Induced Einstein-Kalb-Ramond theory in four dimensions

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(Received 10 December 1991; revised manuscript received 29 May 1992)

We analyze a four-dimensional induced Einstein-Kalb-Ramond theory with a conformally coupled Kalb-Ramond action. We also comment on an *Ansatz* which is inconsistent with the assumption of maximal form invariance imposed on the Kalb-Ramond field due to the cosmological principle. It is argued that the spatial dependences of various fields considered here are inconsistent with the Friedmann-Robertson-Walker metric. One hence justifies the *Ansatz* used in many articles. We also show that the contribution from the Kalb-Ramond action is negligible effectively after the inflationary era. Some of the solutions to the field equations are presented.

PACS number(s): 04.50.+h

I. INTRODUCTION

The ten-dimensional Einstein-Kalb-Ramond action [1,2] given by

$$S=\int d^{10}x\sqrt{g}\,[-R-\tfrac{1}{6}F_{\mu\nu\alpha}F^{\mu\nu\alpha}] \tag{1.1}$$

has attracted a lot of activity lately. The Kalb-Ramond field strength $F_{\mu\nu\alpha}$ is the curvature tensor for the torsion field $A_{\mu\nu}$. Defining a three-form [3]

$$F\equiv F_{\mu\nu\alpha}dx^{\mu}\wedge dx^{\nu}\wedge dx^{\alpha}\ , \tag{1.2}$$

and a two-form

$$A\equiv A_{\mu\nu}dx^{\mu}\wedge dx^{\nu}\ , \tag{1.3}$$

the formal relation between $F_{\mu\nu\alpha}$ and $A_{\mu\nu}$ can be read off from the equation

$$F=dA\ . \tag{1.4}$$

The F^2 term has been studied in many articles, especially in the pointlike limit of the superstring low-energy effective theory, namely, the ten-dimensional (10D) supergravity theory [2] where the F^2 term is known as the Kalb-Ramond action.

Solving the torsion equation, can restrict the torsion field such that the whole effect of the torsion field behaves similar to a scalar field. This property has been the hope of many authors to play the role of inflaton in the inflationary era [4]. Unfortunately, one can show that this effective scalar field will not contribute much to inflation. In fact, one is able to show that the torsion field tends to vanish, in many aspects, as soon as the inflationary process is completed. The complete inflationary action will, however, require an additional inflaton scalar field [5] conformally coupled to both the metric and torsion fields. The inclusion of the scalar field has been studied before in the so-called induced gravity model (without a torsion field) with great success.

On the other hand, scale invariance [6] is one of the key symmetries in obtaining the low-energy effective action for a massless string [7] mode. It is also important in many effective-field theories such as the nonlinear σ model, which has been rather successful in describing low-energy nucleonic interactions. Some even proposed that global scale invariance should be gauged [8].

Therefore, in this paper, we are going to study a four-dimensional induced gravity model with a torsion field conformally coupled to the metric and scalar fields. We are going to argue that the torsion field will not contribute much to the inflationary process and tends to vanish as soon as inflation is completed. This unique property of the torsion field explains in part the reason that the torsion field does not seem to play a role in today's low-energy physics.

This paper is organized as follows. In Sec. II we will solve the torsion-field equation and analyze its properties. In Sec. III the *Ansatz* used in Ref. [1] is justified even if it violates the well-known cosmological principle [9–12] imposed on the $A_{\mu\nu}$ field. We are going to show, in great detail that the radial dependence of all the fields in this model is incompatible with the Friedmann-Robertson-Walker (FRW) metric [9–12] in the asymptotic region. One can further show that the spatial dependence of these fields is inconsistent with the FRW metric. In Sec. IV we show that the contribution of the torsion field is negligible in and after the inflationary process. In Sec. V we solve for some solutions to the field equations and analyze their implications. Finally, in Sec. VI we make a few concluding remarks. We show the conventions used in this paper in the Appendix. The redundancy of the field equations in the FRW metric spaces is also clarified in the Appendix.

II. INDUCED GRAVITY WITH TORSION

In this paper we will study the four-dimensional induced gravity model

$$S=\int d^4x\sqrt{g}\,[-\tfrac{1}{2}\epsilon\phi^2R-\tfrac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\,\partial_{\nu}\phi-V(\phi)$$
$$-(1/6\phi^2)F_{\mu\nu\alpha}F^{\mu\nu\alpha}]\ , \tag{2.1}$$

with a conformally coupled Kalb-Ramond field strength. Here ϕ denotes a real scalar field. Moreover, ϵ is a dimensionless coupling constant. Equation (2.1) also provides a natural explanation for a universe with dimensionful constants such as gravitational "constant" and cosmological "constant."

Indeed, (2.1) is invariant under the scale transformation $g_{\mu\nu}\to s^2 g_{\mu\nu}$ and $\phi\to s^{-1}\phi$, provided that $V(\phi)=(\lambda/8)\phi^4$. Here s denotes a constant scale parameter.

One notes that other possible terms could be present in (2.1) such as higher curvature terms motivated by string theories [13]. In fact, R^2 terms are scale invariant by themselves without the help of the ϕ field. Moreover, the compactification of 10D Einstein-Kalb-Ramond theory also has a contribution from the compactified six-spaces. Our strategy is to take (2.1) as a toy model for simplicity in analyzing the theory in greater detail. The equations of motion (EOM's) from varying the action (2.1) with respect to ϕ, $g_{\mu\nu}$, and $A_{\mu\nu}$ are

$$\epsilon\phi^2 G^{\mu\nu}=\epsilon(D^\mu\partial^\nu-g^{\mu\nu}D_\alpha\partial^\alpha)\phi^2 + \frac{1}{6\phi^2}(F^2 g^{\mu\nu}-6F^\mu_{\alpha\beta}F^{\nu\alpha\beta})+T^{\mu\nu}_\phi \ , \tag{2.2}$$

$$\epsilon R\phi=D_\mu\partial^\mu\phi+\frac{1}{3\phi^2}F^2-\frac{\partial V(\phi)}{\partial\phi} \ , \tag{2.3}$$

$$D_\mu(\phi^{-2}F^{\mu\nu\alpha})=0 \ . \tag{2.4}$$

Here $T^{\mu\nu}_\phi$, the energy-momentum tensor associated with ϕ, is

$$T^{\mu\nu}_\phi=\partial^\mu\phi\,\partial^\nu\phi-g^{\mu\nu}[\tfrac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi+V(\phi)] \ . \tag{2.5}$$

Also, $G^{\mu\nu}\equiv\frac{1}{2}Rg^{\mu\nu}-R^{\mu\nu}$ defines the Einstein tensor.

In fact, (2.4) is equivalent to $\partial_\mu(\sqrt{g}\,\phi^{-2}F^{\mu\nu\alpha})=0$, which can be solved straightforwardly. Indeed, this can be done by observing that the Kalb-Ramond field strength $F_{\mu\nu\alpha}$ is a totally skew-symmetric-type $T(0,3)$ tensor. Therefore we can map it to some type $T(1,0)$ contravariant vector T^δ with the help of the totally skew-symmetric-type $T(0,4)$ Levi-Civita tensor $\epsilon_{\mu\nu\alpha\delta}$. To be more precise, there exists a contravariant vector T^δ such that

$$F_{\mu\nu\alpha}=\epsilon_{\mu\nu\alpha\delta}T^\delta \tag{2.6}$$

for every totally skew-symmetric-type $T(0,3)$ tensor $F_{\mu\nu\alpha}$ defined on our four-dimensional base manifolds. Note that both sides of Eq. (2.6) have exactly the same symmetry among their indices such that all degrees of freedom have been taken into account. One can hence show that (2.4) can be written as

$$\epsilon^{\mu\nu\alpha\delta}\partial_\mu(\phi^{-2}T_\delta)=0 \ . \tag{2.7}$$

By introducing the one-form $T\equiv T_\mu dx^\mu$, one can show that (2.7) is equivalent to the two-form equation

$$^*d(\phi^{-2}T)=0 \tag{2.8}$$

after we multiply (2.7) with $dx^\nu\wedge dx^\alpha$ such that (2.7) becomes $\epsilon^{\mu\nu\alpha\delta}\partial_\mu(\phi^{-2}T_\delta)dx^\nu\wedge dx^\alpha=0$. Here the asterisk is the Hodge star operator [3], which maps a differential n-form into its dual $(d-n)$-form in d-dimensional spaces. Therefore one has

$$d(\phi^{-2}T)=0 \tag{2.9}$$

as a result of the involutive Hodge star operator, namely, $**\mathbb{1}$, the identity map. If we live in a (pseudo-)Riemannian space M that has a trivial first cohomology group [14], namely, $H^1(M)=0$ such that all closed one-forms defined on M are exact, there exists a scalar field χ that satisfies

$$\phi^{-2}T=d\chi \ . \tag{2.10}$$

For example, all simply connected spaces [i.e., $\Pi_1(M)=0$] and contractible spaces belong to the class $H^1(M)=0$. Following the idea of the Mach principle, we hope that all dimensionful coupling constants can be dimensionless somehow. Therefore it sill be convenient for us to write χ as $\chi\equiv\ln\eta^l$ for some dimension-1 real scalar field η, where l denotes some dimensionless constant. The solution to Eq. (2.4) can hence be shown to be

$$F_{\mu\nu\alpha}=l\epsilon_{\mu\nu\alpha\delta}\phi^2\frac{\partial^\delta\eta}{\eta} \ , \tag{2.11}$$

which has the required correct dimension. Moreover, in Sec. III we are going to treat η as an effective-field variable and study its implications. Before going any further, we will need the identities

$$F_{\mu\alpha\beta}F^{\alpha\beta}_\nu=-2l^2\frac{\phi^4}{\eta^2}(g_{\mu\nu}\partial_\alpha\eta-\partial_\mu\eta\,\partial_\nu\eta) \ ; \tag{2.12}$$

hence,

$$F^2=-6l^2\frac{\phi^4}{\eta^2}\partial_\alpha\eta\,\partial^\alpha\eta \ . \tag{2.13}$$

Consequently, (2.2) and (2.3) can be simplified greatly by substituting all $F_{\mu\nu\alpha}$ with the solutions given in (2.12) and (2.13). In order to bring (2.2) and (2.3) into a more comprehensive form, we will define $\phi\equiv e^{\varphi/2}$, $\eta\equiv e^{\theta/2}$, $k_1\equiv 1/4\epsilon$, and $k_2\equiv l^2/2\epsilon$ for later convenience. In terms of these new variables and parameters, (2.2) and (2.3) become

$$G_{\mu\nu}=\partial_\mu\varphi\,\partial_\nu\varphi+D_\mu\partial_\nu\varphi-g_{\mu\nu}(\partial_\alpha\varphi\,\partial^\alpha\varphi+D_\alpha\partial^\alpha\varphi) -T_{\mu\nu}(\varphi)-T_{\mu\nu}(\theta) \ , \tag{2.14}$$

$$R=k_1(\partial_\mu\varphi\,\partial^\mu\varphi+2D^\mu\partial_\mu\varphi)-k_2(\partial_\mu\theta\,\partial^\mu\theta)-\frac{2}{\epsilon}\frac{\partial V}{\partial\varphi}e^{-\varphi} \ , \tag{2.15}$$

after some algebra. Note that we have brought all the ϕ dependence to the right-hand side of the above equations. Moreover, the generalized energy-momentum tensor $T_{\mu\nu}(\varphi)$ and $T_{\mu\nu}(\theta)$ are defined as

$$T_{\mu\nu}(\varphi)\equiv k_1(\tfrac{1}{2}g_{\mu\nu}\partial_\alpha\varphi\,\partial^\alpha\varphi-\partial_\mu\varphi\,\partial_\nu\varphi)+\frac{1}{\epsilon}Ve^{-\varphi}g_{\mu\nu} \ , \tag{2.16}$$

$$T_{\mu\nu}(\theta)\equiv k_2(\tfrac{1}{2}g_{\mu\nu}\partial_\alpha\theta\,\partial^\alpha\theta-\partial_\mu\theta\,\partial_\nu\theta) \ , \tag{2.17}$$

respectively. Note that, by solving (2.4), we have not obtained all informations of the torsion field, but reduced its degrees of freedom from 4 (T^α) to 1 (θ). The remaining degree of freedom will be shown to be constrained by the invisible Bianchi identity hidden in (2.2). This point will be clarified further in Sec. IV.

III. RADIAL DEPENDENCE

The latest observation [9] of microwave background radiation gives us evidence for isotropy at decoupling. This indicates that our Universe is isotropic and homogeneous to a very high degree of precision. We will hence adopt the well-accepted cosmological principle [10–12] in this paper. Therefore, by time foilating our four-dimensional spaces into spacelike three-dimensional (pseudo-)Riemannian spaces, one can prove that the three scalar curvatures of all isotropic and homogeneous manifolds equal some constants. Moreover, it has been shown that all constant-curvature manifolds fall into three different classes denoted by $k=0,\pm 1$. Here $k=-R_3/6$, with R_3 denoting its three-curvature. Accordingly, we can parametrize the corresponding (pseudo-)Riemannian metric tensor as the well-known FRW metric. Any physically acceptable model must, therefore, admit some FRW-type solutions in order to be compatible with the cosmological observations at the cosmological scale. Hence all effective cosmological theories must be consistent with the FRW metric.

People used to conclude that φ and θ have to be functions of t only if φ and θ are assumed to have a maximal form invariance compatible with the FRW metric, namely, $\varphi'(x)=\varphi(x)$ and $\theta'(x)=\theta(x)$, where $\theta'(x)$ denotes the form of the field θ after a coordinate transformation such that $\theta'(x')=\theta(x)$. Furthermore, it can be shown that if the torsion field itself is assumed to be maximal form invariant under spatial rotation and translation, then θ and ϕ have to be functions of t only. Indeed, after an infinitesimal translation $x''^i=x^i+\epsilon^i$, one can show that

$$\epsilon^i\partial_i\theta=0 \ . \tag{3.1}$$

Hence one concludes that θ and, similarly, φ have to be functions of t only.

On the other hand, it was also assumed that all fields involved should have maximal form invariance, including $A_{\mu\nu}$ studied in this paper. It is then straightforward to show that $A_{\mu\nu}$ has to be a function of t only, for example, in $k=0$ FRW spaces or spatial $\mathbb{R}^3$. Indeed, under an infinitesimal translation $x''^i=x^i+\epsilon^i$, one has

$$\partial_\mu\epsilon^i A_{i\nu}-\partial_\nu\epsilon^i A_{i\mu}+\epsilon^i\partial_i A_{\mu\nu}=0 \ . \tag{3.2}$$

If ϵ^i is a constant three-vector (i.e., a translation in spatial $\mathbb{R}^3$), one has $\partial_i A_{\mu\nu}=0$ such that $A_{\mu\nu}$ has to be a function of t only.

Consequently, the only nonvanishing components of the Kalb-Ramond field tensor are F_{0ij} and its permutations. Comparing with (2.11), we find that $F_{0ij}=0$ if $\eta=\eta(t)$. Hence the form invariance of $A_{\mu\nu}$ is inconsistent with the *Ansatz* $\theta=\theta(t)$ unless $\eta=0$. Therefore it seems to indicate that θ_t should not play a role on the FRW manifolds. This also favors somehow our conclusion, which will be shown shortly, that the torsion field should decrease to a negligible value in order to keep the FRW spaces stable.

In fact, (3.2) itself is enough to imply $A_{\mu\nu}=0$ by exhausting all symmetries generated by typical rotational generators $\epsilon^i=\epsilon^i_j x^j$. Here ϵ^{ij} denotes some skew-symmetric constant parameters for rotation in $\mathbb{R}^3$. Indeed, (3.2) becomes

$$\partial_\mu\epsilon^i A_{i\nu}=\partial_\nu\epsilon^i A_{i\mu} \ , \tag{3.3}$$

since $\partial_i A_{\mu\nu}=0$. Hence the $\mu=0$ equation of (3.3) gives $\partial_\nu\epsilon^i A_{i0}=0$, which implies $A_{i0}=0$. On the other hand, $\mu=k$ gives $\partial_k\epsilon^i A_{ij}=\partial_j\epsilon^i A_{ik}$, which implies $\epsilon^i_k A_{ij}=\epsilon^i_j A_{ik}$. This in turn implies $A_{ij}=0$. Note also that $A_{\mu\nu}$ is a skew-symmetric second-rank tensor such that above argument shows that all skew-symmetric second-rank tensors vanish in the $k=0$ FRW spaces if they are assumed to be maximal form invariant.

Note that one does not, however, need to assume that all fields are maximal form invariant except the metric field [15]. Indeed, it is known that the metric is deformed by the contributions from all fields such that the geometry of the system is determined accordingly and passively. It follows that the only object that should respect the various symmetries involved is the generalized energy-momentum tensor [for example, the $T^{\mu\nu}$ given in (A5)] because of its coupling to the metric field via the variational equations.

It is, however, not easy to elaborate all spatial dependence in the field equations. Fortunately, we are going to show that by relaxing one above constraint and allowing a radial deviation (i.e., parametrizing φ and θ as functions of t and r only), the radial deviation is in general incompatible with the FRW metric. It turns out that any small radial deviation will turn on the divergent deviation in the asymptotic region ($r\to\infty$). The field form in its asymptotic region is independent of its angular variations because all fields are assumed to vanish in spatial infinity. Therefore, if one can prove that all fields are divergent in spatial infinity by assuming the radial deviation mentioned above, it is reasonable to conclude that all fields involved in the theory should exclude any spatial dependence in the FRW spaces. Note that r in $k=1$ spaces is, in fact, an angular variable; therefore, our argument above will not be applied. But it seems reasonable to assume that we do not need to consider the angular deviation at all in $k=1$ FRW spaces. It is expected that the cosmological principle in $k=1$ FRW spaces will simply keep all angular variables discriminated equally. Therefore it is natural to assume the function of a t-only conclusion.

Therefore, before accepting the *Ansätze* $\varphi=\varphi(t)$ and $\theta=\theta(t)$, we are going to elaborate a bit about the this tricky radial dependence of θ and φ. Indeed, the ti and ij components of (2.14) read

$$\partial_t\varphi\,\partial_i\varphi+D_t\partial_i\varphi+k_1\partial_t\varphi\,\partial_i\varphi+k_2\partial_t\theta\,\partial_i\theta=0 \ , \tag{3.4}$$

$$\partial_i\varphi\,\partial_j\varphi+(D_i\partial_j\varphi)_3+k_1\partial_i\varphi\,\partial_j\varphi+k_2\partial_i\theta\,\partial_j\theta=Wh_{ij} \ . \tag{3.5}$$

Here $(D_i\partial_j\varphi)_3 \equiv \partial_i\partial_j\varphi - \Gamma^k_{ij}\partial_k\varphi$ denotes that all derivatives are to be evaluated on the three-dimensional spatial slices (M^3, h_{ij}). Also, the right-hand side of Eq. (3.5) denotes all h_{ij} proportional terms. We will be able to show that the details of W are not essential to solving (3.5). Therefore we will keep dumping all h_{ij} proportional terms found on the left-hand side of (3.5) without renaming W for convenience.

Note that

$$h_{ij} = \text{diag}\left[\frac{1}{1-kr^2}, r^2, r^2\sin^2\theta\right],$$

hence, it is straightforward to show that

$$\Gamma^r_{ij} = \delta_{ir}\delta_{jr}\frac{h_{rr}}{r} + h_{ij}\left[kr - \frac{1}{r}\right]. \tag{3.6}$$

Therefore (3.5) can be rewritten as

$$\partial_i\partial_j\varphi - \delta_{ir}\delta_{jr}\frac{h_{rr}}{r}\partial_r\varphi + (1+k_1)\partial_i\varphi\,\partial_j\varphi + k_2\partial_i\theta\,\partial_j\theta = Wh_{ij}, \tag{3.7}$$

where we have dumped $h_{ij}(kr-1/r)\partial_r\varphi$ in Wh_{ij}. It is easy to show that the only nonvanishing component on the left-hand side of (3.7) is the rr component. Therefore the $\theta\theta$ and $\varphi\varphi$ components of (3.7) imply

$$W = 0 . \tag{3.8}$$

Hence the tr equation of (3.4) and the rr equation of (3.7) become

$$\dot{\varphi}' - \dot{\alpha}\varphi' + (1+k_1)\dot{\varphi}\varphi' + k_2\dot{\theta}\theta' = 0 , \tag{3.9}$$

$$\varphi'' - \frac{1}{r(1-kr^2)}\varphi' + (1+k_1)\varphi'^2 + k_2\theta'^2 = 0 . \tag{3.10}$$

Here we have kept only nonvanishing parts of (3.4) and (3.7), while an overdot and prime denote differentiating with respect to t and r, respectively, throughout the rest of this section. We can also write (3.9) and (3.10) slightly different as

$$e^{\alpha-(1+k_1)\varphi}\partial_t(e^{-\alpha+(1+k_1)\varphi}\varphi') = -k_2\dot{\theta}\theta' , \tag{3.11}$$

$$e^{g(r)-(1+k_1)\varphi}\partial_r(e^{-g(r)+(1+k_1)\varphi}\varphi') = -k_2\theta'^2 , \tag{3.12}$$

by introducing the integration factors $e^{-\alpha+(1+k_1)\varphi}$ and $e^{-g(r)+(1+k_1)\varphi}$, respectively, in (3.11) and (3.12). Here $g(r)$ is defined by the differential equation

$$g' \equiv \frac{1}{r(1-kr^2)} , \tag{3.13}$$

up to an arbitrary integration constant. Note that (3.11) and (3.12) can be written as

$$\varphi'\dot{f} = -k_2\dot{\theta}\theta' , \tag{3.14}$$

$$\varphi' f' = -k_2\theta'^2 . \tag{3.15}$$

Here we have introduced a real function $f = f(t,r)$ such that

$$\varphi'(t,r) = e^{\alpha(t)+g(r)-(1+k_1)\varphi(t,r)+f(t,r)} , \tag{3.16}$$

namely,

$$f(t,r) = (1+k_1)\varphi(t,r) - \alpha(t) - g(r) + \ln\varphi'(t,r) . \tag{3.17}$$

If $k_2 \neq 0$, (3.14) and (3.15) imply

$$\dot{f}\theta' = f'\dot{\theta} . \tag{3.18}$$

By introducing one-forms $df \equiv \dot{f}dt + f'dr$ and $d\theta \equiv \dot{\theta}dt + \theta' dr$ defined on M^2, one can rewrite (3.18) as a two-form equation

$$df \wedge d\theta = 0 . \tag{3.19}$$

There implies that f is a function of θ, namely, $f = f(\theta(t,r))$ or vice versa. Consequently, (3.14) and, also, (3.15) imply

$$\varphi' = -k_2\theta'\left[\frac{\partial f}{\partial\theta}\right]^{-1} . \tag{3.20}$$

Since $\partial f/\partial\theta$ is a function of θ too, it is possible to find a function $F(\theta)$ such that

$$\frac{\partial F}{\partial\theta}\frac{\partial f}{\partial\theta} = -k_2 . \tag{3.21}$$

Note that $\partial F/\partial\theta = -k_2(\partial f/\partial\theta)^{-1}$; hence, $\partial F/\partial\theta$ could be trouble at those points where, $\partial f/\partial\theta = 0$. But we will show, shortly, that there will not be any troubles at all by showing that $\partial F/\partial\theta$ is, in fact, finite everywhere except at spatial infinity. Therefore (3.20) becomes

$$\varphi' = \frac{\partial F}{\partial\theta}\theta' = F' , \tag{3.22}$$

such that φ can be integrated directly so that the result reads

$$\varphi = F(\theta(t,r)) + \beta(t) . \tag{3.23}$$

Here β, an integration constant, denotes some arbitrary function of t only. Consequently, (3.16) becomes

$$F' = e^{\tilde{\alpha}(t)+g(r)-(1+k_1)F(\theta)+f(\theta)} . \tag{3.24}$$

Here $\tilde{\alpha}(t) \equiv \alpha(t) - (1+k_1)\beta(t)$. Furthermore, by defining

$$G(\theta) \equiv \frac{\partial F(\theta)}{\partial\theta}e^{(1+k_1)F(\theta)-f(\theta)} , \tag{3.25}$$

which is a function of θ only, (3.24) can be written as

$$G(\theta)\theta' = e^{\tilde{\alpha}(t)+g(r)} . \tag{3.26}$$

Moreover, by introducing $H(\theta)$ such that

$$G(\theta) \equiv \frac{\partial H(\theta)}{\partial\theta} , \tag{3.27}$$

(3.26) can be further written as

$$\frac{\partial H(\theta)}{\partial r} = e^{\tilde{\alpha}(t)+g(r)} . \tag{3.28}$$

Equation (3.28) can thus be integrated directly to give

$$H(\theta)=e^{\bar{\alpha}(t)}\int^r ds\, e^{g(s)}\equiv e^{\bar{\alpha}(t)}Q(r) ; \tag{3.29}$$

i.e., the factor $Q(r)$ defined by

$$Q(r)\equiv\int^r ds\, e^{g(s)} \tag{3.30}$$

can be computed from the definition of $g(r)$ given in (3.13). After some algebra, one has

$$g(r)=\ln g_0\frac{r}{(1-kr^2)^{1/2}} . \tag{3.31}$$

here g_0 is an integration constant, which will be ignored later. We are only interested in the (asymptotic) radial dependence of θ [hence $\varphi=F(\theta)+\beta(t)$]; therefore, we will also turn off the t dependence for later convenience. Furthermore, Eq. (3.29) states that the radial dependence of $\theta(r)$ can be obtained from, and should be similar to, $Q(r)$. Indeed, the radial dependence of Q can be solved directly by integrating (3.30). After some algebra, one has

$$Q_{k=0}(r)=\frac{r^2}{2} , \tag{3.32}$$

$$Q_{k=-1}(r)=(1+r^2)^{1/2} , \tag{3.33}$$

up to some integration constants.

Note that if $k_2=0$, (3.11) and (3.12) can actually be integrated directly [9] to give

$$e^{-\alpha+(1+k_1)\varphi}\varphi'=f_1(r) , \tag{3.34}$$

$$e^{-g(r)+(1+k_1)\varphi}\varphi'=f_2(t) . \tag{3.35}$$

Here $f_1(r)$ and $f_2(t)$ are integration constants in t and r, respectively. One has immediately $e^{-\alpha+g}=f_1/f_2$. A careful analysis shows that

$$e^{-\alpha(t)-g(r)+(1+k_1)\varphi}\varphi'=k_5 , \tag{3.36}$$

since $f_1(r)e^{-g(r)}=f_2(t)e^{-\alpha(t)}$ must equal a real constant $\equiv k_5$. Therefore (3.36) can be written as

$$\partial_r e^{(1+k_1)\varphi}=k_6 e^{\alpha+g} , \tag{3.37}$$

which can be integrated directly to give

$$\varphi(t,r)=\frac{1}{1+k_1}\ln[k_6 e^{\alpha(t)}Q(r)] . \tag{3.38}$$

Here $k_6\equiv(1+k_1)k_5$ is a constant. Note that $\phi^2\equiv e^{\varphi}$.

Note that both $Q_{k=0}$ and $Q_{k=-1}$ diverge badly as $r\to\infty$. Hence θ and φ also diverge in the asymptotic region. Therefore the dynamics of this model in the FRW spaces favors exactly isotropic and homogeneous θ and φ fields. Therefore we have shown that the spatial dependence of θ and φ is inconsistent with the FRW metric in this theory.

IV. CONTRIBUTION FROM THE TORSION FIELD

Note that the G_{ij} equation in (2.14) or, equivalently, (2.15) is, in fact, redundant as a result of the Bianchi identity $D_\mu G^{\mu\nu}=0$ (see the Appendix) [16] and the spherically symmetric property of the FRW metric. Therefore we are left with two equations for three unknowns α, φ, and θ. It is apparent that there must be some hidden information somewhere; otherwise, there should be an extra hidden symmetry to be discovered. Fortunately, it is the right-hand side of (2.14), or $T_{\mu\nu}(\varphi,\theta)$, the generalized energy-momentum tensor, which should obey the current-conservation constant $D_\mu T^{\mu\nu}(\varphi,\theta)=0$. Moreover, one has

$$[D_\mu,D_\nu]\partial_\alpha\varphi=R^{\ \beta}_{\alpha\nu\mu}\partial_\beta\varphi \tag{4.1}$$

from the definition of curvature tensor given in (A1). By taking the trace with $g^{\nu\alpha}$, one obtains

$$[D_\mu,D_\nu]\partial^\nu\varphi=R_{\mu\nu}\partial^\nu\varphi . \tag{4.2}$$

Indeed, the current conservation $D_\mu T^{\mu\nu}(\varphi,\theta)=0$ can be shown to be

$$R_{\mu\nu}\partial^\nu\varphi=(1+k_1)\partial_\mu\varphi\, D_\nu\partial^\nu\varphi-D_\mu\partial_\nu\varphi\,\partial^\nu\varphi+k_2\partial_\mu\theta\, D_\nu\partial^\nu\theta \tag{4.3}$$

after a little bit of algebra. Furthermore, $R_{\mu\nu}=\frac{1}{2}g_{\mu\nu}R-G_{\mu\nu}$ can be used to replace the $R_{\mu\nu}$ in (4.3) with R and $G_{\mu\nu}$ given by (2.15) and (2.14), respectively. Therefore one obtains

$$k_2\partial_\mu\theta(\partial_\nu\theta\,\partial^\nu\varphi+D_\nu\partial^\nu\theta)=0 . \tag{4.4}$$

If $k_2\neq0$ and $\partial_\mu\theta\neq0$ for some μ (or $F_{\mu\nu\alpha}\neq0$), (4.4) gives, simply,

$$\partial_\nu\theta\,\partial^\nu\varphi+D_\nu\partial^\nu\theta=0 . \tag{4.5}$$

We will, from now on, accept the *Anätze* $\varphi=\varphi(t)$ and $\theta=\theta(t)$ along with the FRW metric. As a result, (4.5) is equivalent to

$$\theta_{tt}+3\alpha_t\theta_t+\theta_t\varphi_t=0 . \tag{4.6}$$

Here $f_t\equiv df/dt$, $f_{tt}\equiv d^2f/dt^2$, etc., denote differentiation with respect to t once, twice, etc. In fact, (4.6) can be integrated directly by introducing an integration factor $e^{3\alpha+\varphi}$ such that (4.6) is equivalent to

$$\partial_t(e^{3\alpha+\varphi}\theta_t)=0 . \tag{4.7}$$

Hence the solution to (4.6) is

$$\theta_t=\text{const}\times e^{-3\alpha-\varphi} . \tag{4.8}$$

Note that (4.4) holds for any form of the potential term $V(\varphi)$. Therefore the solution (4.8) states that θ_t (hence the torsion field) is, in fact, negligible if the induced Einstein-Kalb-Ramond theory (2.1) describes an inflationary universe. This is induced by the strong decreasing factor $e^{-3\alpha}$ in (4.8) if e^α [$=a(t)$] is undergoing a strong expansion during the inflationary phase and some reasonable initial conditions imposed on θ_t which fix the proportional constant in (4.8). We can also put it another way: It is the low-energy phenomenon that requires the torsion field to be negligible, which is a reasonable consequence of Eq. (4.8) that accommodates the physics without any exotic boundary conditions. Some explicit solutions will be solved in Sec. V for further details.

V. SOME SOLUTIONS

In this section we are going to study the explicit θ behavior by solving the EOM directly. Note that one can simplify the tt equation in (2.14) as

$$\alpha_t^2+\alpha_t\varphi_t+ke^{-2\alpha}=\frac{k_1}{6}\varphi_t^2+\frac{k_2}{6}\theta_t^2+\frac{V}{3\epsilon}e^{-\varphi}\ , \tag{5.1}$$

when the FRW metric is substituted. Furthermore, the trace equation of (2.14), $g^{\mu\nu}R_{\mu\nu}$, gives

$$R=-3(\partial_\mu\varphi\,\partial^\mu\varphi+D^\mu\partial_\mu\varphi)-k_1\partial_\mu\varphi\,\partial^\mu\varphi-k_2\partial_\mu\theta\,\partial^\mu\theta-\frac{4V}{\epsilon}e^{-\varphi}\ . \tag{5.2}$$

Hence (2.15) and (5.2) show that

$$D_\mu\partial^\mu\varphi+\partial_\mu\varphi\,\partial^\mu\varphi=\frac{2}{(3+2k_1)\epsilon}e^{-\varphi}\left[\frac{\partial V}{\partial\varphi}-2V\right]\ . \tag{5.3}$$

Substituting the FRW metric into (5.3), one arrives at the ordinary differential equation (ODE)

$$\varphi_{tt}+3\alpha_t\varphi_t+\varphi_t^2=\frac{2}{(3+2k_1)\epsilon}e^{-\varphi}\left[2V-\frac{\partial V}{\partial\varphi}\right]\ . \tag{5.4}$$

We will demonstrate the minor contribution from the Kalb-Ramond field by solving two special models with $V(\phi)=(\lambda/8)\phi^4$ and $V=0$, respectively. These two sets of solutions turn out, however, to be not very interesting because they are difficult to be considered as physically acceptable inflationary solutions.

Note that the right-hand side of (5.4) vanishes if $V(\phi)=(\lambda/8)\phi^4$. This is also the scale-invariant potential required by the scale symmetry for the action (2.1). Therefore, in the scale invariant limit, (5.4) reads

$$\varphi_{tt}+3\alpha_t\varphi_t+\varphi_t^2=0\ , \tag{5.5}$$

which can be integrated by introducing the integration factor $e^{3\alpha+\varphi}$ too. The result is

$$\partial_t(e^{3\alpha+\varphi}\varphi_t)=0\ , \tag{5.6}$$

which has the solution

$$\varphi_t=\text{const}\times e^{-3\alpha-\varphi}\ . \tag{5.7}$$

This implies, from the similar structure of (4.8) and (5.7), that

$$\theta_t=k_3\varphi_t\ , \tag{5.8}$$

with k_3 denoting some constant to be determined by the initial condition. Eliminating θ_t in (5.1) by $k_3\phi_t$, one has

$$\alpha_t^2+\alpha_t\varphi_t+ke^{-2\alpha}=\frac{k_4}{6}\varphi_t^2+\frac{\lambda}{24\epsilon}e^{\varphi}\ . \tag{5.9}$$

Here $k_4\equiv k_1+k_2k_3=(1+2k_3l^2)/4\epsilon$ denotes another constant. The EOM's (5.5) and (5.9) have the unique solution.

$$\varphi=\varphi_0\ , \tag{5.10}$$

$$\alpha=\ln\left\{\exp\left[\alpha_0+\left[\frac{\lambda\varphi_0}{24\epsilon}\right]^{1/2}t\right]+\frac{6\epsilon k}{\lambda\varphi_0}\exp\left[-\alpha_0-\left[\frac{\lambda\varphi_0}{24\epsilon}\right]^{1/2}t\right]\right\}\ . \tag{5.11}$$

Note that (5.11) is a solution to $\alpha_t^2+ke^{-2\alpha}=(\lambda/24\epsilon)e^{\varphi_0}$, a result of (5.9) and (5.10). Here φ_0 and α_0 are constants. Note that α_0 is just an arbitrary parameter, not the initial value of α. Note further that any small deviation for φ_t will obey (5.7) and, hence, tends to return to its initial value $\varphi=\varphi_0$ if we are dealing with inflationary physics by applying the action (2.1). This is obvious from (5.7) and is also confirmed from a numerical computation.

The reason for obeying (5.7) comes from the scale symmetry. Equation (5.7) is, in fact, the scale current equation associated with the global scale symmetry. Hence the scale symmetry imposes a fairly strong constraint on the inflationary models that forces the set of solutions (5.10) and (5.11) to be unique up to two unknown parameters φ_0 and α_0.

Unfortunately, the solution (5.11) gives rise to inflation with no graceful exit [17]. Therefore ϕ^4 theory is commonly considered as a physically unacceptable inflationary model.

If $\lambda=0$, this set of solutions will no longer be valid [18]. It turns out that the vanishing cosmological limit ($V=0$) must be treated separately. It also turns out that the theory in this limit can be solved exactly on $k=0$ FRW spaces. For later convenience, we will consider the equivalent set of EOM [Eq. (5.9)] and ij equation of (2.14), namely,

$$\alpha_t^2+\alpha_t\varphi_t=\frac{k_4}{6}\varphi_t^2\ , \tag{5.12}$$

$$2\alpha_{tt}+3\alpha_t^2-\alpha_t\varphi_t=-\frac{k_4}{2}\varphi_t^2\ . \tag{5.13}$$

Note that we have set $\lambda=0=k$. This set of equations is also a good approximation if $k=0$ or $\alpha_t^2\gg e^{-2\alpha}$ such that the $ke^{-2\alpha}$ term in (5.1) is negligible. This approximation turns out to be a good one if we are dealing with an inflationary solution. Note that (5.12) is a simple algebraic equation for φ_t and θ_t. Therefore it can be solved easily to give

$$\alpha_t=-p\varphi_t \tag{5.14}$$

or, equivalently, $\alpha+p\varphi=\text{const}$. Here $p=p_\pm=(1\pm\sqrt{1+(2/3)k_4})$ with $p_\pm$ denoting the roots of $p^2-p=k_4/6$. The solution can hence be obtained rather straightforwardly. The final answer is

$$\varphi=-\frac{\varphi_0}{pq}\ln(1-pq\varphi_{t_0}t)\ , \tag{5.15}$$

$$\alpha=\frac{\alpha_0}{q}\ln(1-pq\varphi_{t_0}t)\ . \tag{5.16}$$

Here $\alpha_0\equiv\alpha(t=0)$, $\varphi_0\equiv\varphi(t=0)$, and $\varphi_{t_0}\equiv\varphi_t(t=0)$ denote initial conditions. Also, $q_\pm=3-1/p_\pm$, the solutions to $q_\pm=2+k_4/6p^2$, denote some constant parameters. Note that $1+\frac{2}{3}k_4 \geq 0$ is required for real solutions in (5.15) and (5.16). Equivalently, $k_3 l^2 \geq -(1+6\epsilon)/2$ should be a constraint on k_3 and l.

Note that the solution (5.16) is, however, impossible to generate enough inflation. Therefore we will turn our attention to a more realistic model which is more relevant to the inflationary process. One notes that the torsion-free induced gravity model has been studied intensively [5,17] before. Various results shown above are well known in the absence of the Kalb-Ramond term. We will show that the contribution from the Kalb-Ramond field is negligible as compared to the torsion-free models if one is dealing with inflationary theories. Moreover, we will show that the torsion field tends to vanish as soon as the inflationary process is completed.

In fact, the torsion-free broken-symmetric potential $=(\lambda/8)(\phi^2-v^2)^2$ has been studied [5,17] intensively before. The minor correction in the theory (2.1) with torsion is indicated by (4.38). Various properties in the theory with torsion are similar to the torsion-free theory. The complete numerical solutions to Eqs. (5.1) and (5.4) can be solved by a Runge-Kutta [19] package in MATHEMATICA. We plot a few graphs in Fig. 1 for the general behavior of $\alpha(t)$ and $\varphi(t)$. In Fig. 1 we have set $k=0$, $\alpha(t=0)=1$, $\varphi_0=-5$, $\varphi_{t_0}=4\times10^{-17}$, $\epsilon=0.01$, and $v=10$, which are taken from the paper of Accetta, Zoller, and Turner [5]. Note that the purpose of this figure is aimed at demonstrating in part the general behavior of $\alpha(t)$ and $\varphi(t)$, while $\theta_t(t)$ can be realized easily from the t dependence of α. Also, note that the solid (dotted) lines draw the t dependence of α and $\phi_2=e^{\varphi}$ in the absence (presence) of the Kalb-Ramond action, respectively. The upper (lower) graphs denote ϕ^2 (α), respectively.

Note that we have also set $l=k_7=1$ (where $\theta_t \equiv k_7 e^{-3\alpha-\varphi}$) for the graphs with a Kalb-Ramond term. Indeed, there are some differences in $\alpha(t)$ in the early expansion era from the contribution of the Kalb-Ramond action. Moreover, it can be shown that $\alpha(t)$ in the $k^7=1000$ case is slightly larger than the $k_7=1$ case. This is reasonable; as indicated by (4.8), the major source of deviations comes from the early interval. Hence the choice of k_7 is not as sensitive as the choice of other parameters.

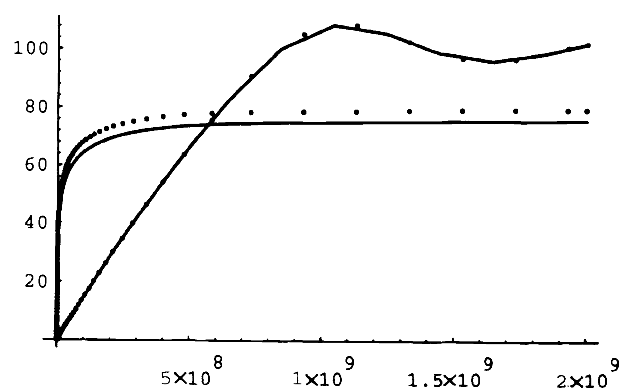

FIG. 1. We set $k=0$, $\alpha(t=0)=1$, $\varphi_0=-5$, $\varphi_{t_0}=0$, $\lambda=4\times10^{-17}$, $\epsilon=0.01$, and $v=10$ in the plot. The solid (dotted) lines draw the t dependence of α and $\phi^2=e^{\varphi}$ in the absence (presence) of the Kalb-Ramond action, respectively. Also, upper (lower) graphs denote $\phi^2(\alpha)$, respectively. In plotting the dotted lines, we have also set $l=k_7=1$. Note that the horizontal axis is the t axis in cosmological units.

One notes further that the ϕ dependence is roughly the same, which also gets oscillated as soon as ϕ approaches its vacuum $\phi=v$. The physics near the end of the inflationary process is hence not much different from previous work.

VI. CONCLUSION

In summary, we have analyzed an induced effective theory with torsion. We found that the torsion field is, in fact, negligible in the inflation era. The *Ansatz* $\theta=\theta(t)$ is found to be inconsistent with the maximal form invariance requirement imposed on the torsion field $A_{\mu\nu}$. It is argued that the well-known cosmological principle imposed on various fields is a strong restriction. We hence argue that, by allowing the radial dependence of various fields, the spatial dependence is inconsistent with the field equation for the scalars φ and θ. The use of the *Ansatz* $\theta(t)$ is hence justified. We also show that the contribution from the Kalb-Ramond action is, in fact, small in the inflationary era in which this theory should remain effective in an energy range slightly lower than the Planck scale. We also give some solutions to the field equations in order to demonstrate the explicit behavior of the torsion field.

ACKNOWLEDGMENTS

The author would like to thank Ue-Li Pen for helpful discussions and especially for showing the author the Runge-Kutta package for solving ODE's. The author wishes to thank Herming Chiueh and S. Y. Lin for assistant in running the computer program. This work was supported in part by the Taiwan NSC.

APPENDIX: NOTATIONS AND REDUNDANCY

Note that the curvature tensor $R^{\ \ \beta}_{\mu\nu\alpha}(g_{\mu\nu})$ used in this paper is defined by the equation

$$[D_\mu, D_\nu]A_\alpha = R^{\ \ \beta}_{\alpha\nu\mu}A_\beta \ , \tag{A1}$$

i.e.,

$$R^{\ \ \beta}_{\mu\nu\alpha} = -\partial_\alpha\Gamma^\beta_{\mu\nu} - \Gamma^\lambda_{\mu\nu}\Gamma^\beta_{\alpha\lambda} - (\nu \leftrightarrow \alpha) \ .$$

Here $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol or spin connection of the covariant derivative, namely, $D_\mu A_\nu \equiv \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu}A_\alpha$. To be more specific,

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) \ .$$

Here we use $\mu,\nu,\alpha=0,1,2,3$ and $i,j,k=1,2,3$ to denote time-space and spatial indices, respectively. Also, the Ricci tensor $R_{\mu\nu}$ is defined as

$$R_{\mu\nu} = R^{\ \ \alpha}_{\mu\nu\alpha} \ . \tag{A2}$$

And the scalar curvature R is defined as the trace of the

Ricci tensor, i.e., $R \equiv g^{\mu\nu}R_{\mu\nu}$. Moreover, the Einstein tensor $G_{\mu\nu}$ is defined as

$$G_{\mu\nu} \equiv \tfrac{1}{2} g_{\mu\nu} R - R_{\mu\nu} \ . \tag{A3}$$

In standard cosmology, we are dealing with a spatially homogeneous and isotropic universe which is indicated by gravitational observations [9] as well as some philosophical considerations. It is known that the three different classes of FRW spaces are the only spatially isotropic and homogeneous spaces. In fact, the FRW metric can be read off from the definition

$$\begin{aligned} ds^2 &\equiv g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 d\Omega \right] \ . \end{aligned} \tag{A4}$$

Here $d\Omega$ is the solid angle $d\Omega = d\theta^2 + \sin^2\theta \, d\chi^2$ and $k = 0, \pm 1$ stand for a flat, closed, or open universe, respectively.

Most gravitational equations containing a piece of metric-field equations read

$$G_{\mu\nu} = T_{\mu\nu} \ . \tag{A5}$$

This is known as the generalized Einstein equation. Here $T_{\mu\nu}$, the generalized energy-momentum tensor, can be a function of $g_{\mu\nu}$ and the rest of fields couple to the metric field. In fact, Eq. (A5) takes the form $H_{\mu\nu} = 0$. Here $H_{\mu\nu} \equiv G_{\mu\nu} - T_{\mu\nu}$.

It can be shown that four (in fact, one, because of the symmetry of the Friedmann-Robertson-Walker metric) out of the generalized Einstein equations are redundant as a result of the Bianchi identity $D_\mu G^{\mu\nu} = 0$. One can show, [14] however, that every equation is equally redundant except the tt component of Eq. (A5), the well-known generalized Friedmann equation. This can be readily understood by observing that the generalized Friedmann equation is, in fact, a first-order ODE in contrast with all other equations, which are second-order ones.

Consequently, the Bianchi identity can be rephrased, on shell, as

$$D_\mu H^{\mu\nu} = 0 \ . \tag{A6}$$

This is because $D_\mu G^{\mu\nu} = 0$ as a result of the Bianchi identity, while $D_\mu T^{\mu\nu} = 0$ is considered an on-shell constraint. Equation (A6) becomes

$$\left[\partial_t + 3 \frac{a_t}{a} \right] H_{00} + 3 a a_t H = 0 \ , \tag{A7}$$

as soon as we substituted the FRW metric into (A6). Here $H \equiv \frac{1}{3} h^{ij} H_{ij}$ and note that $g_{ij} = a^2 h_{ij}$. Hence it is straightforward to show that $H_{ij} = H h_{ij}$ in this theory if φ is spatially independent. The exclusive role of the generalized Friedmann equation (H_{00}) can be easily checked. If $a_t \neq 0$, (A7) states that "$H_{00} = 0$ implies $H = 0$." On the other hand, $H = 0$ implies, instead,

$$\left[\partial_t + 3 \frac{a_t}{a} \right] H_{00} = 0 \ . \tag{A8}$$

Indeed, Eq. (A8) can be integrated directly to obtain

$$a^3 H_{00} = \text{const} \ . \tag{A9}$$

Note that (A9) is not enough by itself to imply the desired result $H_{00} = 0$. Therefore the generalized Friedmann equation is indeed an exclusive equation of motion. Therefore we can ignore any redundant equation but the generalized Friedmann equation.

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