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## Control of AMIRA's ball and beam system via improved fuzzy feedback linearization approach

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### ABSTRACT

This paper first studies the tracking and almost disturbance decoupling problem of nonlinear AMIRA's ball and beam system based on the feedback linearization approach and fuzzy logic control. The main contribution of this study is to construct a controller, under appropriate conditions, such that the resulting closed-loop system is valid for any initial condition and bounded tracking signal with the following characteristics: input-to-state stability with respect to disturbance inputs and almost disturbance decoupling, i.e., the influence of disturbances on the  $L_2$  norm of the output tracking error can be arbitrarily attenuated by changing some adjustable parameters. One example, which cannot be solved by the first paper on the almost disturbance decoupling problem, is proposed in this paper to exploit the fact that the tracking and the almost disturbance decoupling performances are easily achieved by our proposed approach. The simulation results show that our proposed approach has achieved the almost disturbance decoupling performance perfectly.

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## 1. Introduction

Many approaches for nonlinear systems are introduced including feedback linearization, regulation control, nonlinear  $H^\infty$  control, internal model principle and  $H^\infty$  adaptive fuzzy control. An output tracking approach is to utilize the scheme of the output regulation control [1] in which the outputs are assumed to be excited by an exosystem. However, the nonlinear regulation problem requires to solve the difficult solution of partial-differential algebraic equation. Another problem of the output regulation control is that the exosystem states need to be switched to describe changes in the output and this will create transient tracking errors [2]. In general, the nonlinear  $H^\infty$  control has to solve the Hamilton–Jacobi equation, which is a difficult nonlinear partial-differential equation [3–6]. Only for some particular nonlinear systems we can derive a closed-form solution [7]. The control approach based on internal model principle converts the tracking problem to nonlinear output regulation problem [8]. This approach depends on solving a first-order partial-differential equation of the center manifold [1]. For some special nonlinear systems and desired trajectories, the asymptotic solutions of this equation via ordinary differential equations have been developed [9,10]. Recently,  $H^\infty$  adaptive fuzzy control has been proposed to systematically deal with nonlinear systems [11]. The drawback with  $H^\infty$  adaptive fuzzy control is that the complex parameter update law makes this approach impractical. During the past decade significant progress has been made in the research of control approaches for nonlinear systems based on the feedback linearization theory [12–15]. Moreover, feedback linearization approach has been applied successfully to address many real controls. These include the control of electromagnetic suspension system

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[16], pendulum system [17], spacecraft [18], electrohydraulic servosystem [19], car-pole system [20] and bank-to-turn missile system [21].

Almost disturbance decoupling analysis and systematic design are among the most issues to be further addressed. The almost disturbance decoupling problem, that is the design of a controller which attenuates the effect of the disturbance on the output terminal to an arbitrary degree of accuracy, was originally developed for linear and nonlinear control systems by Willems [22] and Marino et al. [23], respectively. Henceforward, the problem has attracted considerable attention and many significant results have been developed for both linear and nonlinear control systems [24–26]. Marino et al. [23] shows that for nonlinear SISO systems the almost disturbance decoupling problem may not be solvable, as the following examples show:

$$\dot{x}_1(t) = \tan^{-1} x_2 + \theta(t), \quad \dot{x}_2(t) = u,$$

$$y = x_1,$$

where  $u, y$  denote the input and output respectively and  $\theta$  is the disturbance. On the contrary, these examples can be easily solved via the proposed approach in this paper.

Fuzzy logic control has been applied not only to cement kiln, subway train but also to industrial processes. Its designing procedure is as follows. First representing the nonlinear system as the famous Takagi–Sugeno fuzzy model offers an alternative to conventional model [27,28]. The control design is carried out based on an aggregation of linear controllers constructed for each local linear element of the fuzzy model via the parallel distributed compensation scheme [29]. For the stability analysis of fuzzy system, a lot of studies are reported (see, e.g., [30–33], and the references therein). The stability and controller design of fuzzy system can be mainly discussed by Tanaka–Sugeno's theorem [30]. However, it's difficult to find the common positive definite matrix  $P$  for linear matrix inequality (LMI) problem [34,35] even if  $P$  is a second order matrix [36]. To overcome the difficulty of finding the common positive definite matrix  $P$  for fuzzy-model approach, we will propose a new method to guarantee that the closed-loop systems is stable and the almost disturbance decoupling performance is achieved. The desired problem for this study is summarized as follows. First, based on the feedback linearization approach a tracking control is designed to guarantee the almost disturbance decoupling property, i.e. the influence of disturbances on the  $L_2$  norm of the output tracking error can be arbitrarily attenuated by changing some adjustable parameters, and the uniform ultimate bounded stability of the control system response within an adjustable global final attractor of the zero state, i.e., such response enters a neighborhood of zero state in finite time and remains within it thereafter. Once the tracking errors are driven to touch the attractor with the desired radius, the conventional fuzzy logic control immediately applied via human expert's knowledge to improve the convergence rate. In order to exploit the significant applicability, this paper also has successfully derived tracking controller with almost disturbance decoupling for a famous AMIRA's ball and beam system.

## 2. Mathematical model of the AMIRA's ball and beam system

Fig. 2.1 is the hardware structure of the AMIRA's ball and beam system. U-type aluminium profiles construct the platform and the organization of the ball and beam system which is covered at the side by four sheets of Plexiglas. The steel ball rolls freely on the beam and its position is measured by a camera unit and the lighting module mounted below a small platform

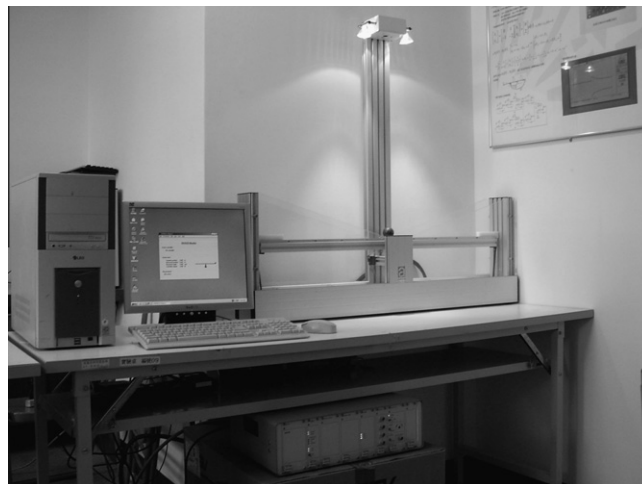


Fig. 2.1. The AMIRA's ball and beam experimental equipment.

on top of the system. The beam is located in the center of the system and driven by a tooth-belt, a tooth wheel and a DC motor. The angle of the beam is measured by an incremental encoder mounted at the rear end of the beam shaft. Two limited switches are located below the beam to detect whether the beam reaches its maximum angle or not. The unmeasurable states, the speed of the ball and the angular speed of the beam, are estimated by a Luenberger reduced order observer. Due to the mounting of the beam, the maximum angle  $\alpha_{\max} \approx 0.24$  rad.

Balancing all the forces acted upon the system, it is easy to evaluate the Kinetic energy, potential energy, dissipative forces, and generalized forces of the system. Inserting them into the Lagrange equation, we can obtain simultaneously the motion equations [37]

$$\left(m + \frac{I_b}{r^2}\right)\ddot{x} + (mr^2 + I_b)\frac{1}{r}\ddot{\alpha} - m\dot{x}\dot{\alpha}^2 = mg(\sin \alpha), \tag{2.1a}$$

$$[m(\dot{x}')^2 + I_b + I_w]\ddot{\alpha} + (2m\dot{x}\dot{x}' + bl^2)\dot{\alpha} + Kl^2\alpha + (mr^2 + I_b)\frac{1}{r}\dot{x}' - mgx'(\cos \alpha) = ul(\cos \alpha) \tag{2.1b}$$

and the nonlinear state equation

$$\dot{x}_1 = x_2, \tag{2.2a}$$

$$\dot{x}_2 = \frac{a_2[(b_2x_1x_2 + b_3)x_4 + b_4x_3 - b_6x_1 \cos(x_3)] + (mx_1^2 + b_1)(a_3 \sin(x_3) + mx_1x_4^2) - a_2l \cos(x_3)u}{a_1(mx_1^2 + b_1) - a_2b_5}, \tag{2.2b}$$

$$\dot{x}_3 = x_4, \tag{2.2c}$$

$$\begin{aligned} \dot{x}_4 = & \frac{-(b_2x_1x_2 + b_3)x_4 - b_4x_3 + b_6x_1 \cos(x_3)}{mx_1^2 + b_1} - \frac{b_5(a_3 \sin(x_3) + mx_1x_4^2)}{a_1(mx_1^2 + b_1) - a_2b_5} \\ & - \frac{a_2b_5[(b_2x_1x_2 + b_3)x_4 + b_4x_3 - b_6x_1 \cos(x_3)]}{(mx_1^2 + b_1)(a_1(mx_1^2 + b_1) - a_2b_5)} + \left[1 + \frac{a_2b_5}{a_1(mx_1^2 + b_1) - a_2b_5}\right] \frac{l \cos(x_3)u}{mx_1^2 + b_1}, \end{aligned} \tag{2.2d}$$

where the abbreviations have the following meanings:  $m$  = mass of the ball,  $r$  = roll radius of the ball,  $I_b$  = inertia moment of the ball,  $a_2 = (mr^2 + I_b)\frac{1}{r}$ ,  $b_2 = 2m$ ,  $b$  = friction coefficient of the drive mechanics,  $l$  = radius of force application,  $b_3 = bl^2$ ,  $I_w$  = radius of beam,  $K$  = stiffness of the drive mechanics,  $b_4 = Kl^2$ ,  $g$  = gravity,  $b_6 = mg$ ,  $I_w$  = inertia moment of the beam,  $b_1 = I_b + I_w$ ,  $a_3 = mg$ ,  $u$  = force of the drive mechanics,  $a_1 = m + \frac{I_b}{r^2}$ ,  $b_5 = (mr^2 + I_b)\frac{1}{r}$ ,  $x_1 = x'$  = position of the ball,  $x_2 = \dot{x}'$  = velocity of the ball,  $x_3 = \alpha$  = angle of the beam to the horizontal,  $\alpha_{\max}$  = maximum angle of the beam to the horizontal,  $x_4 = \dot{\alpha}$  = angular velocity of the beam. Substituting all the physical values  $r = 0.02$  m,  $l = 0.48$  m,  $m = 0.0162$  Kg,  $M = 1.122$  Kg,  $b = 1$  Ns/m,  $K = 0.001$  N/m,  $I_w = 0.5$  m into (2.2), the state equation can be rewritten as follows:

$$\dot{x}_1 = x_2, \tag{2.3a}$$

$$\dot{x}_2 = \frac{a_{12} + a_{34}}{a_5} + \frac{b_{11}}{b_{22}}u + 0.1\theta_1, \tag{2.3b}$$

$$\dot{x}_3 = x_4, \tag{2.3c}$$

$$\dot{x}_4 = \frac{z_{12} + z_{34} + z_{56}}{z_7} + au, \tag{2.3d}$$

where

$$a_{12} = 0.0000561038x_1x_2x_4 + 0.0004157x_4 + 0.000000415757x_3 - 0.00027490882x_1 \cos(x_3)$$

$$a_{34} = 0.0025719x_1^2(\sin(x_3)) + 0.00026244x_1^3x_4^2 + 0.03711015 \sin(x_3) + 0.00378675x_1x_4^2$$

$$a_5 = 0.0015x_1^2 + 0.021642252, \quad b_{11} = -0.000848484 \cos(x_3), \quad b_{22} = 0.0015x_1^2 + 0.021642252$$

$$z_{12} = -0.0000486x_1^3x_2x_4 - 0.00036018x_1^2x_4 - 0.00000036x_1^2x_3 + 0.000238x_1^3 \cos(x_3)$$

$$z_{34} = -0.0007013x_1x_2x_4 - 0.005197x_4 - 0.000005197x_3 + 0.003436x_1 \cos(x_3)$$

$$z_{56} = -0.00000445x_1^2 \sin(x_3) - 0.000000454x_1^3x_4^2 - 0.0000643 \sin(x_3) - 0.00000655x_1x_4^2$$

$$z_7 = 0.0000243x_1^4 + 0.00070126x_1^2 + 0.021645,$$

$$a = \frac{0.000735x_1^2 \cos(x_3) + 0.0106 \cos(x_3)}{0.0000243x_1^4 + 0.00070126x_1^2 + 0.021645},$$

and  $\theta_1 = \sin(t - 8)$  is assumed to be the disturbance item.

2.1. Feedback linearization controller design

In this paper, we consider the following nonlinear control system with disturbances:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2, \dots, x_n) \\ f_2(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{bmatrix} + \begin{bmatrix} g_1(x_1, x_2, \dots, x_n) \\ g_2(x_1, x_2, \dots, x_n) \\ \vdots \\ g_n(x_1, x_2, \dots, x_n) \end{bmatrix} u + \sum_{i=1}^p q_i^* \theta_i, \tag{3.1a}$$

$$y(t) = h(x_1, x_2, \dots, x_n) \tag{3.1b}$$

i.e.,

$$\dot{X}(t) = f(X(t)) + g(X(t))u + \sum_{i=1}^p q_i^* \theta_i,$$

$$y(t) = h(X(t)),$$

where  $X(t) := [x_1(t)x_2(t) \dots x_n(t)]^T \in \mathfrak{R}^n$  is the state vector,  $u \in \mathfrak{R}^1$  is the input,  $y \in \mathfrak{R}^1$  is the output,  $\theta := [\theta_1(t)\theta_2(t) \dots \theta_p(t)]^T$  is a bounded time-varying disturbance vector,  $f, g, q_1^*, \dots, q_p^*$  are smooth vector fields on  $\mathfrak{R}^n$ , and  $h(X(t)) \in \mathfrak{R}^1$  is a smooth function. The nominal system is then defined as follows:

$$\dot{X}(t) = f(X(t)) + g(X(t))u, \tag{3.2a}$$

$$y(t) = h(X(t)). \tag{3.2b}$$

The nominal system (3.2) consists of relative degree  $r$  [38], i.e., there exists a positive integer  $1 \leq r < \infty$  such that

$$L_g L_f^k h(X(t)) = 0, \quad k < r - 1, \tag{3.3}$$

$$L_g L_f^{r-1} h(X(t)) \neq 0 \tag{3.4}$$

for all  $X \in \mathfrak{R}^n$  and  $t \in [0, \infty)$ , where the operator  $L$  is the Lie derivative [12]. The desired output trajectory  $y_d(t)$  and its first  $r$  derivatives are all uniformly bounded and

$$\| [y_d(t), y_d^{(1)}(t), \dots, y_d^{(r)}(t)] \| \leq B_d, \tag{3.5}$$

where  $B_d$  is some positive constant.

Under the assumption of well-defined relative degree, it has been shown [12] that the mapping

$$\phi : \mathfrak{R}^n \rightarrow \mathfrak{R}^n, \tag{3.6}$$

defined as

$$\phi_i(X(t)) := \xi_i(t) = L_f^{i-1} h(X(t)), \quad i = 1, 2, \dots, r \tag{3.7}$$

$$\phi_k(X(t)) := \eta_k(t), \quad k = r + 1, r + 2, \dots, n \tag{3.8}$$

and satisfying

$$L_g \phi_k(X(t)) = 0, \quad k = r + 1, r + 2, \dots, n \tag{3.9}$$

is a diffeomorphism onto image. For the sake of convenience, define the trajectory error to be

$$e_i(t) := \xi_i(t) - y_d^{(i-1)}(t), \quad i = 1, 2, \dots, r \tag{3.10}$$

$$e(t) := [e_1(t)e_2(t) \dots e_r(t)]^T \in \mathfrak{R}^r, \tag{3.11}$$

the trajectory error multiplied with some adjustable positive constant  $\varepsilon$

$$\bar{e}_i(t) := \varepsilon^{i-1} e_i(t), \quad i = 1, 2, \dots, r, \tag{3.12}$$

$$\bar{e}(t) := [\bar{e}_1(t)\bar{e}_2(t) \dots \bar{e}_r(t)]^T \in \mathfrak{R}^r \tag{3.13}$$

and

$$\xi(t) := [\xi_1(t) \xi_2(t) \cdots \xi_r(t)]^T \in \mathfrak{R}^r, \tag{3.14a}$$

$$\eta(t) := [\eta_{r+1}(t) \eta_{r+2}(t) \cdots \eta_n(t)]^T \in \mathfrak{R}^{n-r}, \tag{3.14b}$$

$$q(\xi(t), \eta(t)) := [L_f \phi_{r+1}(t) L_f \phi_{r+2}(t) \cdots L_f \phi_n(t)]^T := [q_{r+1} \ q_{r+2} \ \cdots \ q_n]^T. \tag{3.14c}$$

Define a phase-variable canonical matrix  $A_c$  to be

$$A_c := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \cdots & -\alpha_r \end{bmatrix}_{r \times r}, \tag{3.15}$$

where  $\alpha_1, \alpha_2, \dots, \alpha_r$  are any chosen parameters such that  $A_c$  is Hurwitz and the vector  $B$  to be

$$B := [0 \ 0 \ \cdots \ 0 \ 1]_{r \times 1}^T. \tag{3.16}$$

Let  $P$  be the positive definite solution of the following Lyapunov equation:

$$A_c^T P + P A_c = -I, \tag{3.17}$$

$$\lambda_{\max}(P) := \text{the maximum eigenvalue of } P, \tag{3.18}$$

$$\lambda_{\min}(P) := \text{the minimum eigenvalue of } P. \tag{3.19}$$

**Assumption 1.** For all  $t \geq 0, \eta \in \mathfrak{R}^{n-r}$  and  $\xi \in \mathfrak{R}^r$ , there exists a positive constant  $L$  such that the following inequality holds:

$$\|q_{22}(t, \eta, \bar{e}) - q_{22}(t, \eta, 0)\| \leq M(\|\bar{e}\|), \tag{3.20}$$

where  $q_{22}(t, \eta, \bar{e}) := q(\xi, \eta)$ .

**Assumption 2.** There exists known function  $\beta_2(\cdot) : \mathfrak{R}^n \rightarrow \mathfrak{R}^+$  such that

$$\|K_x^T X\| \leq \beta_2 \|\bar{e}\|, \tag{3.21}$$

where  $K_x := [k_{x1} \ k_{x2} \ \cdots \ k_{xn}]_{n \times 1}^T$  and  $k_{xi}, \ i = 1, 2, \dots, n$ , are real constants. For the sake of stating precisely the investigated problem, define

$$d := L_g L_f^{r-1} h(X(t)), \quad c := L_f^r h(X(t)) \tag{3.22a}$$

and

$$\bar{e} := \alpha_1 \bar{e}_1 + \alpha_2 \bar{e}_2 + \cdots + \alpha_r \bar{e}_r. \tag{3.22b}$$

**Definition 1 ([15]).** Consider the system  $\dot{x} = f(t, x, \theta)$ , where  $f : [0, \infty) \times \mathfrak{R}^n \times \mathfrak{R}^n \rightarrow \mathfrak{R}^n$  is piecewise continuous in  $t$  and locally Lipschitz in  $x$  and  $\theta$ . This system is said to be input-to-state stable if there exists a class  $KL$  function  $\beta$ , a class  $K$  function  $\gamma$  and positive constants  $k_1$  and  $k_2$  such that for any initial state  $x(t_0)$  with  $\|x(t_0)\| < k_1$  and any bounded input  $\theta(t)$  with  $\sup_{t \geq t_0} \|\theta(t)\| < k_2$ , the state exists and satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma \left( \sup_{t_0 \leq \tau \leq t} \|\theta(\tau)\| \right) \tag{3.23a}$$

for all  $t \geq t_0 \geq 0$ . Now we formulate the almost disturbance decoupling problem as follows:

**Definition 2 ([25]).** The tracking problem with almost disturbance decoupling is said to be globally solvable by the state feedback controller  $u$  for the transformed-error system by a global diffeomorphism (3.6), if the controller  $u$  enjoys the following properties:

- (i) It is input-to-state stable with respect to disturbance inputs.
- (ii) For any initial value  $\bar{x}_{e0} := [\bar{e}(t_0) \ \eta(t_0)]^T$ , for any  $t \geq t_0$  and for any  $t_0 \geq 0$

$$|y(t) - y_d(t)| \leq \beta_{11}(\|x(t_0)\|, t - t_0) + \frac{1}{\sqrt{\beta_{22}}} \beta_{33} \left( \sup_{t_0 \leq \tau \leq t} \|\theta(\tau)\| \right) \tag{3.23b}$$

and

$$\int_{t_0}^t [y(\tau) - y_d(\tau)]^2 d\tau \leq \frac{1}{\beta_{44}} \left[ \beta_{55}(\|\bar{x}_{e0}\|) + \int_{t_0}^t \beta_{33}(\|\theta(\tau)\|^2) d\tau \right], \tag{3.23c}$$

where  $\beta_{22}, \beta_{44}$  are some positive constants,  $\beta_{33}, \beta_{55}$  are class  $K$  functions and  $\beta_{11}$  is a class  $KL$  function.

**Definition 3.** Consider the following dynamical system:

$$\dot{z}(t) = f(t, z(t)), \quad z \in \mathfrak{R}^p, \quad z(t_0) := z_0,$$

where  $z \in \mathfrak{R}^p$  is the state and  $f(\cdot)$  is a smooth function. We use  $z(t; t_0, z_0)$  to denote the solution of system with  $z(t_0; t_0, z_0) = z_0$ . A closed set  $S$  is called a global final attractor for the trajectories  $z(\cdot) : [t, \infty) \rightarrow \mathfrak{R}^p, z(t_0) = z_0$ , of the system, if for any initial state  $z_0$ , there exists a finite constant  $T(z_0, S) \in [0, \infty)$  such that

$$z(t; t_0, z_0) \in S, \quad \forall t \geq t_0 + T(z_0, S).$$

**Theorem 1.** Suppose that there exists a continuously differentiable function  $V_0 : \mathfrak{R}^{n-r} \rightarrow \mathfrak{R}^+$  such that the following three inequalities hold for all  $\eta \in \mathfrak{R}^{n-r}$ :

$$(a) \quad \omega_1 \|\eta\|^2 \leq V(\eta) \leq \omega_2 \|\eta\|^2, \quad \omega_1, \omega_2 > 0 \tag{3.24a}$$

$$(b) \quad \nabla_t V + (\nabla_\eta V)^T q_{22}(t, \eta, 0) \leq -2\alpha_x \|\eta\|^2, \quad \alpha_x > 0 \tag{3.24b}$$

$$(c) \quad \|\nabla_\eta V\| \leq \varpi_3 \|\eta\|, \quad \varpi_3 > 0, \tag{3.24c}$$

then the tracking problem with almost disturbance decoupling is globally solvable by the controller defined by

$$u_{feedback} = \left[ L_g L_f^{r-1} h(X(t)) \right]^{-1} \left\{ -L_f^r h(X) + y_d^{(r)} - \varepsilon^{-r} \alpha_1 \left[ L_f^0 h(X) - y_d \right] - \varepsilon^{1-r} \alpha_2 \left[ L_f^1 h(X) - y_d^{(1)} \right] - \dots - \varepsilon^{-1} \alpha_r \left[ L_f^{r-1} h(X) - y_d^{(r-1)} \right] + K_x^T X \right\}, \tag{3.25}$$

where  $K := [k_1 \quad k_2 \quad \dots \quad k_n]$  is some adjustable real matrix, and the influence of disturbances on the  $L_2$  norm of the tracking error can be arbitrarily attenuated by increasing the following adjustable parameter  $NN_2$ :

$$H(\varepsilon) := \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} := \begin{bmatrix} 2\alpha_x - \frac{\omega_2^2}{\omega_1} \|\phi_\eta\|^2 & -\frac{1}{\sqrt{k(\varepsilon)}} \left[ \frac{w_3 M}{\sqrt{2w_1 \lambda_{\min}(P)}} \right] \\ -\frac{1}{\sqrt{k(\varepsilon)}} \left[ \frac{w_3 M}{\sqrt{2w_1 \lambda_{\min}(P)}} \right] & \frac{1}{\varepsilon \lambda_{\max}(P)} - \frac{2\varepsilon^r \|B^T P\| \beta_2 + 2(k(\varepsilon)/\varepsilon) \|\phi_\varepsilon\|^2 \|P\|^2}{\varepsilon \lambda_{\min}(P)} \end{bmatrix}, \tag{3.26a}$$

$$\alpha_s(\varepsilon) := \frac{H_{11} + H_{22} - \left[ (H_{11} - H_{22})^2 + 4H_{12}^2 \right]^{1/2}}{4}, \tag{3.26b}$$

$$N := 2\alpha_s(\varepsilon), \tag{3.26c}$$

$$N_2 := \min \left\{ \omega_1, \frac{k(\varepsilon)}{2} \lambda_{\min}(P) \right\} \tag{3.26d}$$

$$\phi_\varepsilon(\varepsilon) := \begin{bmatrix} \varepsilon \frac{\partial}{\partial X} h q_1^* & \dots & \varepsilon \frac{\partial}{\partial X} h q_p^* \\ \vdots & & \vdots \\ \varepsilon^r \frac{\partial}{\partial X} L_f^{r-1} h q_1^* & \dots & \varepsilon^r \frac{\partial}{\partial X} L_f^{r-1} h q_q^* \end{bmatrix} \tag{3.26e}$$

$$\phi_\eta(\varepsilon) := \begin{bmatrix} \frac{\partial}{\partial X} \phi_{r+1} q_1^* & \dots & \frac{\partial}{\partial X} \phi_{r+1} q_p^* \\ \vdots & & \vdots \\ \frac{\partial}{\partial X} \phi_n q_1^* & \dots & \frac{\partial}{\partial X} \phi_n q_q^* \end{bmatrix}, \tag{3.26f}$$

where  $H$  is positive definite matrix and  $k(\varepsilon) : \mathfrak{R}^+ \rightarrow \mathfrak{R}^+$  is any continuous function satisfies

$$\lim_{\varepsilon \rightarrow 0} k(\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{k(\varepsilon)} = 0. \tag{3.26g}$$

Moreover, the output tracking error of system (2.1) is exponentially attracted into a sphere  $B_r, r = \sqrt{\frac{N_1}{NN_2}}$ , with an exponential rate of convergence

$$\frac{1}{2} \left( \frac{NN_2}{\Delta_{\max}} - \frac{N_1}{\Delta_{\max} L^2} \right) := \frac{1}{2} \alpha^*, \tag{3.26h}$$

where

$$N_1 := \frac{1}{2} \left( \sup_{t_0 \leq \tau \leq t} \|\theta(\tau)\| \right)^2 \tag{3.26i}$$

and

$$\Delta_{\max} = \min \left\{ \omega_2, \frac{k}{2} \lambda_{\max}(P) \right\}. \tag{3.26j}$$

**Proof.** Applying the co-ordinate transformation (3.6) yields

$$\begin{aligned} \dot{\xi}_1(t) &= \frac{\partial \phi_1}{\partial X} \frac{dX}{dt} = \frac{\partial h(X(t))}{\partial X} \left[ f + g \cdot u + \sum_{i=1}^p q_i^* \theta_i \right] \\ &= L_f^1 h(X(t)) + L_g L_f^0 h(X(t)) u + \frac{\partial h(X)}{\partial X} \sum_{i=1}^p q_i^* \theta_i \\ &= L_f^1 h(X(t)) + \frac{\partial h(X)}{\partial X} \sum_{i=1}^p q_i^* \theta_i = \zeta_2(t) + \sum_{i=1}^p \frac{\partial h(X)}{\partial X} q_i^* \theta_i, \end{aligned} \tag{3.27}$$

⋮

$$\begin{aligned} \dot{\xi}_{r-1}(t) &= \frac{\partial \phi_{r-1}}{\partial X} \frac{dX}{dt} = \frac{\partial L_f^{r-2} h(X(t))}{\partial X} \left[ f + g \cdot u + \sum_{i=1}^p q_i^* \theta_i \right] \\ &= L_f^{r-1} h(X(t)) + L_g L_f^{r-2} h(X(t)) u + \frac{\partial L_f^{r-2} h(X(t))}{\partial X} \sum_{i=1}^p q_i^* \theta_i. \\ &= L_f^{r-1} h(X(t)) + \frac{\partial L_f^{r-2} h(X(t))}{\partial X} \sum_{i=1}^p q_i^* \theta_i = \zeta_r(t) + \sum_{i=1}^p \frac{\partial L_f^{r-2} h(X(t))}{\partial X} q_i^* \theta_i \end{aligned} \tag{3.28}$$

$$\begin{aligned} \dot{\xi}_r(t) &= \frac{\partial \phi_r}{\partial X} \frac{dX}{dt} = \frac{\partial L_f^{r-1} h(X(t))}{\partial X} \left[ f + g \cdot u + \sum_{i=1}^p q_i^* \theta_i \right] \\ &= L_f^r h(X(t)) + L_g L_f^{r-1} h(X(t)) u + \frac{\partial L_f^{r-1} h(X(t))}{\partial X} \sum_{i=1}^p q_i^* \theta_i. \\ &= L_f^r h(X) + L_g L_f^{r-1} h(X) u + \sum_{i=1}^p \frac{\partial L_f^{r-1} h(X(t))}{\partial X} q_i^* \theta_i = c + du + \sum_{i=1}^p \frac{\partial L_f^{r-1} h(X(t))}{\partial X} q_i^* \theta_i \end{aligned} \tag{3.29}$$

$$\begin{aligned} \dot{\eta}_k(t) &= \frac{\partial \phi_k(X)}{\partial X} \frac{dX}{dt} = \frac{\partial \phi_k(X)}{\partial X} \left[ f + g \cdot u + \sum_{i=1}^p q_i^* \theta_i \right] \\ &= \frac{\partial \phi_k(X)}{\partial X} f + \frac{\partial \phi_k(X)}{\partial X} g u + \frac{\partial \phi_k(X)}{\partial X} \sum_{i=1}^p q_i^* \theta_i \\ &= L_f \phi_k + \sum_{i=1}^p \frac{\partial \phi_k(X)}{\partial X} q_i^* \theta_i, \quad k = r + 1, r + 2, \dots, n. \end{aligned} \tag{3.30}$$

Since

$$c(\xi(t), \eta(t)) := L_f^r h(X(t)), \tag{3.31}$$

$$d(\xi(t), \eta(t)) := L_g L_f^{r-1} h(X(t)), \tag{3.32}$$

$$q_k(\xi(t), \eta(t)) = L_f \phi_k(X), \quad k = r + 1, r + 2, \dots, n, \tag{3.33}$$

the dynamic equations of system (3.1) in the new co-ordinates are shown as follows:

$$\dot{\xi}_i(t) = \xi_{i+1}(t), \sum_{i=1}^p \frac{\partial}{\partial X} L_f^{i-1} h q_i^* \theta_i, \quad i = 1, 2, \dots, r-1, \tag{3.34}$$

$$\dot{\xi}_r(t) = c(\xi(t), \eta(t)) + d(\xi(t), \eta(t))u + \sum_{i=1}^p \frac{\partial}{\partial X} L_f^{r-1} h q_i^* \theta_i, \tag{3.35}$$

$$\dot{\eta}_k(t) = q_k(\xi(t), \eta(t)), \sum_{i=1}^p \frac{\partial}{\partial X} \phi_k(X) q_i^* \theta_i, \quad k = r+1, \dots, n, \tag{3.36}$$

$$y(t) = \xi_1(t). \tag{3.37}$$

Define

$$v := y_d^{(r)} - \varepsilon^{-r} \alpha_1 [L_f^0 h(X) - y_d] - \varepsilon^{1-r} \alpha_2 [L_f^1 h(X) - y_d^{(1)}] - \dots - \varepsilon^{-1} \alpha_r [L_f^{r-1} h(X) - y_d^{(r-1)}] + K_x^T X. \tag{3.38}$$

According to Eqs. (3.7), (3.10), (3.31), (3.32) and (3.38), the tracking controller can be rewritten as

$$u = d^{-1}[-c + v]. \tag{3.39}$$

Substituting Eq. (3.39) into (3.35), the dynamic equations of system (3.1) can be shown as follows:

$$\begin{bmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \\ \vdots \\ \dot{\xi}_{r-1}(t) \\ \dot{\xi}_r(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_{r-1}(t) \\ \xi_r(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v + \begin{bmatrix} \sum_{i=1}^p \frac{\partial}{\partial X} h q_i^* \theta_i \\ \sum_{i=1}^p \frac{\partial}{\partial X} L_f^1 h q_i^* \theta_i \\ \vdots \\ \sum_{i=1}^p \frac{\partial}{\partial X} L_f^{r-1} h q_i^* \theta_i \end{bmatrix}, \tag{3.40}$$

$$\begin{bmatrix} \dot{\eta}_{r+1}(t) \\ \dot{\eta}_{r+2}(t) \\ \vdots \\ \dot{\eta}_{n-1}(t) \\ \dot{\eta}_n(t) \end{bmatrix} = \begin{bmatrix} q_{r+1}(t) \\ q_{r+2}(t) \\ \vdots \\ q_{n-1}(t) \\ q_n(t) \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^p \frac{\partial}{\partial X} \phi_{r+1} q_i^* \theta_i \\ \sum_{i=1}^p \frac{\partial}{\partial X} \phi_{r+2} q_i^* \theta_i \\ \vdots \\ \sum_{i=1}^p \frac{\partial}{\partial X} \phi_{n-1} q_i^* \theta_i \\ \sum_{i=1}^p \frac{\partial}{\partial X} \phi_n q_i^* \theta_i \end{bmatrix}, \tag{3.41}$$

$$y = [1 \quad 0 \quad \dots \quad 0 \quad 0]_{r \times 1} \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_{r-1}(t) \\ \xi_r(t) \end{bmatrix}_{r \times 1} = \xi_1(t). \tag{3.42}$$

Combining Eqs. (3.10), (3.12), (3.15) and (3.38), it can be easily verified that Eqs. (3.40)–(3.42) can be transformed into the following form:

$$\dot{\eta}(t) = q(\xi(t), \eta(t)) + \phi_\eta \theta := q_{22}(t, \eta(t), \bar{e}) + \phi_\eta, \theta \tag{3.43a}$$

$$\varepsilon \dot{\bar{e}}(t) = A_c \bar{e} + B \varepsilon^r K_x^T X + \phi_\xi \theta, \tag{3.43b}$$

$$y(t) = \xi_1(t). \tag{3.44}$$

We consider  $L(\bar{e}, \eta)$  defined by a weighted sum of  $V(\eta)$  and  $W(\bar{e})$ ,

$$L(\bar{e}, \eta) := V(\eta) + k(\varepsilon)W(\bar{e}), \tag{3.45}$$

as a composite Lyapunov function of the subsystems (3.43a) and (3.43b) [39,40], where  $W(\bar{e})$  satisfies

$$W(\bar{e}) := \frac{1}{2} \bar{e}^T P \bar{e} \tag{3.46}$$

In view of (3.17), (3.20), (3.21), (3.24) and (3.25), the derivative of  $L$  along the trajectories of (3.43a) and (3.43b) is given by



$$\begin{aligned}
 \dot{L} &= [\nabla_t V + (\nabla_\eta V)^T \dot{\eta}] + \frac{k}{2} [(\dot{\bar{e}})^T P \bar{e} + \bar{e}^T P (\dot{\bar{e}})] \\
 &= [\nabla_t V + (\nabla_\eta V)^T \dot{\eta}] + \frac{k}{2} \left\{ \left[ \frac{1}{\varepsilon} A_c \bar{e} + B \varepsilon^{r-1} (K_x^T X) + \frac{1}{\varepsilon} \phi_\xi \theta \right]^T P \bar{e} + \bar{e}^T P \left[ \frac{1}{\varepsilon} A_c \bar{e} + B \varepsilon^{r-1} (K_x^T X) + \frac{1}{\varepsilon} \phi_\xi \theta \right] \right\} \\
 &= [\nabla_t V + (\nabla_\eta V)^T \dot{\eta}] + \frac{k}{2\varepsilon} [\bar{e}^T (A_c^T P + P A_c) \bar{e}] + \frac{k}{2} \varepsilon^{r-1} \{ 2B^T P \bar{e} (K_x^T X) \} + k \left\{ \frac{1}{\varepsilon} \theta^T \phi_\xi^T P \bar{e} \right\} \\
 &\leq [\nabla_t V + (\nabla_\eta V)^T q_{22}(t, \eta(t), 0)] + \|\nabla_\eta V\| \|q_{22}(t, \eta(t), \bar{e}) - q_{22}(t, \eta(t), 0)\| + \|\nabla_\eta V\| \|\phi_\eta\| \|\theta\| - \frac{k}{\varepsilon \lambda_{\max}(P)} W \\
 &\quad + k \varepsilon^{r-1} \{ \|B^T P\| \|\bar{e}\| \|K_x^T X\| \} + \frac{k}{\varepsilon} \|\theta\| \|\phi_\xi\| \|P\| \|\bar{e}\| \\
 &\leq -2\alpha_x V + \omega_3 \|\eta\| M \|\bar{e}\| + \omega_3 \|\eta\| \|\phi_\eta\| \|\theta\| - \frac{k}{\varepsilon \lambda_{\max}(P)} W + k \varepsilon^{r-1} \|B^T P\| \|\bar{e}\| \beta_2 \|\bar{e}\| + \frac{k}{\varepsilon} \|\theta\| \|\phi_\xi\| \|P\| \|\bar{e}\| \\
 &\leq -2\alpha_x V + \omega_3 \|\eta\| M \|\bar{e}\| + \omega_3 \|\eta\| \|\phi_\eta\| \|\theta\| - \frac{k}{\varepsilon \lambda_{\max}(P)} W + k \varepsilon^{r-1} \|B^T P\| \beta_2 \frac{W}{\frac{1}{2} \lambda_{\min}(P)} + \frac{k}{\varepsilon} \|\theta\| \|\phi_\xi\| \|P\| \|\bar{e}\| \\
 &\leq -2\alpha_x V + \omega_3 M \frac{1}{\sqrt{\omega_1}} \sqrt{V} \sqrt{W} \sqrt{\frac{2}{\lambda_{\min}(P)}} + \omega_3^2 \|\eta\|^2 \|\phi_\eta\|^2 + \frac{1}{4} \|\theta\|^2 - \frac{k}{\varepsilon \lambda_{\max}(P)} W + k \varepsilon^{r-1} \|B^T P\| \beta_2 \frac{W}{\frac{1}{2} \lambda_{\min}(P)} \\
 &\quad + \frac{k^2}{\varepsilon^2} \|\phi_\xi\|^2 \|P\|^2 \|\bar{e}\|^2 + \frac{1}{4} \|\theta\|^2 \\
 &\leq -\left( 2\alpha_x - \frac{\omega_3^2}{\omega_1} \|\phi_\eta\|^2 \right) V + 2 \left( \frac{\omega_3 M}{\sqrt{2\omega_1 \lambda_{\min}(P)}} \right) \sqrt{V} \sqrt{W} - \left( \frac{k}{\varepsilon \lambda_{\max}(P)} - \varepsilon^{r-1} \frac{k \|B^T P\| \beta_2}{\frac{1}{2} \lambda_{\min}(P)} - \frac{k^2 \|\phi_\xi\|^2 \|P\|^2}{\frac{\varepsilon^2}{2} \lambda_{\min}(P)} \right) W + \frac{1}{2} \|\theta\|^2 \\
 &= -[\sqrt{V} \quad \sqrt{kW}] H \begin{bmatrix} \sqrt{V} \\ \sqrt{kW} \end{bmatrix} + \frac{1}{2} \|\theta\|^2 \tag{3.47}
 \end{aligned}$$

i.e.

$$\dot{L} \leq -\lambda_{\min}(H)L + \frac{1}{2} \|\theta\|^2, \tag{3.48}$$

where  $\lambda_{\min}(H)$  denotes the minimum eigenvalue of the matrix  $H$ . Utilizing the fact that  $\lambda_{\min}(H) = 2\alpha_s$ , we obtain

$$\begin{aligned}
 \dot{L} &\leq -2\alpha_s L + \frac{1}{2} \|\theta\|^2 \leq -2\alpha_s (V + kW) + \frac{1}{2} \|\theta\|^2 \leq -2\alpha_s \left( \omega_1 \|\eta\|^2 + \frac{k}{2} \lambda_{\min}(P) \|\bar{e}\|^2 \right) + \frac{1}{2} \|\theta\|^2 \\
 &\leq -NN_2 (\|\eta\|^2 + \|\bar{e}\|^2) + \frac{1}{2} \|\theta\|^2. \tag{3.49}
 \end{aligned}$$

Define

$$\bar{e}_{1r} := \begin{bmatrix} \bar{e}_2 \\ \vdots \\ \bar{e}_r \end{bmatrix}. \tag{3.50}$$

Hence

$$\dot{L} \leq -NN_2 (\|\eta\|^2 + \|\bar{e}_1\|^2 + \|\bar{e}_{1r}\|^2) + \frac{1}{2} \|\theta\|^2. \tag{3.51}$$

Utilizing (3.51) easily yields

$$\int_{t_0}^t (y(\tau) - y_d(\tau))^2 d\tau \leq \frac{L(t_0)}{NN_2} + \frac{1}{2NN_2} \int_{t_0}^t \|\theta(\tau)\|^2 d\tau, \tag{3.52}$$

so that statement (3.23c) is satisfied. From (3.49), we get

$$\dot{L} \leq -NN_2 (\|y_{total}\|^2) + \frac{1}{2} \|\theta\|^2, \tag{3.53a}$$

where

$$\|y_{total}\|^2 := \|\bar{e}\|^2 + \|\eta\|^2. \tag{3.53b}$$

By virtue of [15, Theorem 5.2] (3.53a), implies the input-to-state stability for the closed-loop system. Furthermore, it is easy to see that

$$\Delta_{\min}(\|\bar{e}\|^2 + \|\eta\|^2) \leq L \leq \Delta_{\max}(\|\bar{e}\|^2 + \|\eta\|^2) \tag{3.54}$$

i.e.

$$\Delta_{\min}(\|y_{total}\|^2) \leq L \leq \Delta_{\max}(\|y_{total}\|^2), \tag{3.55}$$

where  $\Delta_{\min} := \min \{ \omega_1, \frac{k}{2} \lambda_{\min}(P) \}$  and  $\Delta_{\max} := \min \{ \omega_2, \frac{k}{2} \lambda_{\max}(P) \}$ . From (3.49) and (3.55), we get

$$\dot{L} \leq -\frac{NN_2}{\Delta_{\max}}L + \frac{1}{2} \left( \sup_{t_0 \leq \tau \leq t} \|\theta(\tau)\| \right)^2. \tag{3.56}$$

Hence,

$$L(t) \leq L(t_0)e^{-\frac{NN_2}{\Delta_{\max}}(t-t_0)} + \frac{\Delta_{\max}}{2NN_2} \left( \sup_{t_0 \leq \tau \leq t} \|\theta(\tau)\| \right)^2, \quad t \geq t_0, \tag{3.57}$$

which implies

$$|e_1(t)| \leq \sqrt{\frac{2L(t_0)}{k\lambda_{\min}(P)}}e^{-\frac{NN_2}{2\Delta_{\max}}(t-t_0)} + \sqrt{\frac{\Delta_{\max}}{k\lambda_{\min}(P)NN_2}} \left( \sup_{t_0 \leq \tau \leq t} \|\theta(\tau)\| \right). \tag{3.58}$$

So that statement (3.23b) is proved and then the tracking problem with almost disturbance decoupling is globally solved. Finally, we will prove that the sphere  $B_r$  is a global attractor for the output tracking error of system (3.1). From (3.53a) and (3.26i), we get

$$\dot{L} \leq -NN_2(\|y_{total}\|^2) + N_1. \tag{3.59}$$

For  $\|y_{total}\| > r$ , we have  $\dot{L} < 0$ . Hence any sphere defined by

$$B_r := \left\{ \begin{bmatrix} \bar{e} \\ \eta \end{bmatrix} : \|\bar{e}\|^2 + \|\eta\|^2 \leq r \right\} \tag{3.60}$$

is a global final attractor for the tracking error system of the nonlinear control systems (3.1). Furthermore, it is easy routine to see that, for  $y_{total} \notin B_r$ , we have

$$\frac{\dot{L}}{L} \leq \frac{-NN_2\|y_{total}\|^2 + N_1}{L} \leq \frac{-NN_2\|y_{total}\|^2 + N_1}{\Delta_{\max}\|y_{total}\|^2} \leq \frac{-NN_2}{\Delta_{\max}} + \frac{N_1}{\Delta_{\max}\|y_{total}\|^2} \leq \frac{-NN_2}{\Delta_{\max}} + \frac{N_1}{\Delta_{\max}r^2} := -\alpha^* \tag{3.61}$$

i.e.,

$$\dot{L} \leq -\alpha^*L.$$

According to the comparison theorem [41], we get

$$L(y_{total}(t)) \leq L(y_{total}(t_0)) \exp[-\alpha^*(t - t_0)].$$

Therefore,

$$\Delta_{\max}\|y_{total}\|^2 \leq L(y_{total}(t)) \leq L(y_{total}(t_0)) \exp[-\alpha^*(t - t_0)] \leq \Delta_{\max}\|y_{total}(t_0)\|^2 \exp[-\alpha^*(t - t_0)]. \tag{3.62}$$

Consequently, we get

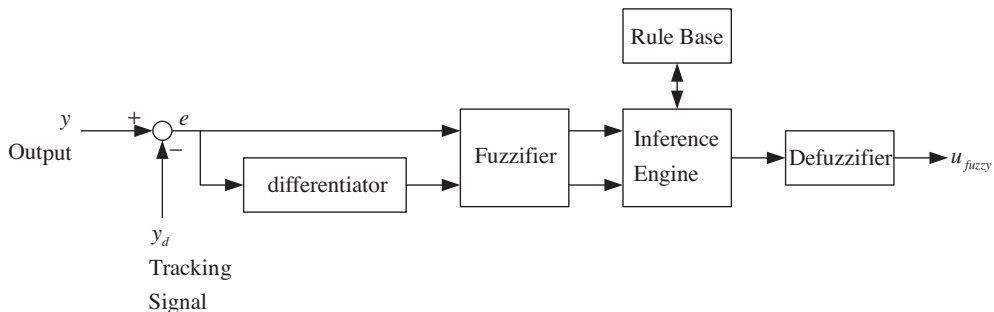


Fig. 3.1. Fuzzy logic controller.

$$\|y_{total}\| \leq \sqrt{\frac{A_{max}}{A_{min}}} \|y_{total}(t_0)\| \exp\left[-\frac{1}{2}\alpha^*(t-t_0)\right]$$

i.e., the convergence rate toward the sphere  $B_r$  is equal to  $\alpha^*/2$ . This completes our proof.  $\square$

### 2.2. Fuzzy controller design

After using feedback linearization control as a guarantee of uniform ultimate bounded stability, the multiple-input/single-output fuzzy control design can be technically applied via human expert’s knowledge to improve the convergence rate of tracking error. The block diagram of the fuzzy control is shown in Fig. 3.1. In general, the tracking error  $e(t)$  and its time derivative  $\dot{e}(t)$  are utilized as the input fuzzy variables of the IF-THEN control rules and the output is the control variable  $u_{fuzzy}$ .

For the sake of easy computation, the membership functions of the linguistic terms for  $e(t)$ ,  $\dot{e}(t)$  and  $u_{fuzzy}$  are all chosen to be the triangular shape function. We define seven linguistic terms: PB (Positive big), PM (Positive medium), PS (Positive small), ZE (Zero), NS (Negative small), NM (Negative medium) and NB (Negative big), for each fuzzy variable, as shown in Fig. 3.2.

Fuzzy control rule table for  $u_{fuzzy}$  is shown in Fig. 3.3. The rule base is heuristically built by the standard Macvicar–Whelan rule base [42] for usual servo control systems. The Mamdani method is used for fuzzy inference. The defuzzification of the output set membership value is obtained by the centroid method. Therefore, we can combine the designs of feedback linearization control and fuzzy control to construct the overall controller as follows:

$$u_{fe+fu} := u_{feedback}u_s(t) + u_{fuzzy}u_s(t-t_1) = \left[L_g L_f^{r-1} h(X(t))\right]^{-1} \{-L_f^r h(X) + y_d^{(r)} - \varepsilon^{-r}\alpha_1 [L_f^0 h(X) - y_d] - \varepsilon^{1-r}\alpha_2 [L_f^1 h(X) - y_d^{(1)}] - \dots - \varepsilon^{-1}\alpha_r [L_f^{r-1} h(X) - y_d^{(r-1)}] + K_x^T X\} u_s(t) + u_{fuzzy}u_s(t-t_1), \tag{3.63}$$

where  $u_s(t)$  denotes the unit step function and  $t_1$  is the time that the tracking error of system touch the global final attractor  $B_r$ .

Applying the Theorem 1 to the AMIRA’s ball and beam system now, we have finished some experiments and achieved the almost disturbance decoupling performance and the goal of finding a tracking controller  $u$  that will steer the angle of beam  $x_3$  and the position of ball  $x_1$ , starting from any initial values, to track the desired zero function (i.e.  $y_d = 0$ ). In order to achieve the goal, we choose  $h(X) = x_1 + x_2 + x_3 + x_4$ . Based on the constraint of hardware,  $h(X) \rightarrow 0$  implies  $x_1 \rightarrow 0$  and  $x_3 \rightarrow 0$ . Let’s arbitrarily choose  $\alpha_1 = 0.007$  and  $K_x = [-0.7 \ -0.7 \ -0.7 \ -0.7]^T$  such that  $A_c = -0.007$  is Hurwitz and  $P = 71.43$ . The

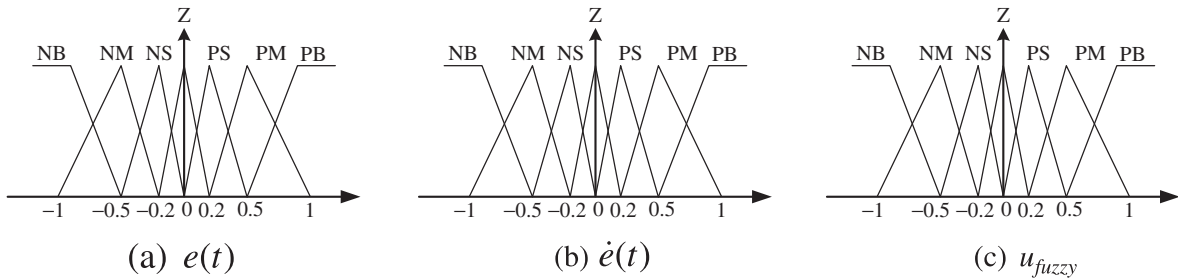


Fig. 3.2. Membership functions for (a)  $e(t)$ , (b)  $\dot{e}(t)$  and (c)  $u_{fuzzy}$ .

$u_{fuzzy}$		$e(t)$						
		NB	NM	NS	ZE	PS	PM	PB
$\dot{e}(t)$	NB	PB	PB	PB	PB	PM	PS	ZE
	NM	PB	PB	PB	PM	PS	ZE	NS
	NS	PB	PB	PM	PS	ZE	NS	NM
	ZE	PB	PM	PS	ZE	NS	NM	NB
	PS	PM	PS	ZE	NS	NM	NB	NB
	PM	PS	ZE	NS	NM	NB	NB	NB
	PB	ZE	NS	NM	NB	NB	NB	NB

Fig. 3.3. Fuzzy control rule base.

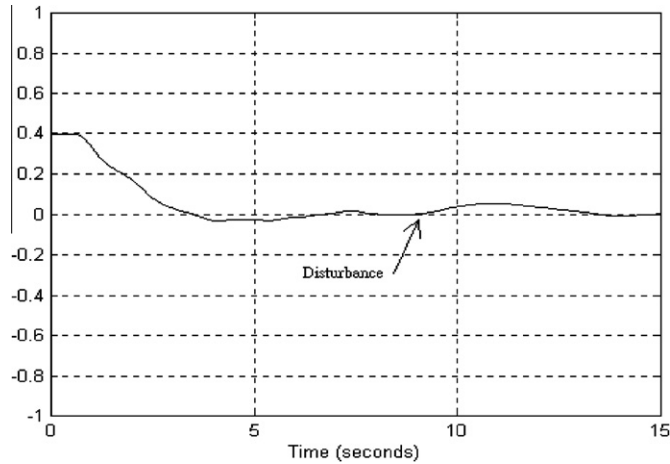


Fig. 3.4. The tracking error dynamic driven by  $u_{fe+fu}$  for (2.3).

AMIRA's ball and beam system is a system of relative degree one. It can be verified that with the choice  $V(\eta) = \eta_2^2 + \eta_3^2 + \eta_4^2$ , conditions (3.24) and (3.26) are satisfied with  $\varepsilon = 0.0025, \beta_2 = 0.7, M = \sqrt{3}, \omega_1 = \omega_2 = 1, \alpha_X = 1, \omega_3 = 2, H_{11} = 2, H_{12} = -1.296, H_{22} = 3.8, N = 1.325$  and  $N_2 = 1$ . From (3.63), we obtain the desired tracking controller

$$u_{fe+fu} = \left(\frac{b_{11}}{b_{22}} + a\right)^{-1} \left\{ -3.5x_1 - 4.5x_2 - 3.5x_3 - 4.5x_4 - \frac{a_{12} + a_{34}}{a_5} - \frac{z_{12} + z_{34} + z_{56}}{z_7} \right\} u_s(t) + u_{fuzzy} u_s(t - t_1). \tag{3.64}$$

Hence the tracking controller will steer the angle of beam  $x_3$  and the position of ball  $x_1$  to track the desired trajectory  $y_d(t) = h(X) = x_1 + x_2 + x_3 + x_4 = 0$  in view of Theorem 1. The tracking errors driven by  $u_{fe+fu}$  for AMIRA ball and beam system (2.3) is depicted in Fig. 3.4.

### 3. Comparative example to some existing approaches

Marino et al. [23] exploits the fact that for nonlinear single-input single-output systems the almost disturbance decoupling problem cannot be solved, as the following example shows:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \tan^{-1}(x_2) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta(t) \tag{4.1a}$$

$$y(t) = x_1(t) := h(X(t)) \tag{4.1b}$$

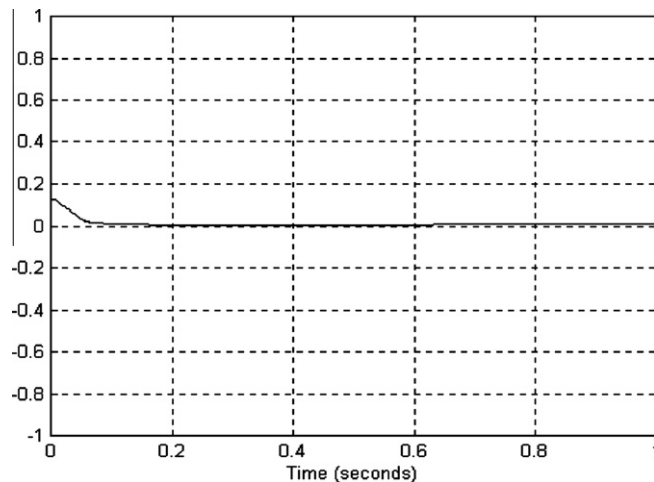


Fig. 4.1a. The tracking error dynamic driven by  $u_{fe+fu}$  for (4.1).

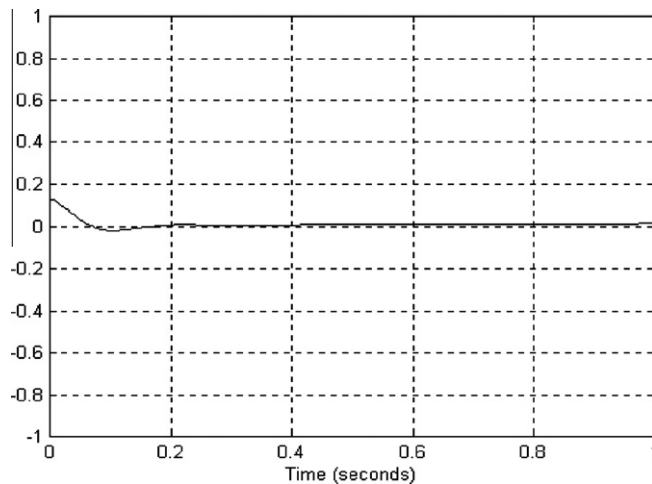


Fig. 4.1b. The tracking error dynamic driven by  $u_{feedback}$  for (4.1).

where  $u$ ,  $y$  denote the input and output respectively,  $\theta(t) = 0.5 \sin t$  is the disturbance. On the contrary, this problem can be easily solved via the proposed approach in this paper. Following the same procedures shown in the demonstrated example, the tracking problem with almost disturbance decoupling problem can be solved by the state feedback controller  $u$  defined as

$$u = (1 + x_2^2)[- \sin t - (0.03)^{-2}(x_1 - \sin t) - (0.03)^{-1}(\tan^{-1} x_2 - \cos t)]u_s(t) + u_{fuzzy}u_s(t - t_1). \quad (4.2)$$

The tracking error dynamics driven by  $u_{fe+fu}$  and  $u_{feedback}$  for (4.1) are depicted in Fig. 4.1a and 4.1b, respectively. It is easy to see that the convergence rate driven by both  $u_{feedback}$  and  $u_{fuzzy}$ , i.e.,  $u_{fe+fu}$ , is better than only by  $u_{feedback}$ .

#### 4. Conclusion

In this paper we have constructed a feedback control algorithm and fuzzy logic control which globally solves the tracking problem with almost disturbance decoupling for AMIRA's ball and beam system. The discussion and practical application of feedback linearization of nonlinear control systems by parameterized co-ordinate transformation have been presented. One comparative example is proposed to show the significant contribution of this paper with respect to some existing approaches. Moreover, a practical example of AMIRA's ball and beam system demonstrated the applicability of the proposed composite Lyapunov approach. Simulation results exploited the fact that the proposed methodology is successfully applied to feedback linearization problem and achieves the almost disturbance decoupling and convergence rate performances of the controlled system.

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