Examining the large-time wellbore flux of constant head test

Hund-Der Yeh,[1] Chih-Tse Wang,[2] and Stefan G. Llewellyn Smith[3]

Received 24 October 2009; revised 9 March 2010; accepted 12 May 2010; published 14 September 2010.

[1] The function $1/[p \ln(p/\lambda)]$ comes from the wellbore flux solution in the Laplace domain for a constant head aquifer test when the Laplace variable p is small. The resulting inverse Laplace transform $\nu(\lambda t)$, representing the large-time wellbore flux, grows exponentially with time, which does not agree with the physical behavior of the wellbore flux. Based on this result, Chen and Stone (1993) asserted that the well-known relationship of small p–large t may fail to yield the correct large-time asymptotic solution. Yeh and Wang (2007) pointed out that the large-time wellbore flux is not $\nu(\lambda t)$ if the inversion of $1/[p \ln(p/\lambda)]$ is subject to the constraint Re $p > \lambda$. Chen (2009) subsequently questioned the necessity of imposing this constraint on the Laplace transform in the inversion of the large-time wellbore flux. Motived by Chen's comment, we reexamine the inversion of $1/[p \ln(p/\lambda)]$ and demonstrate that this contradictory issue originates from a spurious pole introduced when applying the small p–large t correspondence to the Laplace domain solution. We explain why this occurs and why the actual wellbore flux at large times is proportional to the function $N(\lambda t)$, known as Ramanujan's integral. The function $N(\lambda t)$ does decay at large time, which agrees with the steady state wellbore flux of the constant head test.

Citation: Yeh, H.-D., C.-T. Wang, and S. G. Llewellyn Smith (2010), Examining the large-time wellbore flux of constant head test, *Water Resour. Res.*, *46*, W09602, doi:10.1029/2009WR008820.

1. Introduction

[2] The Laplace transform is commonly used to transform linear partial differential equations into ordinary differential equations. The time domain solution can then be obtained after inverting the Laplace domain solution obtained from the ordinary differential equation. To obtain the inverse Laplace transform, one may use tables [e.g., *Abramowitz and Stegun*, 1970; *Sneddon*, 1972; *Oberhettinger and Badii*, 1973; *Hildebrand*, 1976; *O'Neil*, 1991; *Jeffrey*, 2002] in conjunction with rules or methods like the shift theorem, partial fractions, and the convolution theorem. The inverse Laplace transform can also be obtained by Bromwich's integral [e.g., *Peng et al.*, 2002; *Yang et al.*, 2006], occasionally in conjunction with the residue theorem and/or Jordan's lemma [e.g., *Spiegel*, 1971; *Duffy*, 1998].

[3] Numerical inversion methods or approximate solutions may be sought when the Laplace domain solution cannot be inverted analytically. Numerical inversion methods compute the time domain results at specified times [e.g., *Davies*, 2002; *Yeh and Yang*, 2006; *Cohen*, 2007]. In contrast, asymptotic approximations are useful for large or small times. For example, large-time solutions can be obtained from inverting the Laplace domain solutions based on the relationship of small Laplace variable p versus large time t [e.g., *van Everdingen and Hurst*, 1949; *Yeh and Wang*, 2007], sometimes referred to as the small p–large t relation. However, the mathematical conditions to carry out such a relationship rely on knowledge of the functions beyond their transform alone; this is the realm of Abelian and Tauberian theorems [*van der Pol and Bremmer*, 1950].

[4] The constant flux test is usually performed to characterize an aquifer in order to estimate the aquifer parameters. The Laplace domain wellbore flux of the constant head test in a confined aquifer can be expressed as [*Chen and Stone*, 1993, p. 208, equation (7); *Carslaw and Jaeger*, 1959, p. 335, equation (3)]

$$\overline{Q}(r_w, p) = 2\pi r_w s_w T \frac{q K_1(q r_w)}{p K_0(q r_w)}, \tag{1}$$

where r_w is the well radius of test well, S_w is the constant drawdown maintained at the test well, q = $\sqrt{pS/T}$ in which S and T are the storage coefficient and the transmissivity of the confined aquifer, respectively, and $K_0(\cdot)$ and $K_1(\cdot)$ are modified Bessel functions of the second kinds of order zero and order one, respectively. When p is small, with the limiting forms $K_0(x) \sim -[\ln(x/2)+\gamma]$ and $K_1(x) \sim 1/x$, (1) can be reduced to [*Chen and Stone*, 1993; *Yeh and Wang*, 2007]

$$\overline{Q}(r_w, p) \sim -4\pi s_w T \frac{1}{p \ln(p/\lambda)}, \tag{2}$$

where $\lambda = 4T/(e^{2\gamma} r_w^2 S)$ and $\gamma = 0.57722\ldots$ is Euler's constant.

[5] *Chen and Stone* [1993] used an inverse Laplace transform formula listed by *Oberhettinger and Badii* [1973] to

[1] Institute of Environmental Engineering, National Chiao Tung University, Hsinchu, Taiwan.
[2] Taiwan Branch, MWH Americas Inc., Taipei, Taiwan.
[3] Department of Mechanical and Aerospace Engineering, Jacobs School of Engineering, University of California, San Diego, La Jolla, California, USA.

0043-1397/10/2009WR008820

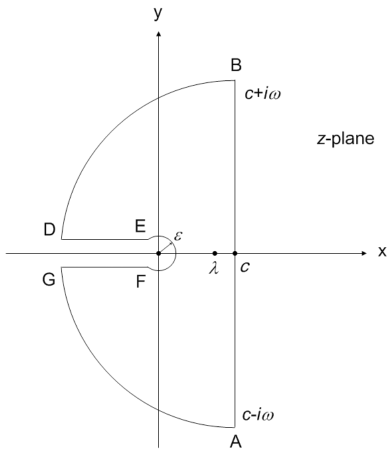

Figure 1. The contour used in Bromwich's integral.

invert $1/[p \ln(p/\lambda)]$ on the right-hand side of (2) and gave the result as

$$Q(r_w, t) \sim -4\pi s_w T \int_0^\infty \frac{(\lambda t)^x}{\Gamma(x+1)} dx \tag{3}$$

in which the integral on the right-hand side of (3) is denoted as $\nu(\lambda t)$. They showed that $\nu(\lambda t)$ tends to infinity as time approaches infinity. Based on this result, they therefore concluded that the use of small p–large t relation fails to retrieve the correct large-time wellbore flux.

[6] *Yeh and Wang* [2007] used the expansion given by *Ritchie and Sakakura* [1956] to invert $1/[p \ln(p/\lambda)]$ at large time and gave the large-time wellbore flux as

$$Q(r_w, t) \sim 4\pi s_w T \cdot \left[\frac{1}{\ln(\lambda t)} - \frac{\gamma}{[\ln(\lambda t)]^2} + \frac{\gamma^2 - \pi^2/6}{[\ln(\lambda t)]^3} - \frac{\gamma^3 - \pi^2\gamma/2 + \xi(3)}{[\ln(\lambda t)]^4} \right], \tag{4}$$

where $\xi(3) = 1.2020569032$ is the Riemann Zeta function. Equation (4) is in fact identical to the formula given in *Carslaw and Jaeger* [1959, p. 336, equation (10)] if the last two terms in the former are neglected. In other words, (4) gives more accurate large-time results than *Carslaw and Jaeger*'s formula [1959, p. 336]. It must be emphasized that (4) tends to zero for large time. *Yeh and Wang* [2007] also argued that the constraint Re $p > \lambda$ should be imposed in deriving the Laplace inverse of $1/[p \ln(p/\lambda)]$ to obtain $\nu(\lambda t)$.

[7] Recently, *Chen* [2009] questioned the necessity of the constraint Re $p > \lambda$. According to his argument, (3) is correct and hence the use of small p–large t relation fails to yield a correct result for the large-time wellbore flux as concluded by *Chen and Stone* [1993]. Therefore, two paradoxical issues need to be clarified. One is whether (3) or (4) is the correct inverse result of (2), while the other is whether the physical character of (2) can represent that of (1). In this note, we will address these two issues.

2. Inverse of $1/[p \ln(p/\lambda)]$

2.1. Laplace Transform

[8] The inverse Laplace transform of $1/[p \ln p]$ can be found in *Bateman* [1954, p. 225, equation 4.26.(1)]. The derivation for $1/[p \ln(p/\lambda)]$ is presented in the following. The inverse Laplace transform of $\Gamma(x+1)/p^{x+1}$ for Re $p > 0$ given in *Oberhettinger and Badii* [1973, p. 21, equation (3.3)] is

$$L^{-1}\left[\frac{\Gamma(x+1)}{p^{x+1}}\right] = t^x, \quad x > -1, \tag{5}$$

where $\Gamma(\cdot)$ is the Gamma function. The linearity property of the Laplace transform gives

$$L^{-1}\left[\frac{1}{(p/\lambda)^{x+1}}\right] = \frac{\lambda}{\Gamma(x+1)}(\lambda t)^x. \tag{6}$$

Integrating both sides of (6) with respect to x from zero to infinity yields

$$L^{-1}\left[\frac{1}{\lambda}\int_0^\infty \frac{1}{(p/\lambda)^{x+1}} dx\right] = \int_0^\infty \frac{(\lambda t)^x}{\Gamma(x+1)} dx. \tag{7}$$

The right-hand side of (7) is indeed the integral on the right-hand side of (3), denoted as $\nu(\lambda t)$. The integral on the left-hand side of (7) becomes

$$\frac{1}{\lambda}\int_0^\infty \frac{1}{(p/\lambda)^{x+1}} dx = \frac{-1}{\lambda \ln(p/\lambda)}\left[\lim_{x\to\infty} \frac{1}{(p/\lambda)^{x+1}} - \frac{1}{(p/\lambda)}\right]. \tag{8}$$

The first term on the right-hand side of (8) vanishes when Re $p > \lambda$. The inverse Laplace transform of $1/[p \ln(p/\lambda)]$ thus gives $\nu(\lambda t)$. The region of validity of the Laplace transform of $1/[p \ln p]$ should be Re $p > 1$ rather than Re $p > 0$, which is the constraint mentioned in *Bateman* [1954, p. 225, equation 4.26.(1)]. By analytic continuation, the function $1/[p \ln(p/\lambda)]$ provides the Laplace transform of $\nu(\lambda t)$ in the whole complex plane minus the point Re $p = \lambda$ and the negative real axis (we take the branch cut for the logarithm along the negative real axis).

[9] The inverse Laplace transform can also be obtained through the use of Bromwich's integral. This approach leads to the result of *Hardy* [1940] and *Llewellyn Smith* [2000]. The Laplace inverse of $1/[p \ln(p/\lambda)]$ is expressed as

$$L^{-1}\left[\frac{1}{p \ln(p/\lambda)}\right] = \frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{c-i\omega}^{c+i\omega} \frac{e^{zt}}{z \ln(z/\lambda)} dz, \tag{9}$$

where $c > \lambda$. The integration of (9) is along the line $x = c$ parallel to the imaginary axis, i.e., the line $(c - i\omega, c + i\omega)$ in the complex plane with real parameter ω ranging from $-\infty$ to ∞. The integral in (9) can be evaluated by the Bromwich contour as shown in Figure 1 which includes a pole contribution from $z = \lambda$ and a branch point at $z = 0$. The former yields $\exp(\lambda t)$. For the latter, one has $z = x\exp(-i\pi)$ on the bottom of the cut (line FG) and $z = x \exp(i\pi)$ on the top (line DE). Furthermore, a small circle of radius ε about the branch point (circle EF) can be used, but the contribution from this circle vanishes. The branch cut integral then gives

$$\frac{1}{2\pi i}\int_0^\infty e^{-xt}\left[-\frac{1}{\ln(e^{-i\pi}x/\lambda)} + \frac{1}{\ln(e^{i\pi}x/\lambda)}\right]\frac{dx}{x} = -\int_0^\infty \left[\frac{e^{-\lambda tx}}{(\ln x)^2 + \pi^2}\right] dx. \tag{10}$$

The integral on the right-hand side of (10) is denoted as $N(\lambda t)$ and is known as Ramanujan's integral [*Llewellyn Smith*, 2000]. Putting this together thus gives the Laplace inverse of $1/[p \ln(p/\lambda)]$ as

$$L^{-1}\left[\frac{1}{p\ln(p/\lambda)}\right] = e^{\lambda t} - N(\lambda t), \tag{11}$$

where $N(\lambda t)$ is defined for positive λt by the integral on the right-hand side of (10).

[10] From the form of $N(\lambda t)$ in (11), it is clear that $N(\lambda t)$ decays at large time. Hence, $v(\lambda t)$ grows exponentially for large time. In the Laplace variable, (11) can be expressed as

$$\overline{N}(p) = \frac{1}{p-\lambda} - \frac{1}{p\ln(p/\lambda)}. \tag{12}$$

2.2. Asymptotic Approximations and Abelian and Tauberian Theorems

[11] The behaviors of $N(\lambda t)$ for small and large t were given by *Llewellyn Smith* [2000], which also included historical background. For small t, one can obtain

$$N(\lambda t) \sim 1 - \frac{1}{\ln(1/\lambda t)} + O\left(\frac{1}{[\ln(1/\lambda t)]^2}\right) \tag{13}$$

which may be developed from using Watson's lemma for $N(\lambda t)$ defined in (10). For large t, the result is

$$N(\lambda t) \sim \frac{1}{\ln(\lambda t)} + O\left(\frac{1}{[\ln(\lambda t)]^2}\right). \tag{14}$$

[12] The approach pioneered by *van Everdingen and Hurst* [1949] uses the relation between the large-time behavior of a function in time domain and the behavior of its Laplace transform for small p. The simplest relation is that $p\overline{f}(p) \sim f(t)$ (this relation also holds for large p–small t). The existence of such a relation is governed by Abelian and Tauberian theorems. The former go from the time variable to the Laplace variable and are simpler than the latter, which go in the opposite direction. In particular, the theorems are only meaningful when a limit exists. The simplest counter example is $\sin(t)$. Its Laplace transform is $p/(p^2+1)$, so that $pL[\sin(t)]$ vanishes for small p. However, $\sin(t)$ does not tend to zero for large time.

[13] Here we have not only $p\overline{v}(p) \sim 0$, but also $p\overline{N}(p) \sim 0$ for small p. This is consistent with the limit of $N(\lambda t)$ for large t, but not for $v(\lambda t)$. The reason for this is that the Tauberian theorem necessary to transform from p to t does not hold for $v(\lambda t)$, in particular because the strip of convergence of the Laplace integral defining $\overline{v}(p)$ does not contain the imaginary axis [*van der Pol and Bremmer*, 1950, section VII.3. Theorem I].

3. Relation Between Large-Time Wellbore Flux and $1/[p \ln(p/\lambda)]$

[14] We have seen that $\nu(\lambda t)$ grows exponentially, which is inconsistent with physical behavior of the wellbore flux. To understand this apparent paradox, we return to the full expression for the wellbore flux (1). Carrying out the Bromwich integral gives

$$Q(r_w, t) = \frac{8T s_w}{\pi}\int_0^\infty \exp\left(-\frac{T}{Sr_w^2}x^2 t\right)\frac{dx}{x\left[J_0^2(x)+Y_0^2(x)\right]}. \tag{15}$$

Clearly, the integral in (15) decays at large time.

[15] The fundamental problem is that in passing from (1) to (2), which is indeed the small p expansion of $\overline{Q}(r_w, p)$, we have introduced a spurious pole at Re $p = \lambda$ in (2). The original form in (1) has no pole at Re $p = \lambda$, and the original vertical Bromwich contour used to obtain (15) can be taken anywhere in the right half plane. However, the Bromwich contour used in obtaining (11) has to be to the right of Re $p = \lambda$. This leads to the exponential term in (11).

[16] Hence, the constraint Re $p > \lambda$ imposed in the development of the integral definition of $\overline{v}(p)$ by *Yeh and Wang* [2007] is appropriate. This removes the pole contribution at λ explicitly from (11). This essentially gives $-N(\lambda t)$, as shown in (12). Alternately, replacing the Bessel functions by their small x expansions simplifies the integral in (15), and the result is proportional to the integral definition of $N(\lambda t)$. In addition, the series obtained from the inverse Laplace transform of $1/[p \ln(p/\lambda)]$ given by *Yeh and Wang* [2007, equation (5)] should be understood as the series expansion of $N(\lambda t)$.

4. Concluding Remarks

[17] The wellbore flux of the constant flux test is usually used in the determination of hydraulic parameters of low-permeability aquifers. The wellbore flux of constant head test at large time can be obtained from the inverse of the Laplace domain wellbore flux for small value of the Laplace parameter p, provided one removes the spurious pole introduced at Re $p = \lambda$ The result is then proportional to $N(\lambda t)$, whose series expansion has been obtained previously [e.g., *Ritchie and Sakakura*, 1956; *Carslaw and Jaeger*, 1959; *Bouwkamp*, 1971; *Llewellyn Smith*, 2000; *Yeh and Wang*, 2007]. The solution $N(\lambda t)$ can then be used to estimate hydraulic parameters when coupled with an optimization approach to analyze the measurement of large-time wellbore flux. The inversion of $1/[p \ln(p/\lambda)]$ without the constraint of Re $p > \lambda$ leads to the incorrect result $\nu(\lambda t)$ on the right-hand side of (7) (i.e., (1) in *Chen*'s [2009] comment). In fact, the physical large-time wellbore flux should be proportional to $N(\lambda t)$ because the pole at Re $p = \lambda$ in $\overline{v}(p)$ has been introduced artificially and must be excluded. This is a good example showing that the application of the small p–large t relation to obtain the large-time solution from the Laplace domain solution should be carried out with care.

[18] **Acknowledgments.** This study was partly supported by the Taiwan National Science Council under the grants NSC 96-2221-E-009-087-MY3 and NSC 99-NU-E-009-001. We are grateful to the Editor, Tissa Illangasekare, and two anonymous reviewers for their helpful comments and suggestions.

References

Abramowitz, M., and I. A. Stegun (1970), *Handbook of Mathematical Functions*, Dover, New York.

Bateman Manuscript Project (1954), *Tables of Integral Transforms*, vol. I, edited by A. Erdélyi, McGraw-Hill, New York.

Bouwkamp, C. J. (1971), Note on an asymptotic expansion, *Indiana Univ. Math. J.*, *21*, 547–549.

Carslaw, H. S., and J. C. Jaeger (1959), *Conduction of Heat in Solids*, 2nd ed., Oxford Univ. Press, London.

Chen, C.-S. (2009), Comment on “Large-time solutions for groundwater flow problems using the relationship of small p versus large t” by Hund-Der Yeh and Chih-Tse Wang, *Water Resour. Res.*, *45*, W12603, doi:10.1029/2009WR008132.

Chen, C. S., and W. D. Stone (1993), Asymptotic calculation of Laplace inverse in analytical solutions of groundwater problems, *Water Resour. Res.*, *29*(1), 207–210.

Cohen, A. M. (2007), *Numerical Methods for Inverse Laplace Transformation*, Springer, New York.

Davies, B. (2002), *Integral Transforms and Their Applications*, 3rd ed., Springer, New York.

Duffy, D. G. (1998), *Advanced Engineering Mathematics*, CRC Press, New York.

Hardy, G. H. (1940), *Ramanujan: Twelve Lectures on Subjects Suggested by His Life and Work*, Cambridge Univ. Press, Cambridge, U.K.

Hildebrand, F. R. (1976), *Advanced Calculus for Applications*, 2nd ed., Prentice Hall, New Jersey.

Jeffrey, A. (2002), *Advanced Engineering Mathematics*, Harcourt, New York.

Llewellyn Smith, S. G. (2000), The asymptotic behavior of Ramanujan’s integral and its application to two-dimensional diffusion-like equations, *Eur. J. Appl. Math.*, *11*, 13–28.

Oberhettinger, F., and L. Badii (1973), *Tables of Laplace Transforms*, Springer, New York.

O’Neil, P. V. (1991), *Advanced Engineering Mathematics*, 3rd ed., Wadsworth, Boston.

Peng, H. Y., H. D. Yeh, and S. Y. Yang (2002), Improved numerical evaluation for the radial groundwater flow equation, *Adv. Water Resour.*, *25*(6), 663–675, doi:10.1016/S0309-1708(02)00030-1.

Ritchie, R. H., and A. Y. Sakakura (1956), Asymptotic expansions of solutions of the heat conduction equation in internally bounded cylindrical geometry, *Appl. Phys.*, *27*(12), 1453–1459.

Sneddon, I. H. (1972), *The Use of Integral Transforms*, McGraw-Hill, New York.

Spiegel, M. R. (1971), *Schaum’s Outline Theory and Problems of Advanced Mathematics for Engineers and Scientists*, McGraw-Hill, New York.

van der Pol, B., and H. Bremmer (1950), *Operational Calculus Based on the Two-Sided Laplace Integral*, Cambridge Univ. Press, Cambridge, U.K.

van Everdingen, A. F., and W. Hurst (1949), The application of the Laplace transformation to flow problems in reservoirs, *Trans. Am. Inst. Min. Metall. Eng.*, *186*, 305–324.

Yang, S. Y., H. D. Yeh, and P. Y. Chiu (2006), A closed form solution for constant flux pumping in a well under partial penetration condition, *Water Resour. Res.*, *42*, W05502, doi:10.1029/2004WR003889.

Yeh, H. D., and C. T. Wang (2007), Large-time solutions for groundwater flow problems using the relationship of small p versus large t, *Water Resour. Res.*, *43*, W06502, doi:10.1029/2006WR005472.

Yeh, H. D., and S. Y. Yang (2006), A novel analytical solution for a slug test conducted in a well with a finite-thickness skin, *Adv. Water Resour.*, *29*(10), 1479–1489, doi:10.1016/j.advwatres.2005.11.002.

S. G. Llewellyn Smith, Department of Mechanical and Aerospace Engineering, Jacobs School of Engineering, University of California, San Diego, 9500 Gilman Dr., Mail Code 0411, La Jolla, CA 92093-0411, USA.

C.-T. Wang, Taiwan Branch, MWH Americas Inc., 5F, 438 Kwang Fu S. Rd., Taipei 106, Taiwan.

H.-D. Yeh, Institute of Environmental Engineering, National Chiao Tung University, No. 1001, University Rd., Hsinchu 300, Taiwan. (hdyeh@mail.nctu.edu.tw)