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Two error-correcting pooling designs from symplectic spaces over a finite field

Zengti Li [a], Tayuan Huang [b], Suogang Gao [a,*]

[a] *Mathematics and Information College, Hebei Normal University, Shijiazhuang, 050016, PR China*
[b] *Department of Applied Mathematics, Chiao-Tung University, Hsinchu 30050, Taiwan*

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ABSTRACT

In this paper, we construct two classes of $t \times n$, s^e-disjunct matrix with subspaces in a symplectic space $\mathbb{F}_q^{(2\nu)}$ and prove that the ratio efficiency t/n of two constructions are smaller than that of D'yachkov et al. (2005) [2].

1. Introduction

The basic problem of group testing is to identify the set of defective items in a large population of items. Suppose we have n items to be tested and that there are at most d defective items among them. Each *test* (or *pool*) is (or contains) a subset of items. We assume some testing mechanism exists which if applied to an arbitrary subset of the population gives a *negative outcome* if the subset contains no positive and *positive outcome* otherwise. Objectives of group testing vary from minimizing the number of tests, limiting number of pools, limiting pool sizes to tolerating a few errors. It is conceivable that these objectives are often contradicting, thus testing strategies are application dependent. A group testing algorithm is *non-adaptive* if all tests must be specified without knowing the outcomes of other tests. A non-adaptive testing algorithm is useful in many areas such as DNA library screening [1,7].

A group testing algorithm is *error tolerant* if it can detect some errors in test outcomes. A mathematical model of error-tolerance designs is an s^e-*disjunct* matrix.

A binary matrix M is said to be s^e-*disjunct* if given any $s + 1$ columns of M with one designated, there are e rows with a 1 in the designated column and 0 in each of the other s columns. An s^1-disjunct

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* Corresponding author.
E-mail address: sggao@heinfo.net (S. Gao).

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matrix is said to be *s-disjunct*. In [3], D'yachkov et al. proposed the concept of fully s^e-disjunct matrices. An s^e-disjunct matrix is *fully s^e-disjunct* if it is not $d^{e'}$-disjunct whenever $d > s$ or $e' > e$.

Macula [5] proposed a novel way of constructing s-disjunct matrices using the containment relation in a structure.

Huang and Weng [4] gave a comprehensive treatment of construction of d-disjunct matrices by using of pooling spaces, which is a significant and important addition to the general theory.

Ngo and Du [6] extended the construction to some geometric structures, such as simplicial complexes, and some graph properties, such as matchings.

D'yachkov et al. [2] claimed that the "containment matrix" method has opened a new door for constructing s-disjunct matrices from many mathematical structures.

In this paper, we construct two classes s^e-disjunct matrix with subspaces in a symplectic space $\mathbb{F}_q^{(2\nu)}$ and exhibit their disjunct properties. Given some fixed items, our goal is to detect the positive items. For a pooling design, the less the number of tests is, the better the pooling design is. In order to discuss easily in the following, we give a new definition. We call the ratio between the number of tests and the number of detected items test efficiency, that is the ratio between the number of rows and the number of columns in the s^e-disjunct matrix, i.e., t/n. We will give some discussions on the ratio t/n and compare them with others, such as in [2].

2. Symplectic space

Let

$$K = \begin{pmatrix} 0 & I^{(\nu)} \\ -I^{(\nu)} & 0 \end{pmatrix}.$$

The *symplectic group* of degree 2ν over $\mathbb{F}_q$, denoted by $Sp_{(2\nu)}(\mathbb{F}_q)$, consists of all $2\nu \times 2\nu$ matrix T over $\mathbb{F}_q$ satisfying $TKT' = K$. The vector space $\mathbb{F}_q^{(2\nu)}$ together with the right multiplication action of $Sp_{2\nu}(\mathbb{F}_q)$ is called the 2ν-dimensional *symplectic space* over $\mathbb{F}_q$ [8]. Let P be an m-dimensional subspace of $\mathbb{F}_q^{(2\nu)}$, denote also by P an $m \times 2\nu$ matrix of rank m whose rows span the subspace P and call the matrix P a matrix representation of the subspace P. An m-dimensional subspace P is said to be of type (m, r), if PKP' is of rank $2r$. In particular, subspaces of type $(m, 0)$ are called m-dimensional *totally isotropic subspaces*. The subspaces of type (m, r) exist if and only if $2r \leqslant m \leqslant \nu + r$. The subspace of type (m, r), which contains subspaces of type (m_1, r), exists if and only if $2r \leqslant m_1 \leqslant m \leqslant \nu + r$. It is known that the number of subspaces of type (m, r), denoted by $N(m, r; 2\nu)$, is given by

$$N(m, r; 2\nu) = q^{2r(\nu+r-m)} \frac{\prod_{i=\nu+r-m+1}^{\nu}(q^{2i}-1)}{\prod_{i=1}^{r}(q^{2i}-1)\prod_{i=1}^{m-2r}(q^{i}-1)}. \tag{1}$$

Let $N(m_1, r; m, r; 2\nu)$ denote the number of subspaces of type (m_1, r) contained in a given subspace of type (m, r). It is known that

$$N(m_1, r; m, r; 2\nu) = q^{2r(m-m_1)} \frac{\prod_{i=m-m_1+1}^{m-2r}(q^{i}-1)}{\prod_{i=1}^{m_1-2r}(q^{i}-1)}. \tag{2}$$

Let $N'(m_1, r; m, r; 2\nu)$ denote the number of subspaces of type (m, r) containing a given subspace of type (m_1, r). It is known that

$$N'(m_1, r; m, r; 2\nu) = \frac{\prod_{i=1}^{\nu+r-m_1}(q^{2i}-1)}{\prod_{i=1}^{\nu+r-m}(q^{2i}-1)\prod_{i=1}^{m-m_1}(q^{i}-1)}. \tag{3}$$

Lemma 2.1. *Let $\mathbb{F}_q^{(2\nu)}$ denote the 2ν-dimensional symplectic space over a finite field $\mathbb{F}_q$ with $2r \leqslant m_0 \leqslant i \leqslant m \leqslant \nu + r$. Fix an (m_0, r)-subspace W_0 of $\mathbb{F}_q^{(2\nu)}$, and an (m, r)-subspace W of $\mathbb{F}_q^{(2\nu)}$ such that $W_0 \subset W$. Then the number of (i, r)-subspace A of $\mathbb{F}_q^{(2\nu)}$, where $W_0 \subset A \subset W$, is $N(i - m_0, 0; m - m_0, 0; 2(\nu + r - m_0))$.*

Proof. Since the symplectic group $Sp_{2\nu}(\mathbb{F}_q)$ acts transitively on each set of subspaces of the same type, we may assume that W has the matrix representation of the form

$$W = \begin{array}{c} \begin{array}{cccccc} r & m_0-2r & \nu+r-m_0 & r & m_0-2r & \nu+r-m_0 \end{array} \\ \left(\begin{array}{cccccc} I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & W_1 & 0 & 0 & W_2 \end{array}\right) \begin{array}{l} r \\ r \\ m_0-2r \\ m-m_0 \end{array} \end{array},$$

where (W_1, W_2) is an $(m-m_0, 0)$-subspace of $\mathbb{F}_q^{2(\nu+r-m_0)}$. By (2), the number of (i, r)-subspace A, where $W_0 \subset A \subset W$, is $N(i-m_0, 0; m-m_0, 0; 2(\nu+r-m_0))$. $\square$

3. Construction I

Definition 3.1. For $2r \leqslant d_0 < d < k \leqslant \nu + r$, assume that P_0 is a fixed (d_0, r)-space of $\mathbb{F}_q^{(2\nu)}$. Let M be a binary matrix whose columns (rows) indexed by all (k, r)-spaces containing P_0 ((d, r)-spaces containing P_0) in $\mathbb{F}_q^{(2\nu)}$ such that $M(A, B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $M_1(\nu, d, k)$.

Theorem 3.1. *Suppose* $2r \leqslant d_0 < d < k \leqslant \nu + r$ *and set* $b = \frac{q(q^{k-d_0-1}-1)}{q^{k-d}-1}$. *Then* $M_1(\nu, d, k)$ *is* s^e*-disjunct for* $1 \leqslant d \leqslant b$ *and*

$$\begin{aligned} e = {} & q^{k-d}N(d-d_0-1, 0; k-d_0-1, 0; 2(\nu+r-d_0)) \\ & -(s-1)q^{k-d-1}N(d-d_0-1, 0; k-d_0-2, 0; 2(\nu+r-d_0)). \end{aligned}$$

Proof. Let $C, C_1, \ldots, C_s$ be $s+1$ distinct columns of $M_1(\nu, d, k)$. To obtain the maximum number of subspaces of type (d, r) which contain P_0 in

$$C \cap \bigcup_{i=1}^{s} C_i = \bigcup_{i=1}^{s} (C \cap C_i),$$

we may assume that each $C \cap C_i$ $(1 \leqslant i \leqslant s)$ is a subspace of type $(k-1, r)$.

Then each $C \cap C_i$ covers $N(d-d_0, 0; k-d_0-1, 0; 2(\nu+r-d_0))$ subspaces of type (d, r) containing P_0 from Lemma 2.1. However, the coverage of each pair of C_i and C_j overlaps at a subspaces of type $(k-2, r)$ containing P_0, where $1 \leqslant i, j \leqslant s$. Therefore, from Lemma 2.1 only C_1 covers the full $N(d-d_0, 0; k-d_0-1, 0; 2(\nu+r-d_0))$ subspaces of type (d, r) containing P_0, while each of $C_2, \ldots, C_s$ can cover a maximum of $N(d-d_0, 0; k-d_0-1, 0; 2(\nu+r-d_0)) - N(d-d_0, 0; k-d_0-2, 0; 2(\nu+r-d_0))$ subspaces of type (d, r) not covered by C_1. By (2), the subspaces of type (d, r) of C not covered by $C_1, C_2, \ldots, C_s$ is at least

$$\begin{aligned} e = {} & N(d-d_0, 0; k-d_0, 0; 2(\nu+r-d_0)) - N(d-d_0, 0; k-d_0-1, 0; 2(\nu+r-d_0)) \\ & -(s-1)(N(d-d_0, 0; k-d_0-1, 0; 2(\nu+r-d_0)) \\ & -N(d-d_0, 0; k-d_0-2, 0; 2(\nu+r-d_0))) \\ = {} & q^{k-d}N(d-d_0-1, 0; k-d_0-1, 0; 2(\nu+r-d_0)) \\ & -(s-1)q^{k-d-1}N(d-d_0-1, 0; k-d_0-2, 0; 2(\nu+r-d_0)). \end{aligned}$$

Since by (2) $\frac{N(d-d_0-1,0;k-d_0-1,0;2(\nu+r-d_0))}{N(d-d_0-1,0;k-d_0-2,0;2(\nu+r-d_0))} = \frac{q^{k-d_0-1}-1}{q^{k-d}-1}$ and $e > 0$, we obtain

$$s < \frac{q(q^{k-d_0-1}-1)}{q^{k-d}-1} + 1.$$

Set

$$b = \frac{q(q^{k-d_0-1} - 1)}{q^{k-d} - 1}.$$

Then $1 \leqslant s \leqslant b$. □

Corollary 3.2. *Suppose that* $2r \leqslant d_0 < d < k \leqslant \nu + r$ *and* $1 \leqslant s \leqslant \min\{b, q+1\}$. *Then* $M_1(\nu, d, k)$ *is not* s^{e+1}*-disjunct, where b and e are as in Theorem 3.1.*

Proof. Let C be a (k, r)-space containing P_0, and E be a fixed $(k-2, r)$-space containing P_0 and contained in C. By Lemma 2.1, we obtain the number of $(k-1, r)$-spaces containing E and contained in C is

$$N(1, 0; 2, 0; 2(\nu + r - k + 2)) = q + 1.$$

For $1 \leqslant s \leqslant \min\{b, q+1\}$, we choose s distinct $(k-1, r)$-subspaces containing E and contained in C, denote these subspaces by Q_i $(1 \leqslant i \leqslant s)$. For each Q_i, we choose a (k, r)-subspace C_i such that $C \cap C_i = Q_i$, where $1 \leqslant i \leqslant s$. Hence each pair of C_i and C_j overlaps at the same $(k-2, r)$-subspace E, where $1 \leqslant i, j \leqslant s$. By Theorem 3.1, it follows that the corollary hold. □

Corollary 3.3. *Suppose that* $d = d_0 + 1$ *and* $1 \leqslant s \leqslant q$. *Then* $M_1(\nu, d, k)$ *is* s^e*-disjunct, but it is not* s^{e+1}*-disjunct, where* $e = q^{k-d_0-2}(q - s + 1)$.

Proof. Setting $d = d_0 + 1$ in the e formula of Theorem 3.1, we obtain

$$e = q^{k-d_0-2}(q - s + 1).$$

The second statement follows directly from Corollary 3.2. □

The following theorem tells us how to choose k so that the test to item ratio is minimized.

Theorem 3.4. *For* $2r \leqslant m_0 < m \leqslant \nu + r$, *the sequence* $N'(m_0, r; m, r; 2\nu)$ *is unimodal and gets its peak at* $m = \left\lfloor \frac{2\nu+2r+m_0}{3} \right\rfloor$ *or* $m = \left\lfloor \frac{2\nu+2r+m_0}{3} \right\rfloor + 1$.

Proof. For $2r \leqslant m_0 \leqslant m_1 < m_2 \leqslant \nu + r$, by (3), we have

$$\begin{aligned} \tfrac{N'(m_0,r;m_1,r;2\nu)}{N'(m_0,r;m_2,r;2\nu)} &= \frac{\prod_{i=m_1-m_0+1}^{m_2-m_0}(q^i - 1)}{\prod_{i=\nu+r-m_2+1}^{\nu+r-m_1}(q^{2i} - 1)} \\ &= \frac{\prod_{i=0}^{m_2-m_1-1}(q^{m_1-m_0+1+i} - 1)}{\prod_{i=0}^{m_2-m_1-1}(q^{2(\nu+r-m_2+1+i)} - 1)} \\ &= \prod_{i=0}^{m_2-m_1-1} \frac{q^{m_1-m_0+1+i} - 1}{q^{2(\nu+r-m_2+1+i)} - 1}. \end{aligned} \tag{4}$$

If $\left\lfloor \frac{2\nu+2r+m_0}{3} \right\rfloor + 1 \leqslant m_1 < m_2 \leqslant \nu + r$, then $\frac{2\nu+2r+m_0}{3} < m_1$. It implies that

$$2m_1 + m_2 > 3m_1 > 2\nu + 2r + m_0. \tag{5}$$

Since $i \leqslant m_2 - m_1 - 1$, by (5) we have

$$m_1 + 2m_2 > 2\nu + 2r + m_0 + 1 + (m_2 - m_1 - 1) \geqslant 2\nu + 2r + m_0 + 1 + i.$$

So

$$m_1 - m_0 + 1 + i > 2(\nu + r - m_2 + 1 + i).$$

It follows that

$$q^{2(\nu+r-m_2+1+i)}-1<q^{m_1-m_0+1+i}-1.$$

Therefore,

$$\frac{q^{m_1-m_0+1+i}-1}{q^{2(\nu+r-m_2+1+i)}-1}>1.$$

From (4) we have

$$N'(m_0,r;m_2,r;2\nu)<N'(m_0,r;m_1,r;2\nu).$$

If $2r\leqslant m_0\leqslant m_1<m_2\leqslant\left\lfloor\frac{2\nu+2r+m_0}{3}\right\rfloor$, then $m_2\leqslant\frac{2\nu+2r+m_0}{3}$. Thus

$$m_1+2m_2<3m_2\leqslant 2\nu+2r+m_0<2\nu+2r+m_0+1+i.$$

It follows that

$$m_1-m_0+1+i<2\nu+2r-2m_2+2+2i=2(\nu+r-m_2+1+i).$$

So

$$q^{m_1-m_0+1+i}-1<q^{2(\nu+r-m_2+1+i)}-1,$$

and hence

$$\frac{q^{m_1-m_0+1+i}-1}{q^{2(\nu+r-m_2+1+i)}-1}<1.$$

From (4) we have

$$N'(m_0,r;m_2,r;2\nu)>N'(m_0,r;m_1,r;2\nu). \qquad \square$$

4. Discussions of test efficiency for construction I

Identifying most positive items with least tests is one of our goals. Therefore, discussing how to make the ratio t/n smaller is significative. In our matrix,

$$t/n=\frac{N'(d_0,r;d,r;2\nu)}{N'(d_0,r;k,r;2\nu)}=\frac{\prod_{i=d-d_0+1}^{k-d_0}(q^i-1)}{\prod_{i=\nu+r-k+1}^{\nu+r-d}(q^{2i}-1)}.$$

We first will explain several facts on the ratio:

(1) Parameter $d_0(\nu,r)$ only appears in the numerator (denominator). It is easy to show that the larger the d_0, ν and r are, the smaller the ratio is.

(2) Noting that the increasing speed of $q^{2i}-1$ is larger than q^i-1, so the smaller the d and k are, the smaller the ratio is.

In [2], D'yachkov et al. constructed with subspaces of $GF(q)$, where q is a prime power, each of the columns(rows) is labeled by an $k(d)$-dimensional space, $m_{ij}=1$ if and only if the label of row i is contained in the label of column j. In order to compare with t/n, we should take the dimension of the space of $GF(q)$ to be $2(\nu+r-d_0)$. Assume that the test efficiency of [2] is t_1/n_1. Then

$$t_1/n_1=\frac{\begin{bmatrix}2(\nu+r-d_0)\\ d\end{bmatrix}_q}{\begin{bmatrix}2(\nu+r-d_0)\\ k\end{bmatrix}_q}=\frac{\prod_{i=d+1}^{k}(q^i-1)}{\prod_{i=2(\nu+r-d_0)-k+1}^{2(\nu+r-d_0)-d}(q^i-1)}.$$

Theorem 4.1. *If* $2d_0 > k-1$, *then* $t/n < q^{d_0(d-k)}t_1/n_1$, *where* $\frac{k-1}{2} < d_0 < d < k$.

Proof

$$\begin{aligned}\frac{t}{n}\Big/\frac{t_1}{n_1} &= \frac{\prod_{i=d-d_0+1}^{k-d_0}(q^i-1)}{\prod_{i=\nu+r-k+1}^{\nu+r-d}(q^{2i}-1)}\Bigg/\frac{\prod_{i=d+1}^{k}(q^i-1)}{\prod_{i=2(\nu+r-d_0)-k+1}^{2(\nu+r-d_0)-d}(q^i-1)}\\
&= \frac{\prod_{i=0}^{k-d-1}(q^{d-d_0+1+i}-1)}{\prod_{i=0}^{k-d-1}(q^{2(\nu+r-k+1+i)}-1)}\Bigg/\frac{\prod_{i=0}^{k-d-1}(q^{d+1+i}-1)}{\prod_{i=0}^{k-d-1}(q^{2(\nu+r-d_0)-k+1+i}-1)}\\
&= \prod_{i=0}^{k-d-1}\frac{q^{d-d_0+1+i}-1}{q^{d+1+i}-1}\prod_{i=0}^{k-d-1}\frac{q^{2(\nu+r-d_0)-k+1+i}-1}{q^{2(\nu+r-k+1+i)}-1}\\
&< \prod_{i=0}^{k-d-1}\frac{q^{d-d_0+1+i}}{q^{d+1+i}}\prod_{i=0}^{k-d-1}\frac{q^{2(\nu+r-d_0)-k+1+i}-1}{q^{2(\nu+r-k+1+i)}-1}\\
&= \prod_{i=0}^{k-d-1}\frac{q^{2(\nu+r)-k+1+i-2d_0}-1}{q^{d_0}(q^{2(\nu+r)-k+1+2i-(k-1)}-1)}.\end{aligned}$$

Since $2d_0 > k-1$, we have $\frac{q^{2(\nu+r)-k+1+i-2d_0}-1}{q^{2(\nu+r)-k+1+2i-(k-1)}-1} < 1$. Therefore,

$$t/n < q^{d_0(d-k)}t_1/n_1,$$

where $\frac{k-1}{2} < d_0 < d < k$. ☐

5. Construction II

Definition 5.1. For $2 \leqslant 2r \leqslant d < k \leqslant \nu + r$, let M be a binary matrix whose columns (rows) indexed by all subspaces of type (k,r) $((d,r))$ in $\mathbb{F}_q^{(2\nu)}$ such that $M(A,B) = 1$ if $A \subseteq B$ and 0 otherwise. This matrix is denoted by $M_2(\nu,d,k)$.

Theorem 5.1. *Suppose* $4 \leqslant 2r+2 \leqslant d < k-1 \leqslant \nu+r-1$. *If* $1 \leqslant s \leqslant q^{2r}$, *then* $M_2(\nu,d,k)$ *is* s^e*-disjunct, where* $e = q^{(k-d-1)d+2r}$.

Proof. Let $C, C_1, \ldots, C_s$ be $s+1$ distinct columns of $M_2(\nu,d,k)$. To obtain the maximum number of subspaces of type (d,r) in

$$C \cap \bigcup_{i=1}^{s} C_i = \bigcup_{i=1}^{s}(C \cap C_i),$$

we may assume that each $C \cap C_i$ is a subspace of type $(k-1,r)$, where $1 \leqslant i \leqslant s$. By (2), the number of the subspaces of type (d,r) of C not covered by $C_1, C_2, \ldots, C_s$ is at least

$$\begin{aligned}&N(d,r;k,r;2\nu) - sN(d,r;k-1,r;2\nu)\\
&\quad = q^{2r(k-d)}\frac{\prod_{i=k-d+1}^{k-2r}(q^i-1)}{\prod_{i=1}^{d-2r}(q^i-1)} - sq^{2r(k-d-1)}\frac{\prod_{i=k-d}^{k-2r-1}(q^i-1)}{\prod_{i=1}^{d-2r}(q^i-1)}\\
&\quad = q^{2r(k-d-1)}\frac{\prod_{i=k-d+1}^{k-2r-1}(q^i-1)}{\prod_{i=1}^{d-2r}(q^i-1)}(q^k - q^{2r} - s(q^{k-d}-1)).\end{aligned}$$

Since $2r+2 \leqslant d < k-1$, we obtain

$$\begin{aligned}\frac{\prod_{i=k-d+1}^{k-2r-1}(q^i-1)}{\prod_{i=1}^{d-2r}(q^i-1)} &= \frac{\prod_{i=0}^{d-2r-2}(q^{i+k-d+1}-1)}{\prod_{i=0}^{d-2r-2}(q^{i+1}-1)}\frac{1}{q^{d-2r}-1}\\ &= \prod_{i=0}^{d-2r-2}\frac{q^{i+k-d+1}-1}{q^{i+1}-1}\frac{1}{q^{d-2r}-1}\\ &= \prod_{i=0}^{d-2r-2} q^{k-d}\frac{q^{i+1}-\frac{1}{q^{k-d}}}{q^{i+1}-1}\frac{1}{q^{d-2r}-1}\\ &> q^{(d-2r-1)(k-d)-(d-2r)}.\end{aligned}$$

Since $1 \leqslant s \leqslant q^{2r}$, and $2r+2 \leqslant d$, we obtain

$$q^k-q^{2r}-s(q^{k-d}-1) \geqslant q^k-q^{2r}-q^{2r}(q^{k-d}-1) = q^{k-d+2r}(q^{d-2r}-1) \geqslant q^{k-d+2r}.$$

Hence $e = q^{(k-d-1)d+2r}$. □

Theorem 5.2. *Suppose* $2 \leqslant 2r \leqslant d < \nu + r$. *Let* $p = \frac{q^{d+1}-q^{2r}}{q-1} - 1$. *If* $1 \leqslant s \leqslant p$, *then* $M_2(\nu, d, d+1)$ *is fully* s^e*-disjunct, where* $e = p - s$.

Proof. By (2), we have $N(d, r; d+1, r; 2\nu) = p+1$. It follows that we can pick $s+1$ distinct subspaces $C, C_1, \ldots, C_s$ of type $(d+1, r)$ such that $C \cap C_i$ and $C \cap C_j$ are two distinct subspaces of type (d, r), where $1 \leqslant i, j \leqslant s$. By the principle of inclusion and exclusion, the number of subspaces of type (d, r) in C but not in each C_i is $p - s + 1$, where $1 \leqslant i \leqslant s$. It follows that $e \leqslant p - s$.

On the other hand, similar to the proof of Theorem 5.4 we obtain

$$e \geqslant N(d, r; d+1, r; 2\nu) - s - 1 = p - s.$$

Hence $e = p - s$. □

The following theorem tells us how to choose k so that the test to item ratio is minimized.

Theorem 5.3. *For m goes from* $2r$ *to* $\nu + r$, *the sequence* $N(m, r; 2\nu)$ *is unimodal and gets its peak at* $m = \left\lfloor \frac{2\nu+2r}{3} \right\rfloor$ *or* $m = \left\lfloor \frac{2\nu+2r}{3} \right\rfloor + 1$.

Proof. For $2r \leqslant m_1 < m_2 \leqslant \nu + r$, by (1), we have

$$\begin{aligned}\frac{N(m_2,r;2\nu)}{N(m_1,r;2\nu)} &= \frac{\prod_{i=\nu+r-m_2+1}^{\nu+r-m_1}(q^{2i}-1)}{\prod_{i=m_1-2r+1}^{m_2-2r}(q^{2r+i}-q^{2r})}\\ &= \frac{\prod_{i=0}^{m_2-m_1-1}(q^{2(\nu+r-m_2+1+i)}-1)}{\prod_{i=0}^{m_2-m_1-1}(q^{m_1+1+i}-q^{2r})}\\ &= \prod_{i=0}^{m_2-m_1-1}\frac{q^{2(\nu+r-m_2+1+i)}-1}{q^{m_1+1+i}-q^{2r}}.\end{aligned}$$

If $\left\lfloor \frac{2\nu+2r}{3} \right\rfloor + 1 \leqslant m_1 < m_2 \leqslant \nu + r$, then $\frac{2\nu+2r}{3} < m_1$. It implies that

$$2m_1 + m_2 > 3m_1 > 2\nu + 2r. \quad (6)$$

Since $i \leqslant m_2 - m_1 - 1$, by (5) we have

$$m_1 + 2m_2 > 2\nu + 2r + 1 + (m_2 - m_1 - 1) \geqslant 2\nu + 2r + 1 + i.$$

Thus

$$m_1 + 1 + i > 2(\nu + r - m_2 + 1 + i).$$

It follows that

$$m_1 + i - 2r \geqslant 2(\nu + r - m_2 + 1 + i) - 2r.$$

So

$$q^{2(\nu+r-m_2+1+i)-2r} \leqslant q^{m_1+i-2r},$$

and hence

$$q^{2(\nu+r-m_2+1+i)-2r} - \frac{1}{q^{2r}} < q^{m_1+i-2r} + [(q-1)q^{m_1+i-2r} - 1] = q^{m_1+1+i-2r} - 1.$$

It follows that

$$\frac{q^{2(\nu+r-m_2+1+i)-2r} - \frac{1}{q^{2r}}}{q^{m_1+1+i-2r} - 1} < 1.$$

Therefore,

$$\frac{q^{2(\nu+r-m_2+1+i)} - 1}{q^{m_1+1+i} - q^{2r}} < 1.$$

From (4) we have

$$N(m_2, r; 2\nu) < N(m_1, r; 2\nu).$$

If $2r \leqslant m_1 < m_2 \leqslant \left\lfloor \frac{2\nu+2r}{3} \right\rfloor$, then $m_2 \leqslant \frac{2\nu+2r}{3}$. Thus

$$m_1 + 2m_2 < 3m_2 \leqslant 2\nu + 2r < 2\nu + 2r + 1 + i.$$

It follows that

$$m_1 + 1 + i < 2\nu + 2r - 2m_2 + 2 + 2i = 2(\nu + r - m_2 + 1 + i).$$

So

$$q^{m_1+1+i} - q^{2r} < q^{2(\nu+r-m_2+1+i)} - q^{2r} < q^{2(\nu+r-m_2+1+i)} - 1.$$

It follows that

$$\frac{q^{2(\nu+r-m_2+1+i)} - 1}{q^{m_1+1+i} - q^{2r}} > 1.$$

From (4) we have

$$N(m_2, r; 2\nu) > N(m_1, r; 2\nu). \quad \square$$

Theorem 5.4. *If $d = 2r,\ k = 2r + 1$, then the test efficiency of construction II is smaller than that of [2].*

Proof. If $d = 2r,\ k = 2r + 1$, then the disjunct matrix of construction II is $M_2(\nu, 2r, 2r + 1)$ and the disjunct matrix of [2] is $M(n, 2r + 1, 2r)$. Let $\frac{t}{n}$ be the test efficiency of $M_2(\nu, 2r, 2r + 1)$ and let $\frac{t_1}{n_1}$ be the test efficiency of $M(n, 2r + 1, 2r)$, respectively. Then

$$\begin{aligned} \tfrac{t}{n} &= \frac{N(d, r; 2\nu)}{N(k, r; 2\nu)} \\ &= \frac{N(2r, r; 2\nu)}{N(2r + 1, r; 2\nu)} \end{aligned}$$

$$\begin{aligned}
&= \frac{q^{2r(\nu+r-2r)}\prod_{i=\nu+r-2r+1}^{\nu}(q^{2i}-1)}{\prod_{i=1}^{r}(q^{2i}-1)\prod_{i=1}^{2r-2r}(q^{i}-1)} \cdot \frac{\prod_{i=1}^{r}(q^{2i}-1)\prod_{i=1}^{2r+1-2r}(q^{i}-1)}{q^{2r(\nu+r-2r-1)}\prod_{i=\nu+r-2r-1+1}^{\nu}(q^{2i}-1)} \\
&= \frac{q^{2r}(q-1)}{q^{2(\nu-r)}-1} \\
&= \frac{q^{2r+1}-q^{2r}}{q^{2\nu-2r}-1},
\end{aligned}$$

and

$$\frac{t_1}{n_1} = \frac{\begin{bmatrix} 2\nu \\ d \end{bmatrix}_q}{\begin{bmatrix} 2\nu \\ k \end{bmatrix}_q} = \frac{\prod_{i=d+1}^{k}(q^{i}-1)}{\prod_{i=2\nu-k+1}^{2\nu-d}(q^{i}-1)} = \frac{q^{2r+1}-1}{q^{2\nu-2r}-1}.$$

Therefore, $\frac{t}{n} < \frac{t_1}{n_1}$. $\square$

6. Conclusion

We construct two classes s^e-disjunct matrix with subspaces in symplectic space $\mathbb{F}_q^{(2\nu)}$. For a pooling design, the less the number of tests is, the better the pooling design is. Assume that the test efficiency in [2] is t_1/n_1. We prove that the test efficiency in construction I is less than $q^{d_0(d-k)}t_1/n_1$, where $\frac{k-1}{2} < d_0 < d < k$, and that the test efficiency in construction II is less than $t_1/n_1 - \frac{q^{2r}-2}{q^{2\nu-2r}-1}$. From Theorem 4.4 of [2], the matrix of construction is s^z-disjunct. To compare the error-correcting capability, we give two tables in the following. Take $s = q,\ d_0 = 2r+1,\ d = 2r+2,\ k = 4r+1,\ \nu = 20$ and $m = 2\nu = 40$, we have Table 1 from Theorem 3.1, 4.1 above and Theorem 4.4 in [2]; similarly, take $s = q,\ d = 2r+1,\ k = 2r+2,\ \nu = 100$ and $m = 2\nu = 200$, we have Table 2 from Theorem 5.1, 5.4 above and Theorem 4.4 in [2]. From Table 1, we know that the error-correcting capability of construction [2] is better than that of ours on some values (for example, $(s,r) = (3,2)$ or $(5,4)$). But in some cases (for example, $(s,r) = (7,5)$ or $(17,6)$), the test efficiency of [2] is not good; whereas on these values (for example, $(s,r) = (7,5)$ or $(17,6)$), the construction I above is feasible. For comparison of construction II with construction [2], their error-correcting capability is better than that of ours from Table 2.

Table 1
Comparison of construction I with D'yachkov et al.

$q = s$	3	5	7	17
r	2	4	5	6
$\frac{t}{n}$	1.9065×10^{-39}	6.3166×10^{-84}	5.2357×10^{-92}	1.7941×10^{-95}
$\frac{t_1}{n_1}$	6.3650×10^{-28}	2.0972×10^{-15}	1.0701×10^{38}	9.0322×10^{175}
e	9	15625	5764801	2.0160×10^{12}
z	675782226	1.1140×10^{49}	2.2283×10^{91}	3.2890×10^{189}

Table 2
Comparison of construction II with D'yachkov et al.

$q = s$	3	5	7	17
r	1	3	4	5
$\frac{t}{n}$	6.0991×10^{-94}	1.5693×10^{-131}	1.9060×10^{-155}	5.2883×10^{-221}
$\frac{t_1}{n_1}$	8.8098×10^{-94}	1.9616×10^{-131}	2.2236×10^{-155}	5.6188×10^{-221}
e	9	15625	5764801	2.0160×10^{12}
z	37	97651	47079201	3.6414×10^{13}

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