J Supercomput (2010) 54: 229–238
DOI 10.1007/s11227-009-0316-3

A family of Hamiltonian and Hamiltonian connected graphs with fault tolerance

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Published online: 10 July 2009

Abstract Processor (vertex) faults and link (edge) faults may happen when a network is used, and it is meaningful to consider *networks* (*graphs*) with faulty processors and/or links. A k-regular *Hamiltonian* and *Hamiltonian connected* graph G is *optimal fault-tolerant Hamiltonian and Hamiltonian connected* if G remains Hamiltonian after removing at most $k-2$ vertices and/or edges and remains Hamiltonian connected after removing at most $k-3$ vertices and/or edges. In this paper, we investigate in constructing optimal fault-tolerant Hamiltonian and optimal fault-tolerant Hamiltonian connected graphs. Therefore, some of the *generalized hypercubes*, *twisted-cubes*, *crossed-cubes*, and *Möbius cubes* are optimal fault-tolerant Hamiltonian and optimal fault-tolerant Hamiltonian connected.

Keywords Fault-tolerance · Hamiltonicity · Hamiltonian connectivity · Generalized hypercube

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1 Introduction

The architecture of an interconnection network is usually represented by a graph. There exist conflicting requirements in designing the topology of interconnection networks. It is almost impossible to design a network which is optimum for all conditions. One has to design a suitable network depending on the required properties. The Hamiltonian property is one of the major requirements in designing the topology of networks. Fault tolerance is also desirable in massive parallel systems that have relatively high probability of failure. Much research have been proposed on the ring embedding problems in fault-tolerant networks [1, 8, 21, 23, 25, 27].

For the graph definitions and notations, we follow [4, 16]. $G = (V, E)$ is a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set*. The *degree* of a vertex v, denoted by $\deg(v)$, is the number of edges incident to v. A graph G is *k-regular* if $\deg(v) = k$ for every vertex in G. A path is a *Hamiltonian path* if its vertices are distinct and they span V. A *cycle* is a path with at least three vertices such that the first vertex is the same as the last one. A cycle is a *Hamiltonian cycle* if it traverses every vertex of G exactly once. A graph G is *Hamiltonian* if it has a Hamiltonian cycle, and G is *Hamiltonian connected* if there exists a Hamiltonian path joining any two vertices of G.

Since vertex faults and edge faults may happen when a network is used, it is practically meaningful to consider faulty networks. A graph G is called *l-fault-tolerant Hamiltonian* (*l-fault-tolerant Hamiltonian connected* respectively) or simply *l-Hamiltonian* (*l-Hamiltonian connected,* respectively) if it remains Hamiltonian (Hamiltonian connected, respectively), after removing at most l vertices and/or edges. The *fault-tolerant Hamiltonicity*, $\mathcal{H}_f(G)$, is defined to be the maximum integer l such that $G - F$ remains Hamiltonian for every $F \subset V(G) \cup E(G)$ with $|F| \le l$ if G is Hamiltonian, and undefined if otherwise. Obviously, $\mathcal{H}_f(G) \le \delta(G) - 2$, where $\delta(G) = \text{minc}\{\deg(v)|v \in V(G)\}$. In establishing their fault-tolerant Hamiltonicity, another parameter called *fault-tolerant Hamiltonian connectivity* is used. The *fault-tolerant Hamiltonian connectivity*, $\mathcal{H}_f^\kappa(G)$, is defined to be the maximum integer l such that $G - F$ remains Hamiltonian connected for every $F \subset V(G) \cup E(G)$ with $|F| \le l$ if G is Hamiltonian connected, and undefined if otherwise. It is not hard to see that $\mathcal{H}_f^\kappa(G) \le \delta(G) - 3$. A regular graph is called *optimal fault-tolerant Hamiltonian and optimal fault-tolerant Hamiltonian connected* if $\mathcal{H}_f(G) = \delta(G) - 2$ and $\mathcal{H}_f^\kappa(G) = \delta(G) - 3$. *Twisted-cubes*, *crossed-cubes*, *Möbius cubes*, and *recursive circulant graphs* are proved to be optimal fault-tolerant Hamiltonian and optimal fault-tolerant Hamiltonian connected [6, 7, 18–20, 26]. All these families of graphs have some good properties in common, including that they can all be recursively constructed.

The *complete graph* K_n is a fully connected network with high performance, but high cost. It has many desirable properties, such as small fixed diameter, maximum connectivity, shortest path routing, parallel paths, optimal fault-tolerant Hamiltonicity, and optimal fault-tolerant Hamiltonian connectivity, due to the high cost with fully connected links/edges between processors/vertices [14]. To reduce the cost and preserve the optimal fault-tolerant Hamiltonian and fault-tolerant Hamiltonian connected properties of the complete graph are the purposes of this paper.

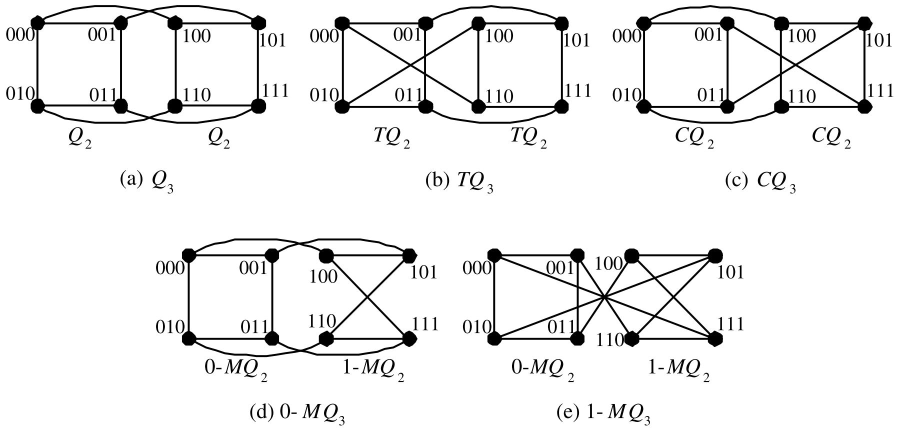


Fig. 1 (**a**) The hypercube Q_3; (**b**) the twisted-cube TQ_3; (**c**) the crossed-cube CQ_3; (**d**) the Möbius cube 0-MQ_3; and (**e**) the Möbius cube 1-MQ_3

Based on the recursively constructed graphs, the *hypercube* Q_n is an important one since it has a simple structure and easy to implement [5, 24]. There are many important variants of the hypercubes appearing in literature, such as *twisted-cubes* [12, 13, 20], *crossed-cubes* [15, 19, 22], and *Möbius cubes* [9, 18]. These variants possess some desirable features of the hypercubes, and even better. For example, the diameter of these variants is around half of that of the hypercube. The ring and path structure embedding into these variants have also been heavily discussed. Many people have studied the problem of the existence of cycles and paths of arbitrary lengths in these networks [2, 6, 11–13, 15, 17–20, 22]. The recursive structures of the hypercubes, twisted-cubes, crossed-cubes, and Möbius cubes are briefly illustrated below. Let X_n be any one of the n-dimensional hypercube, twisted-cube, crossed-cube, or Möbius cube. An X_n is composed of two copies of X_{n-1}s and a specific perfect matching between the two copies of X_{n-1}s. Figure 1 shows the 3-dimensional hypercube, twisted-cube, crossed-cube, and Möbius cube.

The *generalized hypercube* [3] is another variant of the hypercube and also has several good topologies. First, the design of generalized hypercubes is based on the allowable diameter of the network. If the diameter can be increased, a structure with a lower degree of a vertex can be obtained. Secondly, the structures are very general in nature. *Single loop*, *Boolean n-cubes*, *nearest neighbor mesh hypercubes*, and *fully connected systems* can be considered as a part of this generalized structure. Finally, the structures are highly fault-tolerant and they possess a small average message distance and a low traffic density. A generalized hypercube has been recursively defined in [10] as the following. Let $G(m_r, m_{r-1}, \ldots, m_1)$ denote a generalized hypercube graph of size $m_r \times m_{r-1} \times \cdots \times m_1$, where $m_i \geq 2$ for all $1 \leq i \leq r$. There are $N = m_r \times m_{r-1} \times \cdots \times m_1$ vertices in $G(m_r, m_{r-1}, \ldots, m_1)$ which are assigned r-digit identifiers $x_r x_{r-1} \cdots x_1$, where $x_i \in \{0, 1, \ldots, m_i - 1\}$ for all $1 \leq i \leq r$. Two vertices in $G(m_r, m_{r-1}, \ldots, m_1)$ are adjacent if and only if their identifiers differ at exactly one digit position. In Fig. 2, the structure of generalized hypercube $G(4, 2, 2)$ is depicted for illustration. It is clear that $G(m_r, m_{r-1}, \ldots, m_1)$ with $m_i = 2$ for

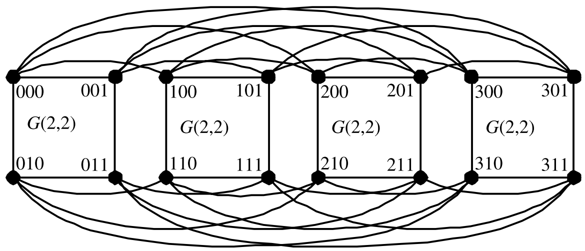


Fig. 2 The generalized hypercube $G(4, 2, 2)$

$1 \le i \le r$ is isomorphic to the hypercube Q_r. In this paper, the twisted-cube, crossed-cube, Möbius cube, and generalized hypercube are elements of the graph family generated by the proposed construction scheme in the next section. The graph family preserves the optimal fault-tolerant Hamiltonian and fault-tolerant Hamiltonian connected properties, and it has lower cost than the complete graph.

The rest of this paper is organized as follows. In Sect. 2, we give a recursively constructed scheme for optimal fault-tolerant Hamiltonian and Hamiltonian connected graphs; and introduce the notations and terminology. In Sect. 3, the optimal fault-tolerant Hamiltonicity and optimal fault-tolerant Hamiltonian connectivity are discussed. Section 4 shows the proof of optimal fault-tolerant Hamiltonicity and Sect. 5 concludes the brief contribution.

2 Construction schemes of fault-tolerant Hamiltonian graphs and some notations

Now, we construct a more generalized graph $G(G_1, G_2, \ldots, G_n, \bigcup_{1 \le i < j \le n} M_{i,j})$ based on the generalized hypercube and complete graph. Let $G_1, G_2, \ldots, G_n$ be k-regular graphs with the same number of vertices. The graph $H = G(G_1, G_2, \ldots, G_n, \bigcup_{1 \le i < j \le n} M_{i,j})$ is defined as follows. Graph H has vertex set $V(H) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n)$, and edge set $E(H) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_n) \cup \bigcup_{1 \le i < j \le n} M_{i,j}$, where $M_{i,j}$ is an arbitrary perfect matching between the vertices of G_i and G_j. See Fig. 3. We call each G_i a *component* for every i. Considering each component G_i as a vertex and each perfect matching $M_{i,j}$ as an edge, then $G(G_1, G_2, \ldots, G_n, \bigcup_{1 \le i < j \le n} M_{i,j})$ is reduced to a *complete graph* K_n. For convenience, we shall abbreviate $G(G_1, G_2, \ldots, G_n, \bigcup_{1 \le i < j \le n} M_{i,j})$ as $G_{(1..n)}$. As an example, generalized hypercubes are essentially constructed in this way. The $G_{(1..n)}$ is a $(k + n - 1)$-regular graph with $\frac{|V|}{2} \times (k + n - 1)$ edges, which has lower cost than that of the complete graph K_n with $\frac{|V|}{2} \times (|V| - 1)$ edges.

In this paper, we show that if all of $G_1, G_2, \ldots, G_n$ are optimal fault-tolerant Hamiltonian and optimal fault-tolerant Hamiltonian connected, then $G_{(1..n)}$ is also optimal fault-tolerant Hamiltonian and optimal fault-tolerant Hamiltonian connected for any arbitrary perfect matchings, $M_{i,j}$, provided $k \ge 5$.

All the proofs of fault-tolerant Hamiltonicity and Hamiltonian connectivity of the aforementioned families of graph are done by induction. We observe that there are certain common phenomena behind the recursive structures so that we may construct

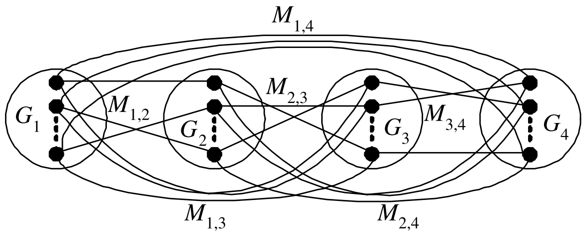


Fig. 3 A schematic diagram of graph $G_{(1..4)}$

other optimal fault-tolerant graphs. In this paper, we try to investigate these phenomena and establish a construction scheme for optimal fault-tolerant Hamiltonian and Hamiltonian connected graphs.

For ease of exposition, we make some convention about our notations. Consider the graph $G_{(1..n)}$. For each component G_i, we use small letters with subscript i to denote the vertices in G_i, e.g., u_i, v_i, etc. For example, u_1 is a vertex in G_1, and u_2 is a vertex in G_2. Let $G_{(i..j)}$ be the graph $G(G_i, G_{i+1}, \ldots, G_j, \bigcup_{i \le p < q \le j} M_{p,q})$. For each $G_{(i..j)}$, we use small letters with subscript $(i..j)$ to denote the vertices in $G_{(i..j)}$, e.g., $u_{(i..j)}, v_{(i..j)}$, etc. For example, $u_{(1..2)}$ is a vertex in $G_{(1..2)}$, and $u_{(2..5)}$ is a vertex in $G_{(2..5)}$. A perfect matching $M_{i,j}$ connecting the vertices of G_i and G_j in pairs, such pairs of vertices are called *matching vertices*, and these edges are called *matching edges*. We shall use the same letter with different subscripts to denote matching vertices of each other; e.g., u_i and u_j are the matching vertices of each other in components G_i and G_j if there is a perfect matching between G_i and G_j.

We shall also consider graphs with some faults. Our objective is to find a fault free Hamiltonian cycle (Hamiltonian path, respectively) and each fault can be a faulty vertex or a faulty edge. If a vertex v is not faulty, we say v is a *healthy vertex*. If an edge e is not faulty, we say e is a *healthy edge*. We call an edge e (respectively, a matching edge e) *super healthy* if both edge e and its two endpoints are not faulty. We use F_i to denote the set of faults in G_i, $F_{(i..j)}$ to denote the set of faults in $G_{(i..j)}$. Let $f_i = |F_i|$ and $f_{(i..j)} = |F_{(i..j)}|$. Given two distinct healthy vertices x and y, we use the x-y *Hamiltonian path* to call a fault free Hamiltonian path joining x and y, HP_i to denote a fault free Hamiltonian path in $G_i - F_i$, and $HP_{(i..j)}$ to denote a fault free Hamiltonian path in $G_{(i..j)} - F_{(i..j)}$ where $i \le j$. A fault free x-y Hamiltonian path in $G_i - F_i$ can be written as $\langle x, HP_i, y\rangle$ and a fault free x-y Hamiltonian path in $G_{(i..j)} - F_{(i..j)}$ can be written as $\langle x, HP_{(i..j)}, y\rangle$. In addition, path $\langle x, HP_i, y\rangle$ and path $\langle x, HP_{(i..j)}, y\rangle$ are cycles if $x = y$.

3 Hamiltonian and Hamiltonian connected graphs with fault tolerance

This section shows *optimal fault-tolerant Hamiltonicity* and *optimal fault-tolerant Hamiltonian connectivity* of the graph $G_{(1..n)}$.

Theorem 1 *For $n \ge 2$ and $k \ge 5$, let $G_1, G_2, \ldots, G_n$ be k-regular, $(k-2)$-Hamiltonian, and $(k-3)$-Hamiltonian connected graphs with the same number of vertices. Then $G(G_1, G_2, \ldots, G_n, \bigcup_{1 \le i < j \le n} M_{i,j}) = G_{(1..n)}$ is $(k-2+n-1)$-Hamiltonian and $(k-3+n-1)$-Hamiltonian connected.*

Proof We prove this theorem by inducting on n. It is proved in [6, 7] that for $n=2$ and $n=3$, the result holds for $G_{(1..2)}$ and $G_{(1..3)}$. So $G_{(1..2)}$ is $(k-2+1)$-Hamiltonian and $(k-3+1)$-Hamiltonian connected for $k \geq 5$. And $G_{(1..3)}$ is $(k-2+2)$-Hamiltonian and $(k-3+2)$-Hamiltonian connected for $k \geq 5$. For the induction step, we divide the proof into the following two lemmas. □

Lemma 1 *For $n \geq 3$ and $k \geq 5$, let $G_1, G_2, \ldots, G_n$ be k-regular, $(k-2)$-Hamiltonian, and $(k-3)$-Hamiltonian connected graphs with the same number of vertices. Then graph $G(G_1, G_2, \ldots, G_n, \bigcup_{1 \leq i < j \leq n} M_{i,j}) = G_{(1..n)}$ is $(k-2+n-1)$-Hamiltonian.*

Lemma 2 *For $n \geq 3$ and $k \geq 5$, let $G_1, G_2, \ldots, G_n$ be k-regular, $(k-2)$-Hamiltonian, and $(k-3)$-Hamiltonian connected graphs with the same number of vertices. Then graph $G(G_1, G_2, \ldots, G_n, \bigcup_{1 \leq i < j \leq n} M_{i,j}) = G_{(1..n)}$ is $(k-3+n-1)$-Hamiltonian connected.*

The proof of Lemma 1 is left in Sect. 4. The proof of Lemma 2 is rather long and similar to the proof of Lemma 1; we omit it here. With the above two lemmas, Theorem 1 is proved.

4 Proof of Lemma 1

Suppose $n \geq 3$. Assume that $G_{(1..d)}$ is $(k-2+d-1)$-Hamiltonian and $(k-3+d-1)$-Hamiltonian connected for all $d \leq n$ where $k \geq 5$. We shall show that graph $G_{(1..n+1)}$ is $(k-2+n)$-Hamiltonian where $k \geq 5$.

To show that the fault-tolerant Hamiltonicity $\mathcal{H}_f(G)$ of $G_{(1..n+1)}$ is $(k-2+n)$ for $k \geq 5$, it suffices to show that $G_{(1..n+1)} - F_{(1..n+1)}$ is Hamiltonian for any faulty set $F_{(1..n+1)} \subset V(G_{(1..n+1)}) \cup E(G_{(1..n+1)})$ with $|F_{(1..n+1)}| = f_{(1..n+1)} = k-2+n$. If $f_{(1..n+1)} < (k-2+n)$, we may arbitrary choose $(k-2+n) - f_{(1..n+1)}$ healthy edges and designate them as faulty, then the total number of faults is exactly $k-2+n$. In $G_{(1..n+1)}$, every component G_i is adjacent to n perfect matchings, $\bigcup_{1 \leq i \neq j \leq n+1} M_{i,j}$. For each $G_i \cup \bigcup_{1 \leq i \neq j \leq n+1} M_{i,j}$ for $1 \leq i \leq n+1$, we may without loss of generality assume that $G_{n+1} \cup \bigcup_{1 \leq j \leq n} M_{n+1,j}$ contains the most number of faults. So the number of faults of it must be greater or equal to $\lceil \frac{k-2+n}{n+1} \rceil \geq \lceil \frac{5-2+n}{n+1} \rceil \geq 2$. Thus, $f_{(1..n)} \leq (k-2+n) - 2 = k-3+n-1$. By induction, $G_{(1..n)} - F_{(1..n)}$ is Hamiltonian connected. Let f_{n+1} be the number of faults of G_{n+1}, this lemma is proved by three cases.

Case 1: $f_{n+1} \leq k-3$.

G_{n+1} is k-regular, so the number of vertices of G_{n+1} is $k+1$ at least. The number of matching edges between $G_{n+1} - F_{n+1}$ and $G_{(1..n)}$ is at least $n(k+1-f_{n+1})$, and the total number of faults is $k-2+n$, so there exists a super healthy matching edge $(u_{n+1}, u_{(1..n)})$. Suppose not, $n(k+1-f_{n+1}) \leq (f_{(1..n+1)} - f_{n+1})$, $n(k+1-f_{n+1}) \leq ((k-2+n) - f_{n+1})$, and $(n-1)(k-f_{n+1}) + 2 \leq 0$, which is a contradiction. Besides, we shall find another super healthy matching edge $(v_{n+1}, v_{(1..n)})$ such that

Fig. 4 Case 1: $f_{n+1} \le k-3$

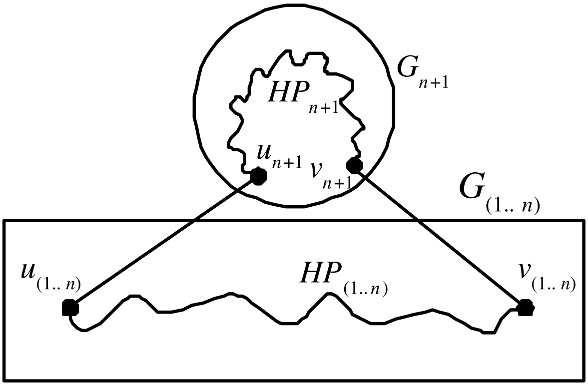
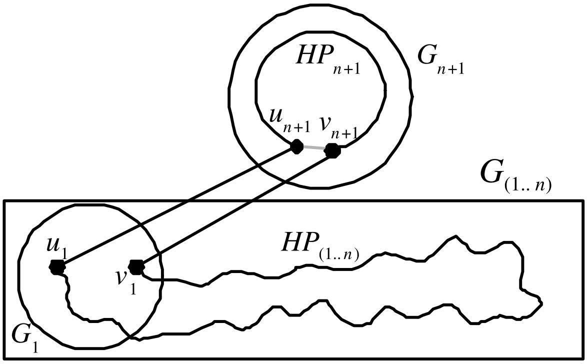


Fig. 5 Case 2: $f_{n+1} = k-2$



$u_{n+1} \ne v_{n+1}$ and $u_{(1..n)} \ne v_{(1..n)}$. Suppose not, the number of matching edges between $G_{n+1} - (F_{n+1} \cup \{u_{n+1}\})$ and $G_{(1..n)}$ is at least $n((k+1) - f_{n+1} - 1)$. Thus, $n((k+1) - f_{n+1} - 1) \le (f_{(1..n+1)} - f_{n+1})$, $n((k+1) - f_{n+1} - 1) \le ((k-2+n) - f_{n+1})$, and $(n-1)(k - f_{n+1} - 1) + 1 \le 0$, which is a contradiction to our assumption. In addition, both $G_{n+1} - F_{n+1}$ and $G_{(1..n)} - F_{(1..n)}$ are Hamiltonian connected. So we have a fault free Hamiltonian cycle $\langle u_{n+1}, HP_{n+1}, v_{n+1}, v_{(1..n)}, HP_{(1..n)}, u_{(1..n)}, u_{n+1}\rangle$ in $G_{(1..n)} - F_{(1..n)}$. See Fig. 4.

Case 2: $f_{n+1} = k-2$.

There is a fault free Hamiltonian cycle, say HC, in $G_{n+1} - F_{n+1}$. Since $|F_{(1..n+1)}| - |F_{n+1}| = n$, we may without loss of generality assume that the number of faults in $G_1 \cup M_{1,n+1}$ is at most 1. The length of the fault free Hamiltonian cycle in $G_{n+1} - F_{n+1}$ is at least 3, so there is an edge (u_{n+1}, v_{n+1}) in HC, such that both matching edges (u_{n+1}, u_1) and (v_{n+1}, v_1) are super healthy. By induction, $G_{(1..n)}$ is $(k-3+n-1)$-Hamiltonian connected where $k \ge 5$. So there is a u_1-v_1 Hamiltonian path in $G_{(1..n)} - F_{(1..n)}$ since $f_{(1..n)} = n$. Therefore, we have a fault free Hamiltonian cycle $\langle u_{n+1}, HP_{n+1}, v_{n+1}, v_1, HP_{(1..n)}, u_1, u_{n+1}\rangle$ in $G_{(1..n)} - F_{(1..n)}$. See Fig. 5.

Case 3: $k-2+1 \le f_{n+1} \le k-2+n$.

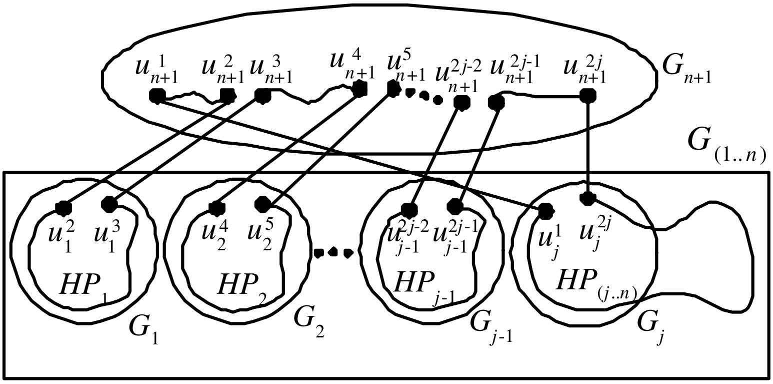


Fig. 6 Case 3: $k-2+1 \leq f_{n+1} \leq k-2+n$

While $f_{n+1} = k-2$, $G_{n+1} - F_{n+1}$ has a fault free Hamiltonian cycle. Now, $f_{n+1} = k-2+i$ where $1 \leq i \leq n$, the fault free Hamiltonian cycle in G_{n+1} may be cut into j paths, where $j \leq i$, say $\langle u^1_{n+1}, \ldots, u^2_{n+1}\rangle$, $\langle u^3_{n+1}, \ldots, u^4_{n+1}\rangle, \ldots, \langle u^{2j-1}_{n+1}, \ldots, u^{2j}_{n+1}\rangle$.

Now, there are $n-i$ faults in $G_{(1..n+1)} - G_{n+1}$. We choose j components in $G_{(1..n)}$, such that the j components have no fault, and the matching edges between the j components and G_{n+1} are healthy. We may without loss of generality say that the j components are $G_1, G_2, \ldots, G_j$. Then we choose $2j$ matching edges (u^2_{n+1}, u^2_1), (u^3_{n+1}, u^3_1), (u^4_{n+1}, u^4_2), $(u^5_{n+1}, u^5_2), \ldots, (u^{2j-2}_{n+1}, u^{2j-2}_{j-1})$, $(u^{2j-1}_{n+1}, u^{2j-1}_{j-1})$, (u^1_{n+1}, u^1_j), and (u^{2j}_{n+1}, u^{2j}_j). There are $n-i$ faults in $G_{(j..n)}$ at most, by induction, $G_{(j..n)}$ is $((n-(j-1)) + k - 3) = (n-j-2+k)$-Hamiltonian connected. And $n-j-2+k > n-i$ since $j \leq i$ and $k \geq 5$. So $G_{(j..n)}$ is fault free Hamiltonian connected. In addition, each component $G_1, \ldots, G_{j-1}$ is Hamiltonian connected. As a result, we have a fault free Hamiltonian cycle as shown in the case of Fig. 6.

This completes the proof of this lemma. □

Corollary 1 *Let $G_1, G_2, \ldots, G_n$ be k-regular optimal fault-tolerant Hamiltonian and Hamiltonian connected graphs with the same number of vertices where $k \geq 5$ and $n \geq 2$. Then graph $G_{(1..n)}$ is also an optimal fault-tolerant Hamiltonian and an optimal fault-tolerant Hamiltonian connected graph.*

As for the case of $k < 5$ in Corollary 1, we conjecture that there have similar results. But the proof is rather long, we left it as an open problem here.

5 Conclusions

The fault-tolerant Hamiltonicity and the fault-tolerant Hamiltonian connectivity are essential parameters of an interconnection network. In this paper, we propose that k-regular Hamiltonian and Hamiltonian connected graphs are optimal fault-tolerant

Hamiltonian and Hamiltonian connected if graph G remains Hamiltonian after removing at most $k-2$ vertices/edges and remains Hamiltonian connected after removing at most $k-3$ vertices/edges. We investigate in constructing optimal fault-tolerant Hamiltonian and Hamiltonian connected graphs with flexibility.

There are many popular interconnection networks which are k-regular graphs. Some of them, e.g., twisted-cubes, crossed-cubes, Möbius cubes, and generalized hypercubes, can be recursively constructed using our construction scheme and, therefore, they are in fact a subclass of our proposed family of graphs, and some of them are optimal fault-tolerant Hamiltonian and Hamiltonian connected.

Acknowledgements The authors would like to express their gratitude to the anonymous referees for their kind suggestions and useful comments on the original manuscript, which have greatly improved the quality of the paper.

References

1. Akers SB, Krishnamurthy B (1989) A group-theoretic model for symmetric interconnection networks. IEEE Trans Comput 38:555–566
2. Ashir YA, Stewart IA (2002) Fault-tolerant embeddings of Hamiltonian circuits in k-ary n-cubes. SIAM J Discrete Math 15:317–328
3. Bhuyan L, Agrawal DP (1984) Generalized hypercube and hyperbus structures for a computer network. IEEE Trans Comput 33:323–333
4. Bondy JA, Murty USR (1980) Graph theory with applications. North Holland, New York
5. Chang CH, Lin CK, Tan JJM, Huang HM, Hsu LH (2009) The super spanning connectivity and super spanning laceability of the enhanced hypercubes. J Supercomput 48:66–87
6. Chen YC, Tsai CH, Hsu LH, Tan JJM (2004) On some super fault-tolerant Hamiltonian graphs. Appl Math Comput 148:729–741
7. Chen YC, Hsu LH, Tan JJM (2006) A recursive construction scheme for super fault-tolerant Hamiltonian graphs. Appl Math Comput 177:465–481
8. Chen YC, Tan JJM (2007) Restricted connectivity for three families of interconnection networks. Appl Math Comput 188:1848–1855
9. Cull P, Larson SM (1995) The Möbius cubes. IEEE Trans Comput 44:647–659
10. Duh DR, Chen GH, Hsu DF (1996) Combinatorial properties of generalized hypercube graphs. Inf Process Lett 57:41–45
11. Dvořák T, Gregor P (2007) Hamiltonian fault-tolerance of hypercubes. Electron Notes Discrete Math 29:471–477
12. Fan J, Lin X, Jia X, Lau RWH (2005) Edge-pancyclicity of twisted cubes. In: Lecture notes in computer science, vol 3827. Springer, Berlin, pp 1090–1099
13. Fan J, Lin X, Pan Y, Jia X (2007) Optimal fault-tolerant embedding of paths in twisted cubes. J Parallel Distrib Comput 67:205–214
14. Fu JS (2008) Conditional fault Hamiltonicity of the complete graph. Inf Process Lett 107:110–113
15. Fu JS, Hung HS, Chen GH (2008) Embedding fault-free cycles in crossed cubes with conditional link faults. J Supercomput. doi:10.1007/s11227-008-0232-y
16. Hsu LH, Lin CK (2008) Graph theory and interconnection networks. CRC Press, Boca Raton, London, New York
17. Hu KS, Yeoh SS, Chen CY, Hsu LH (2007) Node-pancyclicity and edge-pancyclicity of hypercube variants. Inf Process Lett 102:1–7
18. Huang WT, Chuang YC, Tan JM, Hsu LH (2000) Fault-free Hamiltonian cycle in faulty Möbius cubes. J Comput Sist 4:106–114
19. Huang WT, Chuang YC, Hsu LH, Tan JM (2002) On the fault-tolerant Hamiltonicity of Crossed cubes. IEICE Trans Fundam E85-A:1359–1371
20. Huang WT, Tan JM, Hung CN, Hsu LH (2002) Fault-tolerant Hamiltonicity of Twisted cubes. J Parallel Distrib Comput 62:591–604

21. Latifi S, Zheng SQ, Bagherzadeh N (1992) Optimal ring embedding in hypercubes with faulty links. In: Proceedings of the IEEE symposium on fault-tolerant computing, 1992, pp 178–184
22. Ma M, Liu G, Xu JM (2008) Fault-tolerant embedding of paths in crossed cubes. Theor Comput Sci 407:110–116
23. Rowley RA, Bose B (1993) Fault-tolerant ring embedding in de-Bruijn networks. IEEE Trans Comput 12:1480–1486
24. Saad Y, Schultz MH (1988) Topological properties of hypercubes. IEEE Trans Comput 37:867–872
25. Sung TY, Lin CY, Chuang YC, Hsu LH (1998) Fault tolerant token ring embedding in double loop networks. Inf Process Lett 66:201–207
26. Tsai CH, Tan JM, Chuang YC, Hsu LH (2000) Fault-free cycles and links in faulty recursive circulant graphs. In: Proceedings of the 2000 international computer symposium, 2000, pp 74–77
27. Tseng YC, Chang SH, Sheu JP (1997) Fault-tolerant ring embedding in a Star graph with both link and node failures. IEEE Trans Parallel Distrib Syst 8:1185–1195