J Supercomput (2010) 54: 239–251
DOI 10.1007/s11227-009-0317-2

Mutually independent Hamiltonian cycles in dual-cubes

**Yuan-Kang Shih · Hui-Chun Chuang ·
Shin-Shin Kao · Jimmy J.M. Tan**

Published online: 24 July 2009

Abstract The hypercube family Q_n is one of the most well-known interconnection networks in parallel computers. With Q_n, dual-cube networks, denoted by DC_n, was introduced and shown to be a $(n+1)$-regular, vertex symmetric graph with some fault-tolerant Hamiltonian properties. In addition, DC_n's are shown to be superior to Q_n's in many aspects. In this article, we will prove that the n-dimensional dual-cube DC_n contains $n+1$ mutually independent Hamiltonian cycles for $n \geq 2$. More specifically, let $v_i \in V(DC_n)$ for $0 \leq i \leq |V(DC_n)|-1$ and let $\langle v_0, v_1, \ldots, v_{|V(DC_n)|-1}, v_0\rangle$ be a Hamiltonian cycle of DC_n. We prove that DC_n contains $n+1$ Hamiltonian cycles of the form $\langle v_0, v_1^k, \ldots, v_{|V(DC_n)|-1}^k, v_0\rangle$ for $0 \leq k \leq n$, in which $v_i^k \neq v_i^{k'}$ whenever $k \neq k'$. The result is optimal since each vertex of DC_n has only $n+1$ neighbors.

Keywords Hypercube · Dual-cube · Hamiltonian cycle · Hamiltonian connected · Mutually independent

1 Introduction

An *n-dimensional hypercube* Q_n is a graph with the vertex set $\{0, 1\}^n$ and there is an edge between any two vertices that differ exactly in one bit position. The hypercube family is one of the most well-known and popular interconnection networks due to its excellent properties such as the recursive structure, symmetry, small diameter, low degree, easy routing, and so on; see [7, 8, 12, 14, 30].

Y.-K. Shih · J.J.M. Tan
Department of Computer Science, National Chiao Tung University, Hsinchu, 30010 Taiwan, China

H.-C. Chuang · S.-S. Kao (✉)
Department of Applied Mathematics, Chung-Yuan Christian University, Chung-Li, 32023 Taiwan, China
e-mail: skao@math.cycu.edu.tw

The *dual-cube* family DC_n, $n \geq 1$, was first introduced by Li, Peng, and Wu [17]. They make 2^{n+1} copies of Q_n and divide them into two classes, Class 0 and Class 1. Each class consists of 2^n copies of Q_n and each copy is called a *cluster*. By properly adding edges, they connect every pair of clusters from the opposite classes with an edge and prove that DC_n is a $(n+1)$-regular, vertex symmetric graph that contains some properties superior to hypercubes. Notice that the number of vertices of an n-dimensional dual-cube DC_n is equal to the number of vertices of a $(2n+1)$-dimensional hypercube Q_{2n+1}. Each vertex in Q_{2n+1} is adjacent to $2n+1$ neighbors and the total number of edges of Q_{2n+1} is $(2n+1) \times 2^{2n}$. On the other hand, each vertex in DC_n is adjacent to $n+1$ neighbors and the total number of edges of DC_n is $(n+1) \times 2^{2n}$. Although any DC_n has much less edges than Q_{2n+1} with the same number of vertices, the diameter of DC_n, $2n+2$, is of the same order of the diameter of Q_{2n+1}, which is $2n+1$. Other advanced subjects such as fault-tolerant cycle embedding and multiple disjoint paths construction in dual-cubes are also investigated [13, 15–20].

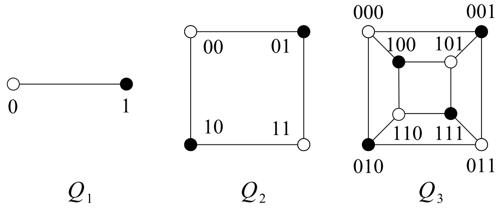
The concept of mutually independent Hamiltonian cycles arises from the following application [22]. If k pieces of data must be sent from a message center u, and the data must be processed at each intermediate receiver (and the process takes time) before they are sent back to the center, then the existence of mutually independent cycles from u guarantees that there will be no waiting time for the parallel processing. Recently, many studies about mutually independent Hamiltonian cycles on hypercubes and their variants are published [22–24]. In this article, we prove that the n-dimensional dual-cube DC_n contains $n+1$ mutually independent Hamiltonian cycles for $n \geq 2$. The result is optimal since DC_n is a $(n+1)$-regular graph. The article is organized as follows. In Sect. 2, we introduce the graph terminologies and notations used in this paper, the precise definition of DC_n and the new labeling of its vertices. In Sect. 3, we prove that DC_n, $n \geq 2$, contains $n+1$ mutually independent Hamiltonian cycles.

2 Preliminaries

For the graph definitions and notations, we follow [3]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{(u, v) \mid (u, v)$ is an unorder pair of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. Two vertices u and v are *adjacent* if $(u, v) \in E$. The total number of vertices of G is denoted by $|V(G)|$. For a vertex u of G, we denote the *degree* of u by $\deg(u) = |\{v \mid (u, v) \in E\}|$. A graph G is *k-regular* if, for every vertex $u \in G$, $\deg(u) = k$.

A *path* is represented by $\langle v_0, v_1, v_2, \ldots, v_k\rangle$, where all vertices are distinct. We also write the path $\langle v_0, v_1, v_2, \ldots, v_k\rangle$ as $\langle v_0, Q_1, v_i, v_{i+1}, \ldots, v_j, Q_2, v_t, \ldots, v_k\rangle$, where Q_1 is the path $\langle v_0, v_1, \ldots, v_{i-1}, v_i\rangle$ and Q_2 is the path $\langle v_j, v_{j+1}, \ldots, v_{t-1}, v_t\rangle$. If a path $P = \langle v_0, v_1, v_2, \ldots, v_{k-1}, v_k\rangle$, then P^{-1} denotes the path $\langle v_k, v_{k-1}, \ldots, v_2, v_1, v_0\rangle$. A *Hamiltonian path* between u and v, where u and v are two distinct vertices of G, is a path joining u to v that visits every vertex of G exactly once. A graph G is

Fig. 1 The hypercubes Q_1, Q_2, and Q_3



Hamiltonian connected if there exists a Hamiltonian path between any two different vertices of G. Two paths $P_1 = \langle u_0, u_1, \ldots, u_m \rangle$ and $P_2 = \langle v_0, v_1, \ldots, v_m \rangle$ from a to b are *independent* [22] if $u_0 = v_0 = a$, $u_m = v_m = b$, and $u_i \neq v_i$ for $1 \leq i \leq m - 1$. Paths with the same number of vertices from a to b are *mutually independent* [22] if every two different paths are independent.

A graph $G = (B \cup W, E)$ is *bipartite* if $V(G)$ is the union of two disjoint sets B and W and $E \subseteq \{(u, v) \mid u \in B, v \in W\}$. It is easy to see that any bipartite graph with at least three vertices is not Hamiltonian connected. A bipartite graph G is *Hamiltonian laceable* if there exists a Hamiltonian path between any two vertices from the opposite partite sets.

A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *Hamiltonian cycle* of G is a cycle that visits every vertex of G exactly once. A *Hamiltonian graph* is a graph with a Hamiltonian cycle. The *length* of a cycle C is the number of edges/vertices in C. Two cycles $C_1 = \langle u_0, u_1, \ldots, u_k, u_0 \rangle$ and $C_2 = \langle v_0, v_1, \ldots, v_k, v_0 \rangle$ beginning at s are *independent* if $u_0 = v_0 = s$ and $u_i \neq v_i$ for $1 \leq i \leq k$ [23]. Cycles beginning at s with equal length are *mutually independent* if every two different cycles are independent. Let G be a graph. We say that G *contains n mutually independent Hamiltonian cycles* if there exist n Hamiltonian cycles in G such that the n cycles begin at the same vertex s and are mutually independent. There are numerous studies in mutually independent Hamiltonian cycles. Readers can refer to [10, 24, 26, 28].

An n-dimensional hypercube, denoted by Q_n, is a graph with 2^n vertices, and each vertex u can be distinctly labeled by an n-bit binary string, $u = u_{n-1}u_{n-2} \cdots u_1u_0$. There is an edge between two vertices if and only if their binary labels differ in exactly one bit position. See Fig. 1 for an illustration. Sun et al. proved that the n-dimensional hypercube Q_n contains n mutually independent Hamiltonian cycles for $n \geq 4$ [28]. Other studies about hypercubes are in [4, 5, 9, 12, 21, 25, 29–31, 33].

The *dual-cube* family DC_n, $n \geq 1$, was first introduced by Li and Peng in 2000 [15]. Its nice structure has drawn the attention of many researchers [11, 15–20]. A dual-cube DC_n is obtained from a *basic component* Q_n as follows. Make 2^{n+1} copies of Q_n and divide them into two classes, Class 0 and Class 1. Each class consists of 2^n copies of Q_n and each copy is called a *cluster*. We shall label the 2^n clusters in each class by $\{0, 1\}^n$, called the *cluster id*. Any vertex $u \in V(DC_n)$ is given a *vertex id*, which is a $(2n + 1)$-bit binary string. Let $u = u_{2n}u_{2n-1} \cdots u_nu_{n-1} \cdots u_0$. If $u_{2n} = 0$, then the next n bits $u_{2n-1} \cdots u_n$ is called the cluster id and the last n bits $u_{n-1} \cdots u_0$ is called the vertex id. If $u_{2n} = 1$, then the next n bits $u_{2n-1} \cdots u_n$ is called

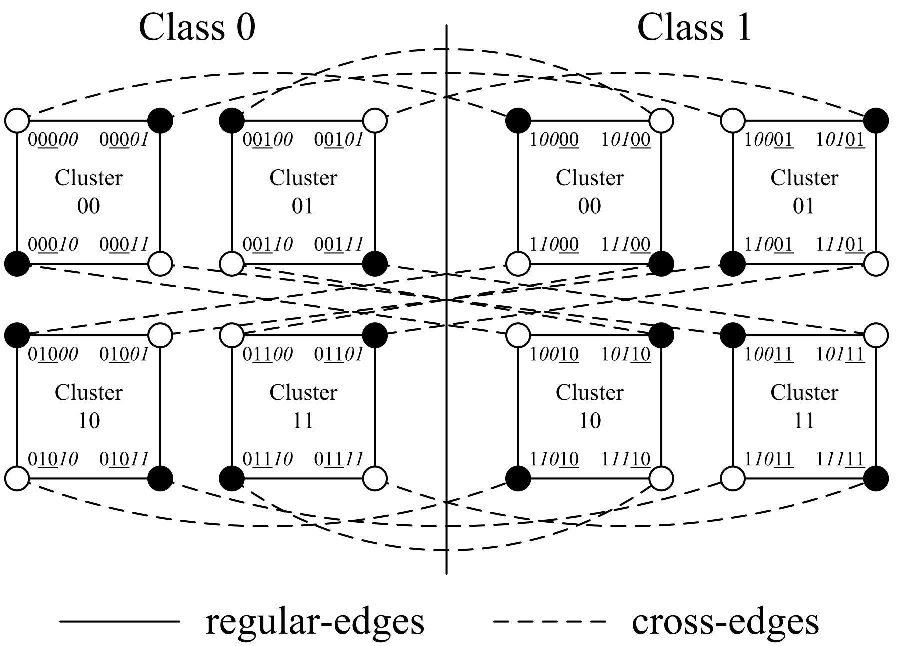


Fig. 2 The graph DC_2. Notice that label of each vertex $u \in V(DC_2)$ consists of 5 bits. The first bit is the class id. The two bits with the underline are called the cluster id, and the other two bits in Italic form are called the vertex id

the vertex id and the last n bits $u_{n-1} \cdots u_0$ is called the cluster id. The following diagram gives an illustration.

$$u \in \text{Class 0:} \quad 0\overbrace{u_{2n-1}u_{2n-2}\cdots u_n}^{\text{Cluster id}}\overbrace{u_{n-1}u_{n-2}\cdots u_0}^{\text{Vertex id in } Q_n};$$

$$u \in \text{Class 1:} \quad 1\overbrace{u_{2n-1}u_{2n-2}\cdots u_n}^{\text{Vertex id in } Q_n}\overbrace{u_{n-1}u_{n-2}\cdots u_0}^{\text{Cluster id}}.$$

Given two vertices $u = u_{2n} \cdots u_0$ and $v = v_{2n} \cdots v_0$, there is an edge between u and v in DC_n if and only if the following conditions are satisfied:

- u and v differ in exactly one bit position i, where $0 \le i \le 2n$;
- if $0 \le i \le n-1$, then $u_{2n} = v_{2n} = 0$;
- if $n \le i \le 2n-1$, then $u_{2n} = v_{2n} = 1$.

By the definition of DC_n and the study of [20], we know that DC_n is an $(n+1)$-regular bipartite graph. Any vertex u in DC_n is adjacent to n vertices in the same cluster and to one vertex in some cluster of the other class. There is no edge between clusters of the same class. The edges within the same cluster are called *regular-edges*, and the edges connecting two clusters of distinct classes are called *cross-edges*. An illustration of DC_2 is given in Fig. 2.

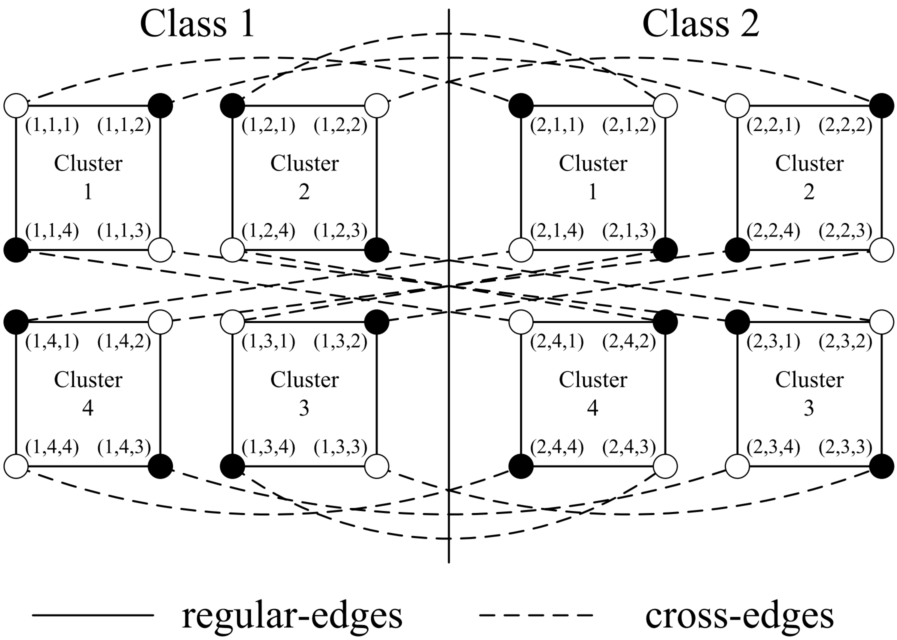


Fig. 3 Using the new labeling scheme which was proposed by Chen and Kao in [6] for $V(DC_2)$, we label each vertex in DC_2 by (i, j, k), where i is the class id, j the cluster id, and k the vertex id

In 2008, Chen and Kao [6] proposed a more convenient new labeling for vertices of dual-cubes. Dual-cube DC_n consists of two *classes*, Class 1 and Class 2. For $i \in \{1, 2\}$, Class i has 2^n copies of Q_n, namely, $G^{i,1}, \dots, G^{i,2^n}$, and each $G^{i,j}$ is called a *cluster*. For $i \in \{1, 2\}$, let $OG^i = \{G^{i,j} \mid 1 \le j \le 2^n$ and j is odd$\}$ and $EG^i = \{G^{i,j} \mid 1 \le j \le 2^n$ and j is even$\}$. Notice that each of OG^1, OG^2, EG^1, and EG^2 consists of 2^{n-1} clusters. We shall label any vertex in $G^{i,j}$ of DC_n by (i, j, k), where k is the *vertex id* in Q_n. Two vertices (i, j, k) and (i', j', k') are adjacent in DC_n if and only if one of the following conditions are satisfied:

(1) $i = i'$, $j = j'$ and the vertices k and k' are adjacent in Q_n;
(2) $|i - i'| = 1$, $j = k'$, and $k = j'$.

The edges satisfying (1) are *regular-edges*. The edges satisfying (2) are *cross-edges*, which connect different pairs of clusters belonging to the two classes. Vertices in a certain cluster use cross-edges to reach vertices in distinct clusters in the opposite class. Therefore, by regarding each cluster as a vertex, DC_n becomes a complete bipartite graph $K_{2^n,2^n}$. Every cross-edge has the corresponding end vertices in the two clusters of the opposite classes. For example, the cross-edge connecting the clusters $G^{1,i}$ and $G^{2,j}$ has end vertices $(1, i, j) \in G^{1,i}$ and $(2, j, i) \in G^{2,j}$. Notice that DC_n is vertex symmetric. Figure 3 depicts DC_2 using the new labeling scheme mentioned above.

3 Mutually independent Hamiltonian cycles in dual-cubes

In this section, we use the new labeling scheme proposed in [6]. Throughout this section, we assume that vertex (i, j, k) is black (resp. white) in DC_n if $i+j+k$ is even (resp. odd), without loss of generality. The following two results were established in [27] and [28].

Lemma 1 [27] *The hypercube Q_n is Hamiltonian laceable for any positive integer n.*

Theorem 1 [28] *The n-dimensional hypercube contains $n-1$ mutually independent Hamiltonian cycles for $n \in \{1, 2, 3\}$ and contains n mutually independent Hamiltonian cycles for $n \geq 4$.*

Assume that $b = (i, j, k)$ is a black vertex, $w = (3-i, k, j)$ is a white vertex, and b and w are connected by a cross-edge in DC_n for $i = 1, 2$ and $1 \leq j, k \leq 2^n$. The following results are true:

- If $i = 1$, j is odd and k is even, then $b \in OG^1$ and $w \in EG^2$.
- If $i = 1$, j is even and k is odd, then $b \in EG^1$ and $w \in OG^2$.
- If $i = 2$, j is odd and k is odd, then $b \in OG^2$ and $w \in OG^1$.
- If $i = 2$, j is even and k is even, then $b \in EG^2$ and $w \in EG^1$.

Therefore, we have the following property.

Property 1 *In DC_n, a black vertex in OG^1, EG^2, EG^1, or OG^2 is adjacent to a white vertex in EG^2, EG^1, OG^2, or OG^1, respectively.*

Theorem 2 *The 2-dimensional dual-cube DC_2 contains 3 mutually independent Hamiltonian cycles.*

Proof Note that DC_2 is vertex symmetric. We assume that any Hamiltonian cycle begins at the vertex $(1, 1, 1)$ without loss of generality. The three required cycles C_1, C_2, and C_3 beginning at $(1, 1, 1)$ are constructed specifically. Please see Appendix A. □

Theorem 3 *The 3-dimensional dual-cube DC_3 contains 4 mutually independent Hamiltonian cycles.*

Proof Since DC_3 is vertex symmetric, we assume that any Hamiltonian cycle begins at the vertex $(1, 7, 1)$ without loss of generality. We construct the four required cycles C_1, C_2, C_3, and C_4 beginning at $(1, 7, 1)$. Please see Appendix B. □

Let a_i's and a'_i's be clusters of DC_n, where $1 \leq i \leq 2^{n+1}$. We say that two cluster sequences $S_1 = \langle a_1, a_2, \ldots, a_j\rangle$ and $S_2 = \langle a'_1, a'_2, \ldots, a'_j\rangle$ are *independent* if $a_i \neq a'_i$ for $1 \leq i \leq j$. Cluster sequences with equal length are *pairwise independent* if any pair of the cluster sequences is independent. For any two positive integers r and d, $[r]_d$ denotes $r(\mathbf{mod}\ d)$.

Lemma 2 *For $n \geq 2$, there exist $2^{n-1} - 1$ pairwise independent cluster sequences of the form $\langle a_1, a_2, \ldots, a_{2^{n+1}-1} \rangle$ in $DC_n - \{G^{1,2^n-1}\}$, where $a_{[i]_4=0} \in OG^1 - \{G^{1,2^n-1}\}$, $a_{[i]_4=1} \in EG^2$, $a_{[i]_4=2} \in EG^1$, and $a_{[i]_4=3} \in OG^2$.*

Proof In DC_n, there are 2^{n-1} clusters in each of EG^1, EG^2, OG^1, and OG^2. So $OG^1 - \{G^{1,2^n-1}\}$ contains $2^{n-1} - 1$ clusters. We divide the cluster sequence $\langle a_1, a_2, \ldots, a_{2^{n+1}-1} \rangle$ into four subsequences. That is, $S_1 = \langle a_1, a_5, \ldots, a_{2^{n+1}-3} \rangle$, $S_2 = \langle a_2, a_6, \ldots, a_{2^{n+1}-2} \rangle$, $S_3 = \langle a_3, a_7, \ldots, a_{2^{n+1}-1} \rangle$, and $S_4 = \langle a_4, a_8, \ldots, a_{2^{n+1}-4} \rangle$.

For $1 \leq i \leq 3$, S_i has 2^{n-1} elements and there exist 2^{n-1} choices for each element. Using the structure of a Latin square with $2^{n-1} \times 2^{n-1}$ entries, we know that there exist 2^{n-1} possible combinations of clusters in S_i, denoted by $\bar{S}_i^k$ for $1 \leq k \leq 2^{n-1}$, such that $\{\bar{S}_i^k\}_{k=1}^{2^{n-1}}$ are pairwise independent cluster sequences.

Similarly, S_4 has $2^{n-1} - 1$ elements and we have $2^{n-1} - 1$ choices for each element. Hence, there exist $2^{n-1} - 1$ possible combinations of clusters in S_4, denoted by $\bar{S}_4^k$ for $1 \leq k \leq 2^{n-1} - 1$, such that $\{\bar{S}_4^k\}_{k=1}^{2^{n-1}-1}$ are pairwise independent cluster sequences.

Therefore, for $n \geq 2$, there exist $(2^{n-1} - 1)$ pairwise independent cluster sequences $\langle a_1, a_2, \ldots, a_{2^{n+1}-1} \rangle$ in $DC_n - \{G^{1,2^n-1}\}$. □

Lemma 3 *Let $\langle a_1, a_2, \ldots, a_{2^{n+1}-1} \rangle$ be a cluster sequence of $DC_n - \{G^{1,2^n-1}\}$, where $a_{[i]_4=0} \in OG^1 - \{G^{1,2^n-1}\}$, $a_{[i]_4=1} \in EG^2$, $a_{[i]_4=2} \in EG^1$, and $a_{[i]_4=3} \in OG^2$. Assume that u is a white vertex in a_1 and v is a black vertex in $a_{2^{n+1}-1}$, then there is a Hamiltonian path $\langle u = x_1, H_1, y_1, x_2, H_2, y_2, \ldots, x_{2^{n+1}-1}, H_{2^{n+1}-1}, y_{2^{n+1}-1} = v \rangle$ between u and v, where x_i is a white vertex, y_i is a black vertex, $\{x_i, y_i\} \in V(a_i)$, and H_i is a Hamiltonian path of a_i joining x_i to y_i for every $1 \leq i \leq 2^{n+1} - 1$.*

Proof By Property 1, a black vertex y_i in a_i is adjacent to a white vertex x_{i+1} in a_{i+1} by a cross-edge for $1 \leq i \leq 2^{n+1} - 2$. Notice that each cluster is a hypercube, $u = x_1$ is a white vertex in a_1 and $v = y_{2^{n+1}-1}$ is a black vertex in $a_{2^{n+1}-1}$. By Lemma 1, there is a Hamiltonian path H_i in cluster a_i joining x_i to y_i for $1 \leq i \leq 2^{n+1} - 1$. Then $\langle u = x_1, H_1, y_1, x_2, H_2, y_2, \ldots, x_{2^{n+1}-1}, H_{2^{n+1}-1}, y_{2^{n+1}-1} = v \rangle$ is the desired Hamiltonian path. □

Theorem 4 *For $n \geq 4$, there are $n + 1$ mutually independent Hamiltonian cycles in DC_n.*

Proof We want to construct $n + 1$ mutually independent Hamiltonian cycles, denoted by $\bar{C}_i$ for $1 \leq i \leq n + 1$, for DC_n. Since DC_n is vertex symmetric, without loss of generality, we assume $\bar{C}_i$ starts at $(1, 2^n - 1, 1)$ for $1 \leq i \leq n + 1$. By Theorem 1 and the fact that each cluster $G^{i,j}$ in DC_n is Q_n, there are n mutually independent Hamiltonian cycles $C_1, C_2, \ldots, C_n$ beginning at the white vertex $(1, 2^n - 1, 1)$ in the cluster $G^{1,2^n-1}$. Without loss of generality, let $C_i = \langle v_1^*, v_2^i, v_3^i, \ldots, v_{2^n-1}^i, v_{2^n}^i, v_1^* \rangle$ for $1 \leq i \leq n$, where $v_1^* = (1, 2^n - 1, 1)$. Please see Fig. 4 for an illustration.

Notice that v_m^i is a white vertex if m is odd and v_m^i is a black vertex if m is even. Besides, $2^{n-1} - 1 \geq n + 1$ when $n \geq 4$. By Lemma 2, we know that there exist

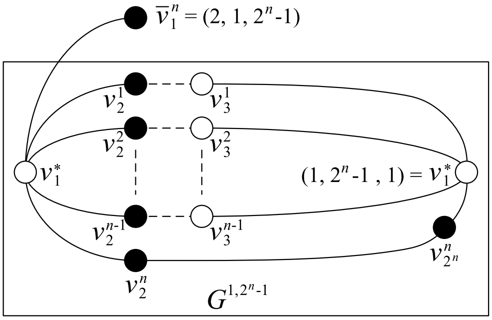


Fig. 4 An illustration for Theorem 4

$2^{n-1} - 1$ pairwise independent cluster sequences for $n \geq 4$. The $2^{n-1} - 1$ pairwise independent cluster sequences are enough for us to construct the $n + 1$ mutually independent Hamiltonian cycles below. There are three cases.

Case 1. To construct $\{\bar{C}_i : 1 \leq i \leq n - 1\}$. We consider the mutually independent Hamiltonian cycles $C_1, C_2, \ldots, C_{n-1}$ in $G^{1,2^n-1}$. Note that v_2^i is adjacent to a white vertex $\bar{v}_2^i$ and v_3^i is adjacent to a black vertex $\bar{v}_3^i$ by cross-edges for $1 \leq i \leq n - 1$. By Lemma 2, there exist $n - 1$ $(\leq 2^{n-1} - 1)$ pairwise independent cluster sequences $T_i = \langle a_1^i, a_2^i, \ldots, a_{2^{n+1}-1}^i \rangle$ in $DC_n - \{G^{1,2^n-1}\}$ for $1 \leq i \leq n - 1$ such that $\bar{v}_2^i \in a_1^i$ and $\bar{v}_3^i \in a_{2^{n+1}-1}^i$. By Lemma 3, there is a Hamiltonian path P_i joining $\bar{v}_2^i$ to $\bar{v}_3^i$ in T_i for $1 \leq i \leq n-1$. Hence, $\bar{C}_i = \langle v_1^*, v_2^i, \bar{v}_2^i, P_i, \bar{v}_3^i, v_3^i, \ldots, v_{2^n}^i, v_1^* \rangle$, $1 \leq i \leq n-1$, are the desired cycles.

Case 2. To construct $\bar{C}_n$. In DC_n, the black vertex $v_{2^n}^n$ is adjacent to the white vertex $\bar{v}_{2^n}^n$ and the white vertex v_1^n is adjacent to the black $\bar{v}_1^n$ by cross-edges. In $DC_n - \{G^{1,2^n-1}\}$, by Lemma 2, there is a cluster sequence $T_n = \langle a_1^n, a_2^n, \ldots, a_{2^{n+1}-1}^n \rangle$, which is pairwise independent with T_i for $1 \leq i \leq n-1$ such that $\bar{v}_{2^n}^n \in a_1^n$ and $\bar{v}_1^* \in a_{2^{n+1}-1}^n$. According to Lemma 3, there is a Hamiltonian path P_n joining $\bar{v}_{2^n}^n$ to $\bar{v}_1^*$ in T_n. Therefore, $\bar{C}_n = \langle v_1^*, v_2^n, \ldots, v_{2^n}^n, \bar{v}_{2^n}^n, P_n, \bar{v}_1^*, v_1^* \rangle$ is the desired cycle.

Case 3. To construct $\bar{C}_{n+1}$. The white vertex v_1^* is adjacent to the black vertex $\bar{v}_1^*$ and the black vertex v_2^n is adjacent to the white vertex $\bar{v}_2^n$ by cross-edges in DC_n. Because of Lemma 2, there is a cluster sequence $T_{n+1} = \langle a_1^{n+1}, a_2^{n+1}, \ldots, a_{2^{n+1}-1}^{n+1} \rangle$ in $DC_n - \{G^{1,2^n-1}\}$, which is pairwise independent with T_i for $1 \leq i \leq n$ such that $\bar{v}_1^* \in a_1^{n+1}$ and $\bar{v}_2^n \in a_{2^{n+1}-1}^{n+1}$. By Lemma 3, there is a Hamiltonian path P_{n+1} joining $\bar{v}_1^*$ to $\bar{v}_2^n$ in T_{n+1}. So $\bar{C}_{n+1} = \langle v_1^*, \bar{v}_1^*, P_{n+1}, \bar{v}_2^n, v_2^n, \ldots, v_{2^n}^n, v_1^* \rangle$ is the desired cycles. Please see the Fig. 5.

By **Case 1**, **Case 2**, and **Case 3**, the $n + 1$ mutually independent Hamiltonian cycles $\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_{n+1}$ in DC_n for $n \geq 4$ are constructed. This completes the proof. □

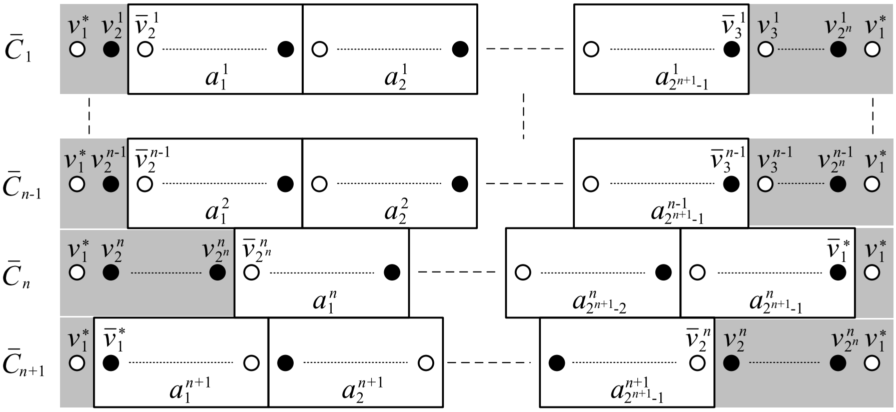


Fig. 5 An illustration for **Case 1**, **Case 2**, and **Case 3** in Theorem 4. Notice that the gray areas are in $G^{1,2^n-1}$

4 Conclusion

In [28], it was shown that the hypercube Q_n, when $n \in \{2, 3\}$, contains only $n - 1$ mutually independent Hamiltonian cycles. However, to our surprise, Theorem 2 and Theorem 3 show that DC_2 contains 3 mutually independent Hamiltonian cycles and DC_3 contains 4 mutually independent Hamiltonian cycles. In addition, according to Theorem 4, there are $n + 1$ mutually independent Hamiltonian cycles in DC_n for $n \geq 4$. Therefore, we have the following result.

Corollary 1 *The n-dimensional dual-cube DC_n contains $n+1$ mutually independent Hamiltonian cycles for $n \geq 2$.*

Due to the fact that each vertex of DC_n is connected to $(n + 1)$ vertices, there are not any more Hamiltonian cycles emerging from the same vertex in addition to the $n+1$ mutually independent Hamiltonian cycles shown above. Therefore, our result is optimal. Another internet architecture that is more cost-effective (more scalable) than the traditional hypercubes and considered as a clever variation to the hypercube is the multi-ring [1, 2, 32]. Various studies such as the fault-tolerant cycle embedding and the existence of mutually independent Hamiltonian cycles in the multi-ring networks will be interesting topics to be explored.

Acknowledgements This research was partially supported by the National Science Council of the Republic of China under contract NSC 96-2115-M-033-002-MY2.

Appendix A: The three required cycles of Theorem 2

$$C_1 = \langle(1,1,1),(1,1,2),(1,1,3),(1,1,4),(2,4,1),(2,4,2),(1,2,4),(1,2,1),$$
$$(2,1,2),(2,1,3),(1,3,1),(1,3,2),(2,2,3),(2,2,4),(2,2,1),(2,2,2),$$

$$(1,2,2),(1,2,3),(2,3,2),(2,3,1),(2,3,4),(2,3,3),(1,3,3),(1,3,4),$$
$$(2,4,3),(2,4,4),(1,4,4),(1,4,3),(1,4,2),(1,4,1),(2,1,4),(2,1,1),$$
$$(1,1,1)\rangle;$$

$$C_2=\langle(1,1,1),(1,1,4),(2,4,1),(2,4,2),(2,4,3),(2,4,4),(1,4,4),(1,4,3),$$
$$(1,4,2),(1,4,1),(2,1,4),(2,1,1),(2,1,2),(2,1,3),(1,3,1),(1,3,4),$$
$$(1,3,3),(1,3,2),(2,2,3),(2,2,4),(2,2,1),(2,2,2),(1,2,2),(1,2,1),$$
$$(1,2,4),(1,2,3),(2,3,2),(2,3,3),(2,3,4),(2,3,1),(1,1,3),(1,1,2),$$
$$(1,1,1)\rangle;$$

$$C_3=\langle(1,1,1),(2,1,1),(2,1,4),(1,4,1),(1,4,2),(2,2,4),(2,2,3),(2,2,2),$$
$$(2,2,1),(1,1,2),(1,1,3),(2,3,1),(2,3,2),(2,3,3),(2,3,4),(1,4,3),$$
$$(1,4,4),(2,4,4),(2,4,3),(1,3,4),(1,3,3),(1,3,2),(1,3,1),(2,1,3),$$
$$(2,1,2),(1,2,1),(1,2,2),(1,2,3),(1,2,4),(2,4,2),(2,4,1),(1,1,4),$$
$$(1,1,1)\rangle.$$

Appendix B: The four required cycles of Theorem 3

$$C_1=\langle(1,7,1),(1,7,2),(1,7,3),(1,7,6),(1,7,7),(1,7,8),(1,7,5),(2,5,7),$$
$$(2,5,2),(2,5,1),(2,5,8),(2,5,5),(2,5,4),(2,5,3),(2,5,6),(1,6,5),$$
$$(1,6,4),(1,6,1),(1,6,8),(1,6,7),(1,6,2),(1,6,3),(1,6,6),(2,6,6),$$
$$(2,6,3),(2,6,2),(2,6,7),(2,6,8),(2,6,1),(2,6,4),(2,6,5),(1,5,6),$$
$$(1,5,3),(1,5,4),(1,5,5),(1,5,8),(1,5,1),(1,5,2),(1,5,7),(2,7,5),$$
$$(2,7,4),(2,7,3),(2,7,6),(2,7,7),(2,7,2),(2,7,1),(2,7,8),(1,8,7),$$
$$(1,8,2),(1,8,3),(1,8,6),(1,8,5),(1,8,4),(1,8,1),(1,8,8),(2,8,8),$$
$$(2,8,1),(2,8,4),(2,8,5),(2,8,6),(2,8,7),(2,8,2),(2,8,3),(1,3,8),$$
$$(1,3,5),(1,3,4),(1,3,3),(1,3,6),(1,3,7),(1,3,2),(1,3,1),(2,1,3),$$
$$(2,1,6),(2,1,5),(2,1,4),(2,1,1),(2,1,8),(2,1,7),(2,1,2),(1,2,1),$$
$$(1,2,8),(1,2,5),(1,2,4),(1,2,3),(1,2,6),(1,2,7),(1,2,2),(2,2,2),$$
$$(2,2,7),(2,2,6),(2,2,3),(2,2,4),(2,2,5),(2,2,8),(2,2,1),(1,1,2),$$
$$(1,1,1),(1,1,8),(1,1,7),(1,1,6),(1,1,5),(1,1,4),(1,1,3),(2,3,1),$$
$$(2,3,2),(2,3,7),(2,3,8),(2,3,5),(2,3,6),(2,3,3),(2,3,4),(1,4,3),$$
$$(1,4,6),(1,4,7),(1,4,2),(1,4,1),(1,4,8),(1,4,5),(1,4,4),(2,4,4),$$

$(2,4,5), (2,4,8), (2,4,1), (2,4,2), (2,4,3), (2,4,6), (2,4,7), (1,7,4), (1,7,1)\rangle$;

$C_2 = \langle(1,7,1), (1,7,4), (2,4,7), (2,4,2), (2,4,3), (2,4,6), (2,4,5), (2,4,8), (2,4,1), (2,4,4), (1,4,4), (1,4,1), (1,4,8), (1,4,7), (1,4,2), (1,4,3), (1,4,6), (1,4,5), (2,5,4), (2,5,5), (2,5,6), (2,5,3), (2,5,2), (2,5,1), (2,5,8), (2,5,7), (1,7,5), (1,7,8), (1,7,7), (1,7,2), (1,7,3), (1,7,6), (2,6,7), (2,6,6), (2,6,3), (2,6,4), (2,6,5), (2,6,8), (2,6,1), (2,6,2), (1,2,6), (1,2,5), (1,2,4), (1,2,3), (2,3,2), (2,3,7), (2,3,8), (2,3,1), (2,3,4), (2,3,5), (2,3,6), (2,3,3), (1,3,3), (1,3,6), (1,3,5), (1,3,4), (1,3,1), (1,3,8), (1,3,7), (1,3,2), (2,2,3), (2,2,6), (2,2,7), (2,2,2), (1,2,2), (1,2,1), (1,2,8), (1,2,7), (2,7,2), (2,7,3), (2,7,4), (2,7,1), (1,1,7), (1,1,8), (1,1,1), (1,1,4), (1,1,5), (1,1,6), (1,1,3), (1,1,2), (2,2,1), (2,2,4), (2,2,5), (2,2,8), (1,8,2), (1,8,3), (1,8,4), (1,8,1), (1,8,8), (1,8,5), (1,8,6), (1,8,7), (2,7,8), (2,7,7), (2,7,6), (2,7,5), (1,5,7), (1,5,2), (1,5,3), (1,5,6), (1,5,5), (1,5,4), (1,5,1), (1,5,8), (2,8,5), (2,8,8), (2,8,7), (2,8,2), (2,8,1), (2,8,4), (2,8,3), (2,8,6), (1,6,8), (1,6,5), (1,6,4), (1,6,3), (1,6,6), (1,6,7), (1,6,2), (1,6,1), (2,1,6), (2,1,3), (2,1,2), (2,1,1), (2,1,4), (2,1,5), (2,1,8), (2,1,7), (1,7,1)\rangle$;

$C_3 = \langle(1,7,1), (1,7,8), (1,7,5), (1,7,4), (1,7,3), (1,7,6), (1,7,7), (2,7,7), (2,7,2), (2,7,3), (2,7,6), (2,7,5), (2,7,4), (2,7,1), (2,7,8), (1,8,7), (1,8,2), (1,8,3), (1,8,6), (1,8,5), (1,8,4), (1,8,1), (1,8,8), (2,8,8), (2,8,5), (2,8,4), (2,8,3), (2,8,6), (2,8,7), (2,8,2), (2,8,1), (1,1,8), (1,1,5), (1,1,4), (1,1,3), (1,1,6), (1,1,7), (1,1,2), (1,1,1), (2,1,1), (2,1,8), (2,1,6), (2,1,4), (2,1,3), (2,1,2), (2,1,7), (2,1,6), (1,6,1), (1,6,8), (1,6,7), (1,6,2), (1,6,3), (1,6,6), (1,6,5), (1,6,4), (2,4,6), (2,4,7), (2,4,2), (2,4,3), (2,4,4), (2,4,1), (2,4,8), (2,4,5), (1,5,4), (1,5,1), (1,5,8), (1,5,5), (1,5,6), (1,5,7), (1,5,2), (1,5,3), (2,3,5), (2,3,8), (2,3,1), (2,3,2), (2,3,7), (2,3,6), (2,3,3), (2,3,4), (1,4,3), (1,4,2), (1,4,7), (1,4,8), (1,4,1), (1,4,4), (1,4,5), (1,4,6), (2,6,4), (2,6,5), (2,6,8), (2,6,1), (2,6,2), (2,6,7), (2,6,6), (2,6,3), (1,3,6),$

$$(1,3,7),(1,3,2),(1,3,3),(1,3,4),(1,3,1),(1,3,8),(1,3,5),(2,5,3),$$
$$(2,5,4),(2,5,1),(2,5,8),(2,5,5),(2,5,6),(2,5,7),(2,5,2),(1,2,5),$$
$$(1,2,8),(1,2,1),(1,2,4),(1,2,3),(1,2,6),(1,2,7),(1,2,2),(2,2,2),$$
$$(2,2,3),(2,2,4),(2,2,1),(2,2,8),(2,2,5),(2,2,6),(2,2,7),(1,7,2),$$
$$(1,7,1)\rangle;$$

$$C_4 = \langle(1,7,1),(2,1,7),(2,1,6),(2,1,3),(2,1,4),(2,1,5),(2,1,8),(2,1,1),$$
$$(2,1,2),(1,2,1),(1,2,8),(1,2,5),(1,2,4),(1,2,3),(1,2,6),(1,2,7),$$
$$(1,2,2),(2,2,2),(2,2,7),(2,2,8),(2,2,1),(2,2,4),(2,2,5),(2,2,6),$$
$$(2,2,3),(1,3,2),(1,3,7),(1,3,8),(1,3,1),(1,3,4),(1,3,5),(1,3,6),$$
$$(1,3,3),(2,3,3),(2,3,6),(2,3,7),(2,3,2),(2,3,1),(2,3,8),(2,3,5),$$
$$(2,3,4),(1,4,3),(1,4,2),(1,4,7),(1,4,8),(1,4,1),(1,4,4),(1,4,5),$$
$$(1,4,6),(2,6,4),(2,6,1),(2,6,8),(2,6,7),(2,6,2),(2,6,3),(2,6,6),$$
$$(2,6,5),(1,5,6),(1,5,3),(1,5,4),(1,5,1),(1,5,2),(1,5,7),(1,5,8),$$
$$(1,5,5),(2,5,5),(2,5,8),(2,5,1),(2,5,4),(2,5,3),(2,5,2),(2,5,7),$$
$$(2,5,6),(1,6,5),(1,6,6),(1,6,3),(1,6,4),(1,6,1),(1,6,2),(1,6,7),$$
$$(1,6,8),(2,8,6),(2,8,7),(2,8,8),(2,8,5),(2,8,4),(2,8,3),(2,8,2),$$
$$(2,8,1),(1,1,8),(1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,1,5),(1,1,6),$$
$$(1,1,7),(2,7,1),(2,7,4),(2,7,5),(2,7,6),(2,7,3),(2,7,2),(2,7,7),$$
$$(2,7,8),(1,8,7),(1,8,6),(1,8,5),(1,8,8),(1,8,1),(1,8,2),(1,8,3),$$
$$(1,8,4),(2,4,8),(2,4,1),(2,4,2),(2,4,3),(2,4,4),(2,4,5),(2,4,6),$$
$$(2,4,7),(1,7,4),(1,7,5),(1,7,6),(1,7,3),(1,7,2),(1,7,7),(1,7,8),$$
$$(1,7,1)\rangle.$$

References

1. Arabnia HR (1993) A transputer-based reconfigurable parallel system. In: Atkins S, Wagner AS (eds) Transputer research and applications (NATUG 6), Vancouver, Canada. IOS Press, Amsterdam, pp 153–169
2. Bhandarkar SM, Arabnia HR (1995) The REFINE multiprocessor theoretical properties and algorithms. Parallel Comput 21(11):1783–1806
3. Bondy JA, Murty USR (1980) Graph theory with applications. North-Holland, New York
4. Caha R, Koubek V (2007) Spanning multi-paths in hypercubes. Discrete Math 307:2053–2066
5. Chang C-H, Lin C-K, Huang H-M, Hsu L-H (2004) The super laceability of the hypercubes. Inf Process Lett 92:15–21

6. Chen S-Y, Kao S-S (2008) The edge-pancyclicity of dual-cube extensive networks. In: 2nd WSEAS international conference on computer engineering and applications (CEA'08), Acapulco, Mexico, Jan 2008, pp 233–236
7. Fink J (2007) Perfect matchings extend to Hamilton cycles in hypercubes. J Comb Theory, B 97:1074–1076
8. Harary F, Hayes JP, Wu H-J (1988) A survey of the theory of the hypercube graphs. Comput Math Appl 15:277–289
9. Hsieh S-Y, Weng Y-F (2009) Fault-tolerant embedding of pairwise independent Hamiltonian paths on a faulty hypercube with edge faults. Theory Comput Syst 45:407–425
10. Hsieh S-Y, Yu P-Y (2007) Fault-free mutually independent Hamiltonian cycles in hypercubes with faulty edges. J Comb Optim 13:153–162
11. Jiang Z, Wu J (2006) A limited-global information model for fault-tolerant routing in dual-cube. Int J Parallel, Emergent Distrib Syst 21:61–77
12. Kobeissi M, Mollard M (2004) Disjoint cycles and spanning graphs of hypercubes. Discrete Math 288:73–87
13. Lai C-J, Tsai C-H (2008) On embedding cycles into faulty dual-cubes. Inf Process Lett 109:147–150
14. Leighton FT (1992) Introduction to parallel algorithms and architecture: arrays, trees, hypercubes. Morgan Kaufmann, San Mateo
15. Li Y, Peng S (2000) Dual-cubes: a new interconnection network for high-performance computer clusters. In: Proceedings of the 2000 international computer symposium, workshop on computer architecture, ChiaYi, Taiwan, Dec 2000, pp 51–57
16. Li Y, Peng S (2001) Fault-tolerant routing and disjoint paths in dual-cube: a new interconnection network. In: International conference on parallel and distributed systems (ICPADS 2001), Kyongju, Korea, Jun 2001, pp 315–322
17. Li Y, Peng S, Chu W (2002) Hamiltonian cycle embedding for fault tolerance in dual-cube. In: IASTED international conference on networks, parallel and distributed processing and applications, Tsukuba, Japan, Oct 2002, pp 1–6
18. Li Y, Peng S, Chu W (2004) Efficient collective communications in dual-cube. J Supercomput 28:71–90
19. Li Y, Peng S, Chu W (2004) Binomial-tree fault tolerant routing in dual-cubes with large number of faulty nodes. In: International symposium on computational and information sciences (CIS'04), Shanghai, China, Dec 2004, pp 51–56
20. Li Y, Peng S, Chu W (2005) Fault-tolerant cycle embedding in dual-cube with node faults. Int J High Perform Comput Netw 3:45–53
21. Li T-K, Tsai C-H, Tan JJM, Hsu L-H (2003) Bipanconnectivity and edge-fault-tolerant bipancyclic of hypercubes. Inf Process Lett 21:107–110
22. Lin C-K, Huang H-M, Hsu L-H, Bau S (2005) Mutually Hamiltonian paths in star networks. Networks 46:110–117
23. Lin C-K, Huang H-M, Tan JJM, Hsu L-H (2009) The mutually Hamiltonian cycles of the pancake networks and the star networks. Discrete Math (accepted)
24. Lin C-K, Shih Y-K, Tan JJM, Hsu L-H (2009) Mutually independent Hamiltonian cycles in some graphs. Ars Comb (accepted)
25. Saad Y, Schultz MH (1988) Topological properties of hypercubes. IEEE Trans Comput 37:867–872
26. Shih Y-K, Lin C-K, Frank Hsu D, Tan JJM, Hsu L-H (2009) The construction of MIH cycles in bubble-sort graphs. Int J Comput Math (accepted)
27. Simmons G (1978) Almost all n-dimensional rectangular lattices are Hamilton laceable. Congr Numer 21:103–108
28. Sun C-M, Lin C-K, Huang H-M, Hsu L-H (2006) Mutually independent Hamiltonian paths and cycles in hypercubes. J Interconnect Netw 7:235–255
29. Tsai C-H (2004) Linear array and rring embeddings in conditional faulty hypercubes. Theor Comput Sci 314:431–443
30. Tsai C-H (2007) Cycles embedding in hypercubes with node failures. Inf Process Lett 102:242–246
31. Tsai C-H, Jiang S-Y (2007) Path bipancyclicity of hypercubes. Inf Process Lett 101:93–97
32. Arif Wani M, Arabnia HR (2003) Parallel edge-region-based segmentation algorithm targeted at reconfigurable multi-ring network. J Supercomput 25(1):43–63
33. Xu J-M, Du Z-Z, Xu M (2005) Edge-fault-tolerant edge-bipancyclicity of hypercubes. Inf Process Lett 96:146–150