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Boundary influence on the entropy of a Lozi-type map

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ABSTRACT

Let T be a Henon-type map induced from a spatial discretization of a reaction–diffusion system. With the above-mentioned description of T, the following open problems were raised in [V.S. Afraimovich, S.B. Hsu, Lectures on Chaotic Dynamical Systems, AMS International Press, 2003]. Is it true that, in general, $h(T)=h_D(T)=h_N(T)=h_{\ell_{(1)},\ell_{(2)}}(T)$? Here $h(T)$ and $h_{\ell_{(1)},\ell_{(2)}}(T)$ (see Definitions 1.1 and 1.2) are, respectively, the spatial entropy of the system T and the spatial entropy of T with respect to the lines $\ell_{(1)}$ and $\ell_{(2)}$, and $h_D(T)$ and $h_N(T)$ are spatial entropy with respect to the Dirichlet and Neuman boundary conditions. If it is not true, then which parameters of the lines $\ell_{(i)}$, $i=1,2$, are responsible for the value of $h(T)$? What kind of bifurcations occurs if the lines $\ell_{(i)}$ move? In this paper, we show that this is in general not always true. Among other things, we further give conditions for which the above problem holds true.

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1. Introduction

Consider a discrete version of the reaction–diffusion

$$u_j(n+1)-u_j(n)=\beta\big(u_{j+1}(n)-2u_j(n)+u_{j-1}(n)\big)+\alpha h\big(u_j(n)\big),$$

where $j\in\mathbb{Z}$ is a spatial coordinate, $n\in\mathbb{Z}_+$ is the discrete time, $\alpha>0$ and $\beta>0$ are parameters and the nonlinearity is of the form

$$h(u)=u(u-a)(1-u),\quad 0<a<1.$$

The steady-state solutions $u_j(n)=u_j$, $j\in\mathbb{Z}$, satisfy the equation

$$0=\beta(u_{j+1}-2u_j+u_{j-1})+\alpha h(u_j) \tag{1.1a}$$

or

$$u_{j+1}=2u_j-\frac{\alpha}{\beta}h(u_j)-u_{j-1}. \tag{1.1b}$$

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If we set $u_{j-1} = x_j$, $u_j = y_j$, then (1.1a,b) can be re-written as the trajectory of a two-dimensional map T of the form

$$(x_{j+1}, y_{j+1}) = T(x_j, y_j) = \left(y_j, 2y_j - \frac{\alpha}{\beta} h(y_j) - x_j \right).$$

A map T of the form

$$T(x, y) = \big(y, F(y) - b\big), \tag{1.2}$$

where $F(y)$ is a polynomial of degree n with negative leading coefficient and distinct real roots, will henceforth be called an nth-degree Henon-type map. If $F(y)$ is replaced by n piecewise affine terms, then the corresponding map T is called an nth-degree Lozi-type map. Any bounded trajectory

$$\ldots, (x_j, y_j),\ (x_{j+1}, y_{j+1}), \ldots$$

of T corresponds to a bounded solution

$$\ldots, u_j(= y_j),\ u_{j+1}(= y_{j+1}), \ldots$$

of Eq. (1.1a,b). System (1.1a,b) is infinitely extended, i.e., $-\infty < j < \infty$. We shall next consider solutions $(u_j)_{j=1}^n$ of (1.1a,b) on a finite lattice, $1 \leqslant j \leqslant n$, where

$$u_{j+1} = 2u_j - \frac{\alpha}{\beta} h(u_j) - u_{j-1}, \quad 1 \leqslant j \leqslant n. \tag{1.3a}$$

We impose the Robin's boundary conditions of the form

$$u_1 = m_1 u_0 + k_1, \tag{1.3b}$$

and

$$u_{n+1} = m_2 u_n + k_2. \tag{1.3c}$$

Here m_i and k_i, $i = 1, 2$, are real numbers. Note that $(m_1, k_1) = (\infty, 0)$ and $(m_2, k_2) = (0, 0)$ (resp., $(m_1, k_1) = (m_2, k_2) = (1, 0)$) correspond to the Dirichlet (resp., Neumann) boundary conditions. It is then natural to ask how the behavior of solutions of (1.3a,b,c) is related to that of solutions of (1.1a,b). One quantity measuring such relationship is "spatial entropy". We next define the spatial entropy of the infinite system as well as that of the finite system. The following notion of the entropy of the system (1.1a,b) was introduced by Mallet-Paret and Chow [1]. Let $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ be given, set $\Gamma_{n,k}(T)$ to be the number of elements in the solution set $S_{n,k}$, where $S_{n,k} = \{\{u_i\}_{i=k}^{n+k-1}\colon \{u_i\}_{i=-\infty}^{\infty}$ is a bounded steady-state solution of (1.1a,b)$\}$. Note that if $\{u_i\}_{i=-\infty}^{\infty}$ is a steady state solution of (1.1a,b), so is $\{u_{i+k}\}_{i=-\infty}^{\infty}$ for any $k \in \mathbb{Z}$. Hence $\Gamma_{n,k}(T)$ is independent of the choice of k. Thus, we may write $\Gamma_{n,k}(T)$ as $\Gamma_n(T)$.

Definition 1.1. The spatial entropy $h(T)$ of system (1.1a,b) or the map T is defined to be the limit

$$h(T) = \overline{\lim_{n \to \infty}} \frac{\ln \Gamma_n(T)}{n}.$$

Inspired by the open problems raised in [2], we are led to define the following notion of the spatial entropy for the finite system. To this end, we first define the line $\ell_{(m,k)}$ as

$$\ell_{(m,k)} = \big\{(x, y) \in \mathbb{R}^2\colon\ y = mx + k\big\}. \tag{1.4a}$$

Here

$$\ell_{(\infty,k)} \text{ is interpreted as } \big\{(x, y) \in \mathbb{R}^2\colon\ x = k\big\}. \tag{1.4b}$$

Denote by $\mathcal{N}(n, \ell_{(m_1,k_1)}, \ell_{(m_2,k_2)}, T)$ the number of points on the intersection of

$$T^n \ell_{(m_1,k_1)} \cap \ell_{(m_2,k_2)}. \tag{1.5}$$

Should no ambiguity arise, we will write $\ell_{(m_i,k_i)}$ as $\ell_{(i)}$. We next elaborate on the meaning of the intersection $T^n \ell_{(m_1,k_1)} \cap \ell_{(m_2,k_2)}$. If $(s, t) \in T^n \ell_{(m_1,k_1)} \cap \ell_{(m_2,k_2)}$, then there exists a point $(u_0, u_1) \in \ell_{(m_1,k_1)}$ such that $(s, t) = T^n(u_0, u_1) := (u_n, u_{n+1}) \in \ell_{(m_2,k_2)}$. Moreover, the finite sequence $\mathbf{u}_n = (u_i)_{i=0}^{n+1}$ satisfies (1.3). Thus, each distinct point in $T^n \ell_{(m_1,k_1)} \cap \ell_{(m_2,k_2)}$ generates a distinct and finite pattern satisfying (1.3). Hence, $\mathcal{N}(n, \ell_{(m_1,k_1)}, \ell_{(m_2,k_2)}, T)$ denotes the number of solutions satisfying (1.3). We are now in a position to define the entropy of the finite system.

Definition 1.2.

(1) The spatial entropy of a finite n-lattice system (1.1a,b) is defined to be

$$h_{n,\ell_{(1)},\ell_{(2)}}(T)=\frac{\ln\mathcal{N}(n,\ell_{(1)},\ell_{(2)},T)}{n}.$$

(2) The spatial entropy $h_{\ell_{(1)},\ell_{(2)}}(T)$ of T with respect to lines $\ell_{(1)}$ and $\ell_{(2)}$ is defined to be the limit of the entropy of the finite system, that is,

$$h_{\ell_{(1)},\ell_{(2)}}(T)=\overline{\lim_{n\to\infty}}\frac{\ln\mathcal{N}(n,\ell_{(1)},\ell_{(2)},T)}{n}. \tag{1.6}$$

Notation 1.1. In the case of Dirichlet (resp., Neumann) boundary conditions, we write $h_{\ell_{(1)},\ell_{(2)}}(T)=h_D(T)$ (resp., $h_N(T)$).

In case that the growth rate of $\mathcal{N}(n,\ell_{(1)},\ell_{(2)},T)$ is super exponential, $h_{\ell_{(1)},\ell_{(2)}}(T)$ is defined to be ∞. Suppose T is a local holomorphic mapping, preserving the origin, and two lines $\ell_{(1)}$ and $\ell_{(2)}$ passing the origin. Suppose all the images $T^n\ell_{(1)}$ are smooth [3] or that everything is algebraic (see [4,5]). Then $h_{\ell_{(1)},\ell_{(2)}}(T)$ exists and is finite. In our case, $\mathcal{N}(n,\ell_{(1)},\ell_{(2)},T)\leqslant 3^n$ (see Section 2). The following open problems were then raise in [2].

(P1) Is it true that, in general, $h(T)=h_D(T)=h_N(T)=h_{\ell_1,\ell_2}(T)$?
(P2) If it is not true, then which parameters m_i and k_i, $i=1,2$, are responsible for the values of $h(T)$. What kind of bifurcations occurs if the lines $\ell_{(1)}$ and $\ell_{(2)}$ move?

The problems (P1) and (P2) are not easy to answer. It is then natural to replace the cubic nonlinearity in $h(u)$ by piece-wise linearity. Consequently, a Henon-type map T becomes a Lozi-type map T. Specifically, we consider a two-dimensional map T of the form

$$T(x,y)=\bigl(y,F(y)-bx\bigr), \tag{1.7a}$$

where

$$F(y)=\begin{cases} a_1y+\bar{a}_1, & \text{if } y\geqslant 1,\\ a_0y+\bar{a}_0, & \text{if } |y|\leqslant 1,\\ a_{-1}y+\bar{a}_{-1}, & \text{if } y\leqslant -1,\end{cases} \tag{1.7b}$$

where $\bar{a}_1$, $\bar{a}_0$ and $\bar{a}_1$ are so defined that $F(y)$ is continuous.

The purpose of this paper is then to study spatial entropies of a cubic Lozi-type map T, which, in turn, answer some questions related to two problems (P1) and (P2) for T being given as in (1.7). Specifically, under some mild conditions, we show that for any $\ell_{(1)}$ and $n\in\mathbb{N}$, except possibly a few pieces of $T^n\ell_{(1)}$, $T^n\ell_{(1)}$ is contained in an N-shaped tube for which its boundary points are ω-limit points of $T(\ell_{(1)})$. Moreover, we show under a stronger condition, as in (3.4), that the entropy $h_{\ell_{(1)},\ell_{(2)}}(T)$ of T with respect to $\ell_{(1)}$ and $\ell_{(2)}$ is independent of the choice of $\ell_{(1)}$. It is also shown that $h_D(T)=h_N(T)=\ln 3$, and that $h_{\ell_{(1)},\ell_{(2)}}(T)(=h_{\ell_{(2)}}(T))$ take on two distinct values $\ln 3$ and 0. The necessary and sufficient conditions on $\ell_{(2)}$ for which $h_{\ell_{(2)}}(T)=\ln 3$ are also obtained. Those results are recorded in Section 3. We remark that the problem of the asymptotic behavior of the number of points on the intersection $f^kL_1\cap L_2$, where L_1,L_2 are submanifolds of a smooth manifold, and f is a smooth map, is said to be a problem of dynamics of the intersection. These problems arise from various branches of analysis. There are some general results [3] obtained for such topics. However, no approaches are available to solve specific problems.

To conclude the introductory section, we describe some rich dynamical behaviors of Lozi maps [6]. A proof of the existence of strange attractions for Lozi maps was first obtained by Misiurewicz [7]. The existence of the Bowen–Ruelle measure for a class of Lozi-type maps were independently studied by Young [8], and Collet and Levy [9]. More recently, Soma and Kiriki proved that Lozi family has some shadowing property near its strange attractors [10].

2. Dynamics of certain maps induced from $T^n\ell_{(m,k)}$

To address the problems (P1) and (P2), we need to solve the dynamics of intersections as in (1.5). The first step, is to understand certain dynamics of $\ell_{(m,k)}$ under T. To this end, we begin with the calculation of $T\ell_{(m,k)}$. Now, for $m\neq 0$,

$$T\bigl(x',mx'+k\bigr)=\bigl(mx'+k,F\bigl(mx'+k\bigr)-bx'\bigr).$$

Setting $x = mx' + k,\ y = F(mx' + k) - bx'$, we obtain that

$$y = F(x) - \frac{b(x-k)}{m} = \begin{cases} (a_1 - \frac{b}{m})x + (\bar{a}_1 + \frac{bk}{m}), & \text{if } x \geqslant 1, \\ (a_0 - \frac{b}{m})x + (\bar{a}_0 + \frac{bk}{m}), & \text{if } |x| \leqslant 1, \\ (a_{-1} - \frac{b}{m})x + (\bar{a}_{-1} + \frac{bk}{m}), & \text{if } x \leqslant -1. \end{cases} \tag{2.1}$$

From (2.1), we see immediately that T consists of three dynamics. Each dynamics acts on the following regions:

$$R_1 = \{(x, y)\colon x \geqslant 1\}, \qquad R_0 = \{(x, y)\colon |x| \leqslant 1\} \quad \text{and} \quad R_{-1} = \{(x, y)\colon x \leqslant -1\}. \tag{2.2}$$

The dynamics on regions R_i, $i = 1, 0, -1$, are to be termed the ith dynamics, respectively. Given a straight line/line segment/half-line ℓ, the image of ℓ consists of possibly two half-lines/line segments ℓ_1 and ℓ_{-1} and one line segment ℓ_0. That is,

$$T\ell = \begin{cases} \ell_1, & \text{if } (x, y) \in \ell \text{ and } (x, y) \in R_1, \\ \ell_0, & \text{if } (x, y) \in \ell \text{ and } (x, y) \in R_0, \\ \ell_{-1}, & \text{if } (x, y) \in \ell \text{ and } (x, y) \in R_{-1}. \end{cases} \tag{2.3a}$$

Note that some of ℓ_1, ℓ_0, or ℓ_{-1} could be empty. We then define the notations $\ell_{i_1,i_2,\ldots,i_{n-1},i_n}$, $i_j \in \{-1, 0, 1\}$, $j = 1, 2, \ldots, n$, inductively as follows.

$$T\ell_{i_1,i_2,\ldots,i_{n-1}} = \begin{cases} \ell_{i_1,i_2,\ldots,i_{n-1},1}, & \text{if } (x, y) \in \ell_{i_1,i_2,\ldots,i_{n-1}} \text{ and } (x, y) \in R_1, \\ \ell_{i_1,i_2,\ldots,i_{n-1},0}, & \text{if } (x, y) \in \ell_{i_1,i_2,\ldots,i_{n-1}} \text{ and } (x, y) \in R_0, \\ \ell_{i_1,i_2,\ldots,i_{n-1},-1}, & \text{if } (x, y) \in \ell_{i_1,i_2,\ldots,i_{n-1}} \text{ and } (x, y) \in R_{-1}. \end{cases} \tag{2.3b}$$

Therefore, $T^n\ell_{(m,k)} = T^n\ell$ contains possibly 3^n pieces of half-lines and line segments. Using the above notations, we have

$$T^n\ell = \{\ell_{i_1,i_2,\ldots,i_n}\colon u_j \in \{-1, 0, 1\},\ j = 1, 2, \ldots, n\}. \tag{2.4}$$

To understand $T^n\ell$, it is natural to first consider the cases that $i_1 = i_2 = \cdots = i_n$. That is the case that ℓ has been applied by same dynamics repeatedly. To this end, we define, via (2.1), the following two-dimensional maps of the form

$$G_i(x, y) = \left(a_i - \frac{b}{x}, \bar{a}_i + \frac{b}{x}y\right) =: \big(g_{i,1}(x), g_{i,2}(x, y)\big). \tag{2.5}$$

We call $g_{i,1}(x)$, $i = 1, 0, -1$, the slope maps of T. Since $g_{i,1}(x)$, $i = 1, 0, -1$, denote, respectively, the slopes of ℓ_i. Here $\ell = \ell_{(x,y)}$. Moreover, $g_{i,2}(x, y)$ are to be termed the intercept maps. Because if we let $\ell_{(x,y)} = \ell$, then $g_{i,2}(x, y)$ denote, respectively, $i = -1, 0, 1$, the y-intercepts of ℓ_i. We next consider the dynamics of the slope and intercept maps $g_{i,1}$ and $g_{i,2}$, $i = 1, 0, -1$.

Proposition 2.1. *Let $b > 0$, $a_i > 2\sqrt{b}$, $i = 1, -1$ and $-a_0 > 2\sqrt{b}$. Then*

(i) *For $i = 1, -1$, $m^{\pm}_{i,\infty} := \frac{a_i \pm \sqrt{a_i^2 - 4b}}{2}$ are two fixed points of the slope maps $g_{i,1}$. For $i = 0$, $m^{-}_{0,\infty} := \frac{a_0 + \sqrt{a_0^2 - 4b}}{2}$ and $m^{+}_{0,\infty} := \frac{a_0 - \sqrt{a_0^2 - 4b}}{2}$ are two fixed points of the slope map $g_{0,1}$.*

(ii) *Moreover, the attracting interval of $m^{+}_{i,\infty}$, $i = 1, 0, -1$, is $R - \{m^{-}_{i,\infty}\}$. That is to say if $x \in R - \{m^{-}_{i,\infty}\}$, then, for $i = 1, 0, -1$, $\lim_{n\to\infty} g^n_{i,1}(x) = m^{+}_{i,\infty}$.*

(iii) *Suppose $a_i = 2\sqrt{b}$. Then $m^{+}_{i,\infty} = m^{-}_{i,\infty}$ is the globally attracting fixed point of $g_{i,1}$, $i = 1, 0, -1$.*

(iv) *If $x \notin (m^{-}_{i,\infty}, m^{+}_{i,\infty})$, $i = 1, -1$ (resp., $x \notin (m^{+}_{0,\infty}, m^{-}_{0,\infty})$), then $g_{i,1}(x)$, $i = 1, 0, -1$, converge to $m^{+}_{i,\infty}$ uniformly. That is, given $\varepsilon > 0$, there exists an N_ε, independent of x, such that $|g^n_{i,1}(x) - m^{+}_{i,\infty}| < \varepsilon$ whenever $n \geqslant N_\varepsilon$.*

Proof. We illustrate only $i = 1$. Clearly, two fixed points of $g_{1,1}$ are $m^{\pm}_{1,\infty}$. The attracting interval of $g_{1,1}$ can be easily concluded by using graphical analysis in Fig. 2.1. To prove (iv), let $x = a_1$. For $\varepsilon > 0$, then there exists an N such that $|g^n_{1,1}(a_1) - m^{+}_{1,\infty}| < \varepsilon$ whenever $n \geqslant N$. Let $x \in (m^{+}_{1,\infty}, a_1)$, clearly, for $\varepsilon > 0$, $|g^n_{1,1}(x) - m^{+}_{1,\infty}| < |g^n_{1,1}(a_1) - m^{+}_{1,\infty}| < \varepsilon$ whenever $n \geqslant N$. Now for $x \in (-\infty, m^{-}_{1,\infty}) \cup (a_1, \infty)$, we see that $g^3_{1,1}(x) \in (m^{+}_{1,\infty}, a_1)$. Thus, the assertion of Proposition 2.1(iv) for $i = 1$ holds by choosing $N_\varepsilon = N + 3$. The other part of the proof is similar and is thus omitted. □

Remark 2.1. Given $\ell_{(m,k)} = \ell$, we see from Fig. 2.1, that the slopes of $\ell_{i_1,i_2,\ldots,i_n}$ remain positive (resp., negative) for all $n \geqslant 3$. Here $i_1, i_2, \ldots, i_n \in \{-1, 1\}$ (resp., $\in \{0\}$).

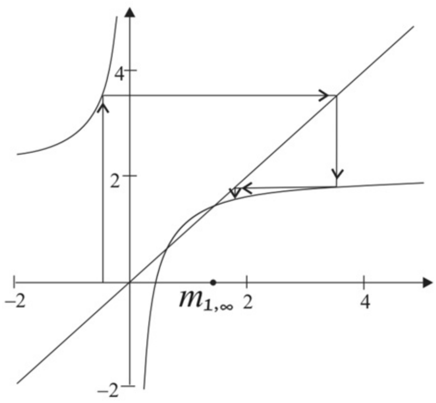

Fig. 2.1.

Proposition 2.2. *Suppose*

$$b > 0, \qquad a_i > 1 + b, \quad i = 1, -1 \quad \text{and} \quad -a_0 > 1 + b. \tag{2.6}$$

For fixed $x = m^+_{i,\infty}$, $i = 1, 0, -1$, the intercept maps $g_{i,2}(m^+_{i,\infty}, y)$ have fixed points $k_{i,\infty} := \frac{m^+_{i,\infty}\bar{a}_i}{m^+_{i,\infty} - b}$, which are globally attracting.

Proof. It suffices to show that $0 < \frac{b}{m^+_{i,\infty}} < 1$, $i = 1, -1$, and $-1 < \frac{b}{m^+_{0,\infty}} < 0$. We illustrate only $i = 1$. Now,

$$0 < \frac{b}{m^+_{1,\infty}} = \frac{2b}{a_1 + \sqrt{a_1^2 - 4b}} = \frac{a_1 - \sqrt{a_1^2 - 4b}}{2} < 1. \tag{2.7}$$

The last inequality is justified by the fact that $a_1 > 1 + b \geqslant 2\sqrt{b} > 0$. $\square$

Theorem 2.1. *Suppose* (2.6) *holds.*

(i) *The two-dimensional map G_i, as defined in* (2.5)*, $i = 1, 0, -1$, have two fixed points $(m^{\pm}_{i,\infty}, \frac{m^{\pm}_{i,\infty}\bar{a}_i}{m^{\pm}_{i,\infty} - b}) =: A^{\pm}_i$.*

(ii) *Moreover, the attracting regions of A^+_i, $i = 1, 0, -1$, are $\mathbb{R}^2 - \{(x, y)\colon\ x = m^-_{i,\infty}\}$. That is to say, for any $(m, k) \in \mathbb{R}^2 - \{(x, y)\colon\ x = m^-_{i,\infty}\}$, $i = 1, 0, -1$, $\lim_{n\to\infty} G^n_i(m, k) = A^+_i$.*

Proof. We only illustrate $i = 1$. The cases for $i = 0, -1$ are similar. Set $g^n_{1,1}(m) = m_{1,n}$. Let $m \neq m^-_{1,\infty}$. Given $\varepsilon > 0$, there exists an $N_\varepsilon \in \mathbb{N}$ such that for every $n \geqslant N_\varepsilon$, we have

$$m^+_{1,\infty} - \varepsilon < m_{1,n} < m^+_{1,\infty} + \varepsilon. \tag{2.8}$$

It follows from (2.8) that for any $k \in \mathbb{R}$, and n sufficiently large,

$$\min\left\{\bar{a}_1 + \frac{bk}{m^+_{1,\infty} - \varepsilon}, \bar{a}_1 + \frac{bk}{m^+_{1,\infty} + \varepsilon}\right\} < \bar{a}_1 + \frac{bk}{m_{1,n}} < \max\left\{\bar{a}_1 + \frac{bk}{m^+_{1,\infty} - \varepsilon}, \bar{a}_1 + \frac{bk}{m^+_{1,\infty} + \varepsilon}\right\}. \tag{2.9}$$

It follows from (2.7) and Proposition 2.2 that for all sufficiently small $\varepsilon > 0$, $\lim_{n\to\infty} g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k)$ exists and equals to

$$\frac{\bar{a}_1(m^+_{1,\infty} \pm \varepsilon)}{m^+_{1,\infty} \pm \varepsilon - b} =: k_{1\pm\varepsilon,\infty}. \tag{2.10}$$

Using (2.9), we see inductively that

$$\min\{g^n_{1,2}(m^+_{1,\infty} + \varepsilon, k), g^n_{1,2}(m^+_{1,\infty} - \varepsilon, k)\} < g^n_{1,2}(m_{1,n}, k) < \max\{g^n_{1,2}(m^+_{1,\infty} + \varepsilon, k), g^n_{1,2}(m^+_{1,\infty} - \varepsilon, k)\}. \tag{2.11}$$

However, it is easy to see that the single limits $\lim_{n\to\infty} g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k)$ and $\lim_{\varepsilon\to 0} g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k)$ exist and the convergence of $\lim_{n\to\infty} g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k)$ is uniform for all sufficiently small $\varepsilon > 0$. So the double limit and both iterated limits of $g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k)$ exist and all three limits are equal. However, $\lim_{\varepsilon\to 0}\lim_{n\to\infty} g^n_{1,2}(m^+_{1,\infty} \pm \varepsilon, k) = \lim_{\varepsilon\to 0} k_{1\pm\varepsilon,\infty} = k_{1,\infty}$.

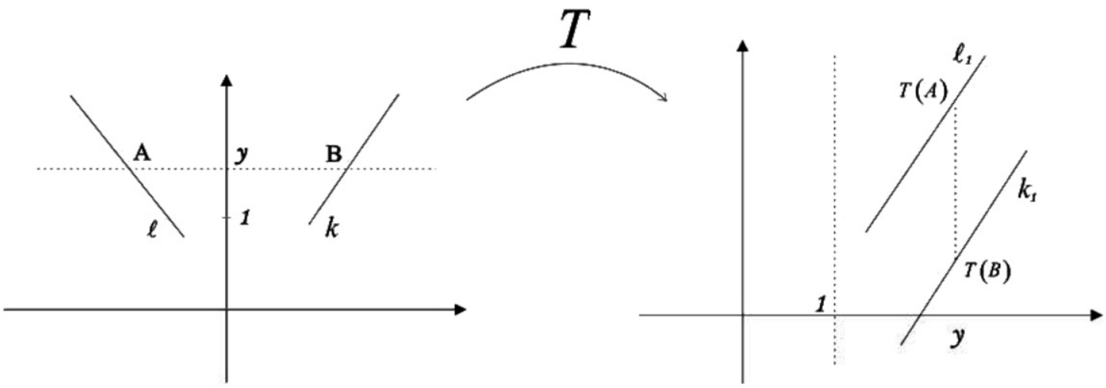

Fig. 3.1.

Taking the double limit on (2.11), we see that the double limit of $g^n_{1,2}(m^+_{1,n}, k)$ exists and equals to $k_{1,\infty}$. It is then easy to see that, for $(m, k) \in \mathbb{R}^2 - \{(x, y)\colon\ x = m^-_{i,\infty}\}$, $\lim_{n\to\infty} G^n_1(m, k) = (m^+_{1,\infty}, k_{1,\infty})$. We thus complete the proof of theorem. □

It then follows from Theorem 2.1 that for any $\ell = \ell_{(m,k)}$, $m \neq m^-_{i,\infty}$, $i = 1, 0, -1$, the notations $\ell_{i,i,\dots,i,\dots}$, applying the ith dynamics on ℓ infinitely many times, are well defined. The resulting images are denoted by ℓ_{i_∞}, where, for $i = 1, 0, -1$,

$$\ell_{i_\infty} = \ell_{(m',k')}, \quad \left(m', k'\right) = A^+_i. \tag{2.12}$$

3. Boundary influence on the spatial entropy of the finite system

In Section 2, we are able to understand the behavior of $\ell_{(m,k)}$ under the same dynamics. However, $T^n\ell_{(m,k)}$, as indicated in (2.4), consists of more images that are obtained via combinations of various dynamics. In this section, we begin with establishing some comparison principles, as in Proposition 3.1, to get a better understanding of $T^n\ell_{(m,k)}$. The following lemma is very useful in determining how the order of the line segments and half-lines $\ell_{i_1,i_2,\dots,i_n}$, $i_j \in \{-1, 0, 1\}$, $j = 1, 2, \dots, n$, is. The proof is trivial and, thus, skipped.

Lemma 3.1. *Let $b > 0$. For fixed y, if $x_1 \geqslant x_2$, then the y-coordinate of $T(x_1, y)$ is no greater than that of $T(x_2, y)$.*

Remark 3.1. Since our objective here is to study how the number of points in the intersection $T^n\ell_{(m_1,m_k)} \cap \ell_{(m_2,k_2)}$ grows as n increases, we may assume from here on by Remark 2.1 that the slopes of ℓ_1 and ℓ_{-1} are positive and that of ℓ_0 is negative.

Proposition 3.1 *(Comparison principles). Suppose* (2.6) *holds.*

(1) *Let ℓ and k be lines or line segments or half-lines, and $\ell \cap k = \emptyset$. Suppose k is to the right of ℓ. Then so are k_i to ℓ_i, for $i = 1, -1$. But, ℓ_0 is to the right of k_0. Here k_i, ℓ_i, $i = 1, 0, -1$ are defined in* (2.3a)*.*
(2) *Let $\{i_m\}^n_{m=1}$ and $\{j_m\}^n_{m=1}$ be two distinct finite sequences, i_m and $j_m \in \{1, 0, -1\}$, $m = 1, 2, \dots, n$. Suppose k is the first index such that $i_\ell = j_\ell$ for all $\ell \geqslant k$. Then $\ell_{i_1,i_2,\dots,i_n}$ is to the right of $\ell_{j_1,j_2,\dots,j_n}$ provided that either* (a) *or* (b) *holds.*
   (a) *$i_k = 1$ or -1, and $i_{k-1} > j_{k-1}$.*
   (b) *$i_k = 0$ and $i_{k-1} < j_{k-1}$.*

Proof. Using Remark 3.1, Lemma 3.1 and the fact that T is one-to-one, we have the first comparison principle of the proposition. See Fig. 3.1 for one special case. The second comparison principle of the proposition follows inductively from the first comparison principle. □

Note that the reverse of the ordering in k_0 and ℓ_0 is due to the fact that, in R_0, $F(y)$ has a negative slope. We next show that all $\ell_{i_1,i_2,\dots,i_n}$, except 6 possibly line segments/half-lines, lie in an N-shaped tube whose boundaries are given in Fig. 3.2, where ℓ_{i_∞}, $i = 1, 0, -1$, are defined in (2.12) and $\ell_{i_\infty,j}$, $j = 1, 0, -1$, mean the jth dynamics being applied to ℓ_{i_∞}.

Proposition 3.2. *Suppose* (2.6) *holds. For any straight line ℓ and $n \in \mathbb{N}$, we have that $\ell_{i_1,i_2,\dots,i_n}$, $i_j \in \{-1, 0, 1\}$, $j = 1, 2, \dots, n$, lie in the N-shaped tube, except possibly for those $\ell_{1,\dots,1,i_n}$, $\ell_{-1,\dots,-1,i_n}$, $i_n = 1, 0$, or -1. Here the boundaries of the N-shaped tube are ℓ_{1_∞}, $\ell_{1_\infty,0}$, $\ell_{1_\infty,-1}$, ℓ_{-1_∞}, $\ell_{-1_\infty,0}$ and $\ell_{-1_\infty,1}$. See Fig. 3.2.*

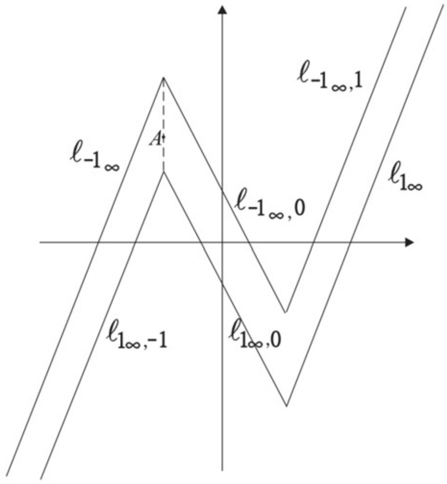

Fig. 3.2.

Proof. Assume $\ell_{i_1,i_2,\dots,i_{n-1},1} \neq \ell_{1,1,\dots,1}$. Let j be the index for which $i_j \neq 1$ and $i_k = 1$, for all $j < k \leqslant n-1$. Then $\ell_{i_1,i_2,\dots,i_{j-1},i_j} \in R_{-1}$ or R_0. Thus, $\ell_{i_1,i_2,\dots,i_j}$ is to the left of ℓ_{1_∞}. Therefore, it follows from the second comparison principle that $\ell_{i_1,i_2,\dots,i_j,i_{j+1},\dots,i_{n-1},1}$ is to the left of $\ell_{1_\infty,1,\dots,1} = \ell_{1_\infty}$. Hence, all $\ell_{i_1,i_2,\dots,i_{n-1},1}$, $i_j \in \{1,0,-1\}$, $1 \leqslant j \leqslant n-1$, are to the left of ℓ_{1_∞} except possibly $\ell_{1,1,\dots,1}$. The proof for the other parts of (2) is similar. $\square$

We note that the boundary points of the N-shaped tube are ω-limit points $\omega(\ell_1;T)$ of $T(\ell_1)$. That is, if $B \in \omega(\ell_1;T)$, then there exist an $A \in \ell_1$, and a sequence $\{n_k\}_{k=1}^{\infty}$, $n_k \in \mathbb{N}$, such that $T^{n_k}(A) \to B$ as $k \to \infty$.

To ensure that each of $\ell_{i_1,i_2,\dots,i_n}$, $i_j \in \{1,0,-1\}$, $1 \leqslant j \leqslant n$, is nonempty, we need the following lemma.

Lemma 3.2. *Let*

$$b > 0, \qquad \min\{a_1, a_{-1}\} \geqslant -a_0 > 1 + 4b, \quad \textit{and} \quad \bar{a}_0 \textit{ is sufficiently small.} \tag{3.1}$$

Then the y-coordinate $(\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})_y$ of $(\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})$ is less than -1, and $(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y > 1$.

Proof. We illustrate only $(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y > 1$. The other assertion is similarly obtained. Note that the equation of the line ℓ_{1_∞} is $y = m^+_{1,\infty}x + k_{1,\infty}$. Letting $y = -1$, we have that $x = \frac{-k_{1,\infty}-1}{m^+_{1,\infty}}$. Clearly,

$$(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y = \text{the } y\text{-coordinate of } T\left(\frac{-k_{1,\infty}-1}{m^+_{1,\infty}}, -1\right) = -a_0 + \bar{a}_0 + \frac{b(k_{1,\infty}+1)}{m^+_{1,\infty}} =: t. \tag{3.2}$$

Now, taking $\bar{a}_0 = 0$, we have

$$0 < \frac{-k_{1,\infty}-1}{m^+_{1,\infty}} < \frac{-k_{1,\infty}}{m^+_{1,\infty}} = \frac{a_1 - a_0}{m^+_{1,\infty} - b} \leqslant \frac{2a_1}{m^+_{1,\infty} - b} \leqslant \frac{4a_1}{a_1 + (\sqrt{a_1^2 - 4b} - 2b)} \leqslant 4.$$

Thus, for $\bar{a}_0 = 0$, we have

$$t \geqslant -a_0 - 4b > 1. \tag{3.3}$$

We just completed the proof the lemma. $\square$

Remark 3.2. Since our objective is to study the spatial entropy of the system, without loss of generality, we may assume, via Proposition 3.2 and Lemma 3.2, that all $\ell_{i_1,i_2,\dots,i_n}$, $i_j \in \{1,0,-1\}$, $1 \leqslant j \leqslant n$, are nonempty provided that (3.1) holds.

We next give stronger conditions on a_i, $i = 1, 0$, or -1, so as to ensure that for any fixed $j \in \mathbb{N}$, $\ell_{i_1,i_2,\dots,i_j,\dots,i_{n+j}}$, where $i_j = \cdots = i_{j+n} = 1$ or $i_j = \cdots = i_{j+n} = -1$ become unbounded as n grows larger.

Lemma 3.3. *Suppose*

$$b > 0, \qquad \frac{1}{2}\min\{a_1, a_{-1}\} \geqslant -a_0 \geqslant 3 + 4b \quad \textit{and} \quad \textit{that } \bar{a}_0 \textit{ is sufficiently small.} \tag{3.4}$$

Let A, as indicated in Fig. 3.2, be any point in the line segment for which its both endpoints are $\ell_{-1_\infty} \cap \ell_{-1_\infty,0}$ and $\ell_{1_\infty,-1} \cap \ell_{1_\infty,0}$ (resp., $\ell_{1_\infty} \cap \ell_{1_\infty,0}$ and $\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1}$). Then the limit of both coordinates of $T^n(A)$ approaches to $+\infty$ (resp., $-\infty$).

Proof. We first note that T has a fixed point $B = (\frac{a_1-a_0-\bar{a}_0}{a_1-1-b}, \frac{a_1-a_0-\bar{a}_0}{a_1-1-b})$ in $\mathbb{R}^2$ for which its stable (resp., unstable) direction is $(1, \frac{a_1-\sqrt{a_1^2-4b}}{2})$ (resp., $(1, \frac{a_1+\sqrt{a_1^2-4b}}{2})$). Since $(\ell_{-1_\infty} \cap \ell_{-1_\infty,0})_y > (\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})_y > 1$, it suffices to show, via Lemma 3.2, that $T^n(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0}) \to (+\infty,+\infty)$ as $n \to \infty$. To this end, we need to show that $T(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0}) = T(-1,t)$, t as given in (3.2), lies on the upper half of the stable line

$$\left(y - \frac{a_1-a_0-\bar{a}_0}{a_1-1-b}\right) = m^-_{1,\infty}\left(x - \frac{a_1-a_0-\bar{a}_0}{a_1-1-b}\right),$$

or, equivalently,

$$F(t) + b - \frac{a_1-a_0-\bar{a}_0}{a_1-1-b} - m^-_{1,\infty}t + m^-_{1,\infty}\frac{a_1-a_0-\bar{a}_0}{a_1-1-b} =: h(\bar{a}_0) > 0.$$

Now,

$$h(0) = a_1t + a_0 - a_1 - \frac{a_1-a_0}{a_1-1-b} + b - m^-_{1,\infty}t + m^-_{1,\infty}\frac{a_1-a_0}{a_1-1-b}. \tag{3.5}$$

We also have that with $\bar{a}_0 = 0$,

$$\begin{aligned} b - m^-_{1,\infty}t = b - \frac{2b}{a_1+\sqrt{a_1^2+4b}}t &\geqslant b - \frac{2b}{a_1+\sqrt{a_1^2+4b}}(-a_0) \\ &\geqslant b - \frac{b(-a_0)}{a_1} > 0, \end{aligned} \tag{3.6}$$

and

$$\frac{a_1-a_0}{a_1-1-b} \leqslant a_1 - a_0. \tag{3.7}$$

It then follows from (3.3), (3.5), (3.6) and (3.7) that

$$h(0) > a_1(-a_0-4b) + 2(a_0-a_1) = a_1(-a_0-4b-2) + 2a_0 \geqslant a_1 + 2a_0 \geqslant 0.$$

We thus complete the proof of the lemma. □

The first main results of the paper are stated in the following.

Theorem 3.1. *Let* (3.4) *hold.*

(i) *Suppose $a_1 > a_{-1}$. Let $\ell_{(2)}$ be a line satisfying the following*:
   (a) $(\ell_{(2)} \cap \ell_{(\infty,1)})_y \leqslant (\ell_{1_\infty} \cap \ell_{1_\infty,0})_y$, *where $\ell_{(\infty,1)}$ is defined in* (1.4b).
   (b) $m^+_{-1,\infty} \leqslant m \leqslant m^+_{1,\infty}$, *where $m^+_{-1,\infty}$ and $m^+_{1,\infty}$ are defined in Proposition* 2.1 *and m is the slope of $\ell_{(2)}$.*
   Then $h_{\ell_{(1)},\ell_{(2)}}(T) = 0$ for any $\ell_{(1)}$; otherwise, $h_{\ell_{(1)},\ell_{(2)}}(T) = \ln 3$.

(ii) *Suppose $a_1 < a_{-1}$. Let $\ell_{(2)}$ be a line satisfying the following*:
   (c) $(\ell_{(2)} \cap \ell_{(\infty,-1)})_y \geqslant (\ell_{-1_\infty} \cap \ell_{-1_\infty,0})_y$.
   (d) $m^+_{1,\infty} \leqslant m \leqslant m^+_{-1,\infty}$.
   Then $h_{\ell_{(1)},\ell_{(2)}}(T) = 0$ for any $\ell_{(1)}$; otherwise, $h_{\ell_{(1)},\ell_{(2)}}(T) = \ln 3$.

(iii) *Suppose $a_1 = a_{-1}$. Let $\ell_{(2)} = \ell_{(a_1,\bar{k})}$ with $\bar{k} \geqslant k_{-1,\infty}$ or $\bar{k} \leqslant k_{1,\infty}$, where $k_{1,\infty}$ and $k_{-1,\infty}$ are defined in Proposition* 2.2. *Then $h_{\ell_{(1)},\ell_{(2)}}(T) = 0$ for any $\ell_{(1)}$; otherwise, $h_{\ell_{(1)},\ell_{(2)}}(T) = \ln 3$.*

(iv) *$h_{\ell_{(1)},\ell_{(2)}}(T)$ is independent of the choice of $\ell_{(1)}$.*

Proof. Let $a_1 > a_{-1}$. We will break down $\ell_{(2)}$ into four cases.

(1) $(\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})_y > (\ell_{(2)} \cap \ell_{(\infty,1)})_y > (\ell_{1_\infty} \cap \ell_{1_\infty,0})_y$. See Fig. 3.3.
(2) $(\ell_{(2)} \cap \ell_{(\infty,1)})_y \geqslant (\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})_y$.

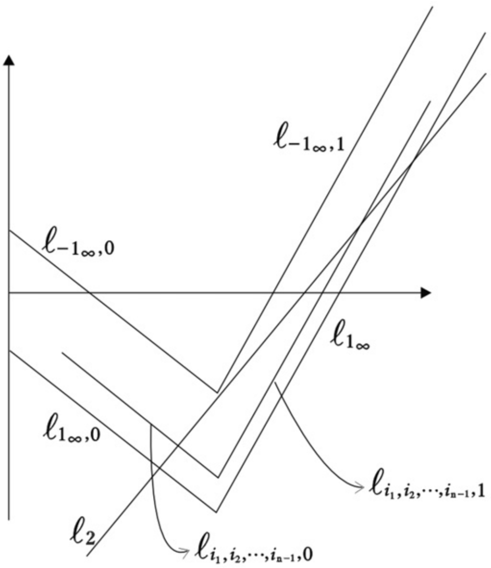

Fig. 3.3.

(3) $(\ell_{(2)} \cap \ell_{(\infty,1)})_y \leqslant (\ell_{1_\infty} \cap \ell_{1_\infty,0})_y$ and $m > m^+_{1,\infty}$ or $m < m^+_{-1,\infty}$.
(4) $\ell_{(2)}$ satisfies (a) and (b).

For the first case, we note, via Proposition 2.1(iv) and comparison principles, that for N sufficiently large, $\ell_{0,i_2,\dots,i_N}$, where $i_2 = i_3 = \cdots = i_N = 1$, we have that

$$(\ell_{(2)} \cap \ell_{(\infty,1)})_y > (\ell_{0,i_2,\dots,i_N} \cap \ell_{(\infty,1)})_y \quad \text{for any } \ell. \tag{3.8}$$

In particular, the natural number N is independent of the choice of ℓ. Therefore, for any $n \geqslant N$, by identifying $\ell = \ell_{j_1,j_2,\dots,j_{n-N}}$, we see that

$$\begin{aligned} &\ell_{j_{n-N+1},\dots,j_n} = \ell_{j_1,j_2,\dots,j_{n-N+1},\dots,j_n}, \quad \text{where } j_k \in \{1,0,-1\},\ \text{for } 1 \leqslant k \leqslant n-N, \\ &j_{n-N+1} = 0 \quad \text{and} \quad j_{n-N+2} = \cdots = j_n = 1, \quad \text{satisfying (3.8)}. \end{aligned} \tag{3.9}$$

Hence, $\ell_{(2)}$ must intersect with $\ell_{j_1,j_2,\dots,j_{n-1},0}$, where $j_1,\dots,j_{n-1}$ are given as in (3.9). See Fig. 3.3 for clarification. Thus, for any $n \geqslant N$, the number $\mathcal{N}(n,\ell_{(1)},\ell_{(2)},T)$ of intersections of $T^n\ell_{(1)} \cap \ell_{(2)}$ satisfies

$$3^{n-N} \leqslant \mathcal{N}(n,\ell_{(1)},\ell_{(2)},T) \leqslant 3^n.$$

It then follows that $h_{\ell_{(1)},\ell_{(2)}}(T) = \ln 3$. If case (2) or case (3) holds, then $\ell_{(2)}$ must intersect with ℓ_{-1_∞} and $\ell_{1_\infty,-1}$ or $\ell_{-1_\infty,0}$ and $\ell_{1_\infty,0}$ or $\ell_{-1_\infty,1}$ and ℓ_{1_∞}. Upon using Lemma 3.3, we conclude that $h_{\ell_1,\ell_2}(T) = \ln 3$. For the last case, we see immediately, via Proposition 3.2, $\mathcal{N}(n,\ell_{(1)},\ell_{(2)},T)$ is smaller or equal than 6. Consequently, $h_{\ell_{(1)},\ell_{(2)}}(T) = 0$. The proof for $a_1 < a_{-1}$ is similar and thus omitted. The case for $a_1 = a_{-1}$ is obvious and thus omitted. The last part of the theorem is a direct consequence of the first three parts of the theorem. □

4. Spatial entropy of the infinite system

In this section, we will study the entropy of system (1.1a,b). We begin with the following lemma. To make sure that $a_1, a_{-1}, -a_0$ and b are all positive, we shall assume that $a > 1$.

Lemma 4.1. *Suppose*

$$b > 0, \qquad -a_0 > 2 + 4b, \qquad \min\{a_1, a_{-1}\} > \max\{-a_0, 4 + 7b\} \quad \textit{and} \quad \bar{a}_0 \textit{ is sufficiently small}. \tag{4.1}$$

Then $T(-1,t) = T(\ell_{1_\infty,-1} \cap \ell_{1_\infty,0})$ (resp., $T(\ell_{-1_\infty,0} \cap \ell_{-1_\infty,1})$) stays above (resp., below) the line $y = x$, where t is given as in (3.2).

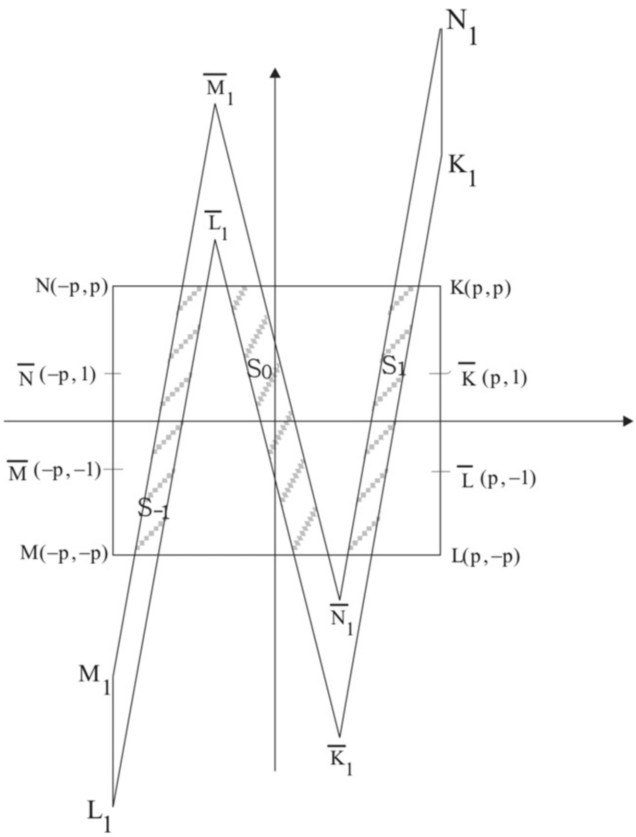

Fig. 4.1. Here we denote by $K_1 = T(K)$. We use similar notations to denote points under the first iteration of T.

Proof. We only illustrate the proof of the first assertion of the lemma. To this end, we first note that (4.1) implies (3.1). We then need to show that

$$F(t) + b > t. \tag{4.2}$$

Letting $\bar{a}_0 = 0$, (4.2) becomes

$$(a_1 - 1)t + a_0 - a_1 + b := L > 0. \tag{4.3}$$

Using (3.3), we see, via (4.1), that

$$L \geqslant (a_1 - 1)(-a_0 - 4b) + a_0 - a_1 + b = (-a_0 - 1 - 4b)(a_1 - 2) - 2 - 7b > 0.$$

The proof of the theorem is thus complete. □

Let S be a square defined as

$$S = \{(x, y) \in \mathbb{R}^2 \colon |x| \leqslant p,\ |y| \leqslant p\},$$

where $p > 1$. Then $T(S) \cap S = S_{-1} \cup S_0 \cup S_1$. See Fig. 4.1.

Inductively, we see that $T^n(S) \cap S$ consists of 3^n nested pieces of $S_{i_1,i_2,\dots,i_n}$, $i_j = 1, 0, -1$, $j = 1, 2, \dots, n$. Likewise, backward iterations: $T^{-n}(S) \cap S$ will produce 3^n nested pieces of $\bar{S}_{i_1,i_2,\dots,i_n}$, $i_j = 1, 0, -1$, $j = 1, 2, \dots, n$ with each piece $\bar{S}_{i_1,i_2,\dots,i_n}$ crossing the east and west sides of the rectangle S. Let A be a point in $\mathbb{R}^2$. $T(A)$ is denoted by A_1. Let $K = (p, p)$, $\bar{L} = (p, -1)$, $\bar{N} = (-p, 1)$ and $M = (-p, -p)$. If

$$K_1 \text{ and } \bar{L}_1 \text{ stay above } y = p, \tag{4.4a}$$

and

$$\bar{N}_1 \text{ and } M_1 \text{ stay below } y = -p, \tag{4.4b}$$

then each of $S_{i_1,i_2,\dots,i_n}$ is nonempty. The following lemma gives a sufficient condition on the parameters a_i, $i = 1, 0, -1$ and b so that (4.4) holds.

Lemma 4.2. *Suppose*

$$b>0,\qquad (-a_0-1-b)\big(\min\{a_1,a_{-1}\}-2(1+b)\big)>2(1+b)^2\quad \textit{and}\quad \bar{a}_0 \textit{ is sufficiently small}. \tag{4.5}$$

Then there exists a $p>1$ such that the following results hold.

$$F(p)-bp>p, \tag{4.6a}$$

$$F(1)+bp<-p, \tag{4.6b}$$

$$F(-1)-bp>p, \tag{4.6c}$$

and

$$F(-p)+bp<-p. \tag{4.6d}$$

Proof. Eq. (4.6) is equivalent to

$$\min\left\{\frac{-a_0+\bar{a}_0}{1+b},\frac{-a_0-\bar{a}_0}{1+b}\right\}>p>\max\left\{\frac{a_1-a_0-\bar{a}_0}{a_1-1-b},\frac{a_{-1}-a_0+\bar{a}_0}{a_{-1}-1-b}\right\}. \tag{4.7}$$

Letting $\bar{a}_0=0$, (4.7) reduces to

$$\frac{-a_0}{1+b}>p>\max\left\{\frac{a_1-a_0}{a_1-1-b},\frac{a_{-1}-a_0}{a_{-1}-1-b}\right\}. \tag{4.8}$$

Clearly, $-\frac{a_0}{1+b}>1$. Thus, if

$$\frac{-a_0}{1+b}>\max\left\{\frac{a_1-a_0}{a_1-1-b},\frac{a_{-1}-a_0}{a_{-1}-1-b}\right\}, \tag{4.9}$$

then there exists a $p>1$ such that (4.8) holds. However, the assumptions (4.5) would yield (4.9). The proof of the lemma is thus complete. □

Remark 4.1. Note that $T^{-1}(x,y)=(\frac{1}{b}F(x)-\frac{y}{b},x)$. Replacing a_i with $\frac{a_i}{b}$, b with $\frac{1}{b}$ and $\bar{a}_0$ with $\frac{\bar{a}_0}{b}$, we see that (4.5) is invariant. That is $(-\frac{a_0}{b}-1-\frac{1}{b})(\min\{\frac{a_1}{b}\frac{a_{-1}}{b}\}-2(1+\frac{1}{b}))>2(1+\frac{1}{b})^2$ and $\frac{\bar{a}_0}{b}$ is sufficiently small if and only if (4.5) holds. Thus, (4.5) is not only to ensure that each of $S_{i_1,i_2,\dots,i_n}$ is nonempty but also that each of $\bar{S}_{i_1,i_2,\dots,i_n}$ is nonempty.

To show that T has a Smale–Horseshoe, we need to show that each of $S_{i_1,i_2,\dots,i_n}$ or $\bar{S}_{i_1,i_2,\dots,i_n}$ shrinks to a line segment as $n\to\infty$. To this end, we first need the following notations. Let the slope and intercept pairs of two straight lines ℓ and $\bar{\ell}$ are (m,k) and $(m,\bar{k})$, respectively. Here $k\neq\bar{k}$. We further assume that the slope and intercept pairs of ℓ_i and $\bar{\ell}_i$, $i\in\{1,0,-1\}$, are, respectively, (m_i,k_i) and $(m_i,\bar{k}_i)$. Define

$$d_{0,i}=|k-\bar{k}|,\quad i=1,0,-1, \tag{4.10a}$$

and

$$d_{1,i}=|k_{1,i}-\bar{k}_{1,i}|,\quad i=1,0,-1. \tag{4.10b}$$

Lemma 4.3. *Let $m>m^+_{i,\infty}$, $i=1,-1$, and $m<m^+_{0,\infty}$, respectively. Suppose*

$$b>0,\qquad a_i\geqslant 1+2b,\quad i=1,-1,\quad \textit{and}\quad -a_0\geqslant 1+2b,\quad \textit{respectively}. \tag{4.11}$$

Then, respectively,

$$d_{1,i}\leqslant\frac{1}{2}d_{0,i},\quad i=1,0,-1. \tag{4.12}$$

Consequently, each of $S_{i_1,i_2,\dots,i_n}$, $i_j\in\{1,0,-1\}$, $1\leqslant j\leqslant n$, shrinks to a line segment as $n\to\infty$.

Proof. We illustrate only the case $i=1$. Using (2.5), we see that

$$d_{n+1,1}=\frac{b}{m_{n,1}}d_{n,1}\leqslant\frac{b}{m^+_{1,\infty}}d_{n,1}.$$

The inequality above is justified by the fact that $m_{n,1}>m^+_{1,\infty}$ for all $n\in\mathbb{N}$. See Fig. 2.1 for clarification. However,

$$\frac{b}{m^+_{1,\infty}} = \frac{2b}{a_1 + \sqrt{a_1^2 - 4b}} \leqslant \frac{2b}{1 + 2b + \sqrt{1 + 4b^2}} = \frac{2}{\frac{1}{b} + 2 + \sqrt{\frac{1}{b^2} + 4}} \leqslant \frac{1}{2},$$

we thus complete the proof of the first part the lemma. It then follows from Remarks 2.1 and 3.1, and (4.12) that the size of each of $S_{i_1,i_2,\dots,i_n,}$ shrinks by a factor no greater than $\frac{1}{2}$ as one applies the ith dynamics, $i = 1, 0, -1$, on them. □

Remark 4.2. Replacing a_i with $\frac{a_i}{b}$, b with $\frac{1}{b}$, and substituting the corresponding quantities into (4.11), we have that the resulting inequalities become

$$b > 0, \qquad a_i \geqslant 2 + b, \quad i = 1, -1 \quad \text{and} \quad -a_0 \geqslant 2 + b. \tag{4.13}$$

Thus, as in Remark 4.1, (4.13) ensures that each of $\bar{S}_{i_1,i_2,\dots,i_n}$ shrinks to a line segment as $n \to \infty$. Consequently, if

$$\min\{a_1, a_{-1}, a_0\} \geqslant \max\{1 + 2b, 2 + b\}, \tag{4.14}$$

then the size of each of $S_{i_1,i_2,\dots,i_n}$ or $\bar{S}_{i_1,i_2,\dots,i_n}$ shrinks to a line segment as $n \to \infty$.

We are now in the position to state the second main results of the paper.

Theorem 4.1.

(i) *Suppose* (3.1) *holds. Then* $h_D(T) = \ln 3$.
(ii) *Suppose* (4.1) *holds. Then* $h(T) = h_N(T) = h_D(T) = \ln 3$.

Proof. Suppose (3.1) holds. The first assertion of the theorem follows from Lemma 3.1. Suppose (4.1) holds. Let $\Gamma_n =$ the number of intersections points of $\ell_{i_1,i_2,\dots,i_n}$, $i_j \in \{1, -1\}$, $1 \leqslant j \leqslant n$, and the line $y = x$. We see, via Lemma 4.1, that $3^n - 4 \leqslant \Gamma_n \leqslant 3^n$. Thus, $h_N(T) = \ln 3$. To prove $h(T) = \ln 3$, we first note that if (4.1) holds, then (4.5) and (4.14) are satisfied. Applying Lemmas 4.2 and 4.3, we see that $\bigcap_{n=-\infty}^{\infty} T^n(S) \cap S =: \Lambda_3$ is a Cantor set of infinite points. Let Σ_3 be the space the two sided sequences of 0's, 1's and -1's. Define the itinerary map $i : \Lambda_3 \to \Sigma_3$

$$i(P) = (\dots s_{-2}s_{-1}s_0s_1s_2\dots),$$

where

$$P \in \Lambda_3 \quad \text{and} \quad s_j = k \quad \text{if and only if} \quad T^j(P) \in S_k.$$

Impose a metric on Σ_3 by defining

$$d\big[(s_i)_{i=-\infty}^{\infty}, (t_i)_{i=-\infty}^{\infty}\big] = \sum_{i=-\infty}^{\infty} \frac{|s_i - t_i|}{3^{|i|}}.$$

Define the shift map σ by

$$\sigma\big((s_i)_{i=-\infty}^{\infty}\big) = (t_i)_{i=-\infty}^{\infty}, \quad \text{where } t_i = s_{i+1}.$$

It is then not difficult to show, as in e.g., Devaney [11] and Robinson [12], that the dynamics of T on the invariant set Λ_3 is conjugate to the shift map on Σ_3. Note that any trajectory of T in Λ_3 is a bounded solution of (1.1a,b). We just complete the proof of the theorem. □

In summary, it is shown that the problem raised by Afraimovich and Hsu [2] is in general not true. Sufficient conditions under which the problem holds true are also given. We conclude the paper with the following remarks.

(1) Is the open problem still true whenever the spatial entropy $h_{\ell_{(1)},\ell_{(2)}}(T)$ of the finite lattice system with respect to Robin's boundary conditions is positive?
(2) It is also of interest to study the boundary influence on the spatial entropies of Henon-type maps, such as when F, defined in (1.2), is a cubic polynomial or a quadratic map for which the resulting T is a Henon map.

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