0$ following similar argument. Indeed, if $w'(x_H) \geq 0$ one should have $w(x_H)^2 + x_H^2 - 1 \geq 0$. Furthermore, there must exist at least a local maxima of w somewhere since w will have to turn its direction in order to approach zero at spatial infinity. Note also that w has to attain this maxima before it reaches the region $w \geq 1$. Otherwise w will keep increasing once it reaches the region $w \geq 1$ according to Lemma one. Hence there must exist a local maxima of w at $x_0 \in (x_H, 1)$ such that $1 \geq w(x_0) > w(x_H)$. Note that one has $w(x_0)^2 + x_0^2 - 1 < 0$ at the local maxima according to equation (51). Hence one has a contradictory result $x_H^2 > x_0^2$ since one has (i) $w(x_H)^2 + x_H^2 - 1 > 0$ and (ii) $w(x_0) > w(x_H)$ as a local maxima. Hence one reaches the conclusion that (i) $w(x_H) < 1$ and (ii) $w'(x_H) < 0$.

Note that $w'(x_H) < 0$ and $a \equiv w(x_H) > 0$ indicate that

$$1 - a^2 > x_H^2 \tag{53}$$

following Eq. (52). Therefore, the necessary condition for the existence of a non-trivial w non-abelian black hole is

$$x_H < 1, \tag{54}$$

in agreement with our previous result. Moreover, one has

$$C'(x_H) = \frac{1}{x_H} - \frac{(1-a^2)^2}{2\epsilon x_H^3} - \frac{a^2}{\epsilon x_H} > 0 \tag{55}$$

from Eq. (48). Equivalently, one has

$$a^4 - 2(1-x_H^2)a^2 - 2\epsilon x_H^2 + 1 \leq 0 \tag{56}$$

which admits real solution for a only if

$$x_H^2 - 2(1-\epsilon) \geq 0. \tag{57}$$

Hence one has (i) $x_H \geq \sqrt{2(1-\epsilon)}$ if $1/2 < \epsilon < 1$; (ii) magnetically charged black hole can not exist if $0 < \epsilon \leq 1/2$; (iii) there is no constraint at all for $\epsilon \geq 1$.

Finally, one notes that the inequalities (53) and (56) saturate for extremal black hole such that $C'(x_H) = 0$. Hence one has

$$x_H = \sqrt{2(1-\epsilon)} \tag{58}$$

if $1/2 < \epsilon < 1$.

In summary, one has shown that (a) w can be an oscillatory function in the domain $x \in (x_H, 1)$; (b) w has to approach 0 monotonically in the domain $x \geq 1$; (c) $w(x_H) < 1$; (d) $w'(x_H) < 0$; (d) $1 > x_H \geq \sqrt{2(1-\epsilon)}$ if $1/2 < \epsilon < 1$; (e) magnetically charged black hole can not exist if $0 < \epsilon \leq 1/2$; (iii) there is no constraint at all for $\epsilon \geq 1$. Note that these constraints will be very helpful for setting correct initial conditions for numerical solutions. Note that equations (48-50) can be reduced to

$$C' = \frac{1}{x}(1 - C) - \frac{1}{\epsilon x}\left[Cw'^2 + \frac{(1-w^2)^2}{2x} + w^2\right], \tag{59}$$

$$Cw'' + C'w' + \frac{Cw'^3}{\epsilon x^3} + \frac{w}{x}(1 - w^2 - x^2) = 0 \tag{60}$$

after eliminating B(z). Note that $C(x)$ and $w(x)$ can be expanded as

$$C(x) = c(x - x_H) + O((x - x_H)^2) + \cdots, \tag{61}$$

$$w(x) = a + b(x - x_H) + O((x - x_H)^2) + \cdots \tag{62}$$

near x_H. Therefore one has, from recurrence relation,

$$c = \frac{1}{x_H} - \frac{(1-a^2)}{2\epsilon x_H^3} - \frac{a^2}{\epsilon x_H}, \tag{63}$$

$$b = -\frac{a}{cx_H^2}(1 - a^2 - x_H^2), \tag{64}$$

with a confined by

$$1 - x_H^2 - x_H\sqrt{x_H^2 - 2(1-\epsilon)} \le a^2 \le 1 - x_H^2. \tag{65}$$

Note that $x_H = \sqrt{2(1-\epsilon)}$ and a = $\sqrt{2\epsilon - 1}$ for extremal black holes. Note that similar stable solutions has been solved numerically in reference [13].

V. Conclusion

We have shown that regular SO_3 non-abelian spherically symmetric monopole solution does not exist in an induced Einstein-Yang-Mills-Higgs model. Hence one turns our attention to charged black hole solutions. It is then also shown that Higgs scalar hair can not exist in the presence of a black hole event horizon in this model. It is also shown that existence of a non-trivial monopole-charged black hole solution in the induced EYMH model imposes a number of constraints on the field parameters $\epsilon, w(x_H), w'(x_H)$ as well as the location of the event horizon x_H. In particular, it is shown that non-trivial non-abelian monopole-charged black hole solution exists only if the radius of the event horizon is smaller than the characteristic radius of the classical monopole, i.e., $r_H < 1/(ev)$. We have also analyzed the global behavior of nontrivial w such that large distance behavior of w is available without solving the field equation directly. These analysis may be helpful for related studies. One would like to emphasize again that our analysis is valid only under the assumption of the spherically symmetric ansatz adopted throughout this paper.

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Appendix

We will show that the generalized energy momentum tensor is indeed a conserved quantity here. In fact, $D^\mu \mathcal{T}_{\mu\nu} = 0$ can be shown to give

$$R_{\mu\nu}\partial^\nu\varphi = \frac{2}{\varphi}(\partial_\mu\varphi\nabla_\nu\partial^\nu\varphi - \partial^\nu\varphi\nabla_\mu\partial_\nu\varphi) + \frac{1}{\epsilon\varphi^2}\left[\frac{1}{4}\partial_\mu\varphi F^a_{\rho\sigma}F^{a\rho\sigma} - \partial_\nu\varphi F^a_{\mu\rho}F^{a\nu\rho}\right]$$
$$+\frac{1}{2\epsilon}\left[\frac{1}{\varphi^2}(2\partial_\mu\varphi V(\varphi) - \partial_\mu\varphi\partial_\nu\varphi\partial^\nu\varphi) + \frac{1}{\varphi}\left(\partial_\mu\varphi\nabla_\nu\partial^\nu\varphi - \partial_\mu\varphi\frac{\partial V(\varphi)}{\partial\varphi}\right)\right. \tag{66}$$
$$\left.+\varphi D\mu\tilde{\varphi}^a D_\nu D^\nu\tilde{\phi}^a\right],$$

with help of the identity $[D_\mu, D_\nu]\partial^\nu\varphi = R_{\mu\nu}\partial^\nu\varphi$ which follows from the definition of the curvature tensor, and the Bianchi identity for $F^a_{\mu\nu}$, i.e., $D_\mu F^a_{\nu\rho} + D_\nu F^a_{\rho\mu} + D_\rho F^a_{\mu\nu} = 0$.

Moreover, Eq. (11) and Eq. (15) gives

$$R_{\mu\nu}\partial^\nu\varphi = \frac{2}{\varphi}(\partial_\mu\varphi\nabla_\nu\partial^\nu\varphi - \partial^\nu\varphi\nabla_\mu\partial_\nu\varphi) + \frac{1}{\epsilon\varphi^2}\left[\frac{1}{4}\partial_\mu\varphi F^a_{\rho\sigma}F^{a\rho\sigma} - \partial_\nu\varphi F^a_{\mu\rho}F^{a\nu\rho}\right]$$
$$+\frac{1}{2\epsilon}\left[\frac{1}{\varphi^2}(2\partial_\mu\varphi V(\varphi) - \partial_\mu\varphi\partial_\nu\varphi\partial^\nu\varphi) + \frac{1}{\varphi}\left(\partial_\mu\varphi\nabla_\nu\partial^\nu\varphi - \partial_\mu\varphi\frac{\partial V(\varphi)}{\partial\varphi}\right)\right. \quad (67)$$
$$\left. -2\partial^\nu\varphi D_\nu\tilde{\phi}^a D_\mu\tilde{\phi}^a\right],$$

Comparing with Eq. (66), one has

$$2\partial_\nu\varphi D^\nu\tilde{\phi}^a D_\mu\tilde{\phi}^a + \varphi D_\mu\tilde{\phi}^a D_\nu D^\nu\tilde{\phi}^a = 0, \quad (68)$$

Indeed, one notes that above equation is exactly Eq. (18). Hence one proves the claim $D^\mu\mathcal{T}_{\mu\nu} = 0$

References

[1] P. Breitenlohner, P. Forgacs, and D. Maison, Nucl. Phys. B383, 357 (1992).

[2] K. Lee, V. P. Nair, and E. J. Weinberg, Phys. Rev. D45, 2751 (1992); K. Lee, V. P. Nair, and E. J. Weinberg, Gen. Rel. Gra. **24(12)**, 1203 (1992).

[3] M. Ortiz, Phys. Rev. D45, R2586 (1992).

[4] P. C. Aichelburg and P. Bizon, Phys. Rev. **D48**, 607 (1993).

[5] G. 't Hooft, Nucl. Phys. **B79,** 276 (1974); A. Polyakov, JETP Lett. 20, 194 (1974); P. Goddard and 0. I. Olive, Rep. Prog. Phys. 41, 1357 (1978).

[6] P.A.M. Dirac, Proc. R. Soc. London A133, 60 (1931).

[7] E. W. Kolb and M. S. Turner, The Early Universe, (Addison-Wesley, California, 1990).

[8] C. W. Minser, K. Thorne, and J. A. Wheeler, Gravitation (Freeman, San Francisco, 1973).

[9] B. Carter, Phys. Rev. Lett. 26, 331 (1971); J. D. Bekenstein, Phys. Rev. 5, 1239, 2403 (1972); C. Teitelboim, Phys. Rev. 5, 2941 (1972); R. H. Price, Phys. Rev. 5, 2419, 2439 (1972).

[10] A. Zee, Phys. Rev. Lett. 42, 417 (1979).

[11] F. S. Accetta, A. J. Zoller, and M. S. Turner, Phys. Rev. D31, 3046 (1985); A. B. Kaganovich, Phys. Lett. B222, 364 (1989); W. F. Kao, Phys. Rev. D46, 5421 (1992).

[12] S. Weinberg, Gravitation and Cosmology, (John Wiley and Sons, New York, 1972).

[13] B. S. Balakrishna and K. C. Wali, Phys. Rev. D46, R5228 (1992); N. A. Viet and K. C. Wali, Phys. Rev. D51, 1664 (1995).

[14] The finiteness of B(r) at the horizon has been discussed carefully, e.g., by M. Visser, Phys. Rev. D46, 2445 (1992).

[15] S. Wolfram, Mathematica: A System for Doing Mathematics by Computer, (Addison-Wesley, California, 1988).