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Coexistence of invariant sets with and without SRB measures in Hénon family*

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Abstract
Let $\{f_{a,b}\}$ be the (original) Hénon family. In this paper, we show that, for any b near 0, there exists a closed interval J_b which contains a dense subset J' such that, for any $a \in J'$, $f_{a,b}$ has a quadratic homoclinic tangency associated with a saddle fixed point of $f_{a,b}$ which unfolds generically with respect to the one-parameter family $\{f_{a,b}\}_{a\in J_b}$. By applying this result, we prove that J_b contains a residual subset $A_b^{(2)}$ such that, for any $a \in A_b^{(2)}$, $f_{a,b}$ admits the Newhouse phenomenon. Moreover, the interval J_b contains a dense subset $\tilde{A}_b$ such that, for any $a \in \tilde{A}_b$, $f_{a,b}$ has a large homoclinic set without SRB measure and a small strange attractor with SRB measure simultaneously.

Mathematics Subject Classification: 37C29, 37G25, 37G30, 37G40

1. Introduction

In [6] Hénon studied numerically the dynamics of the diffeomorphisms $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f_{a,b}(x, y) = (1 - ax^2 + y, bx) \tag{1.1}$$

for $b \neq 0$ and presented a supporting evidence for the existence of a strange attractor of $f_{a,b}$ when $a = 1.4$ and $b = 1.3$. We call these maps *Hénon maps* or more strictly *original* Hénon maps consciously to distinguish them from Hénon-like maps. Subsequently, Fornæss and

* Dedicated to the memory of Floris Takens (12 November 1940–20 June 2010).

Gavosto [3] gave a computer assisted proof that, for some $a_0 = 1.3924198\ldots$ and any fixed b_0 near 0.3, f_{a_0,b_0} has a generic unfolding homoclinic tangency. Furthermore, applying the results of Mora and Viana [10] to the Hénon maps, one has that $f_{a,b}$ exhibits a small strange attractor for parameters (a, b) arbitrarily close to (a_0, b_0).

Let M be a surface and $\mathrm{Diff}^r(M)$ the space of C^r diffeomorphisms with C^r topology for an integer $r \geqslant 2$. We say that a connected subset A of $\mathrm{Diff}^r(M)$ has *persistent homoclinic tangencies* if a continuation of basic sets $\Lambda(f)$ of $f \in A$ is well defined and each $\Lambda(f)$ has a *homoclinic tangency*. That is, there exist points $x, y \in \Lambda(f)$, possibly $x = y$, such that $W^u(x)$ and $W^s(y)$ have a tangency. Newhouse [11, 12] (see also [14]) showed that, if $f \in \mathrm{Diff}^r(M)$ has a homoclinic tangency associated with a dissipative saddle point, then for any neighbourhood $U(f)$ of f in $\mathrm{Diff}^r(M)$, there is an element $g \in U(f)$ some connected neighbourhood N of which has persistent homoclinic tangencies. Moreover, there exists a residual subset of N each element of which is a diffeomorphism admitting infinitely many sinks. The condition with infinitely many sinks is called the *Newhouse phenomenon*. Robinson [16] detected the phenomenon in the context of certain one-parameter families in $\mathrm{Diff}^r(M)$. One of our aims in this paper is to show that the original Hénon family admits the Newhouse phenomenon.

Let Λ_1 and Λ_2 be basic sets for $f \in \mathrm{Diff}^r(M)$. Then, we say that Λ_1 is *homoclinically related* to Λ_2 if either $\Lambda_1 = \Lambda_2$ or there are points $x_1, y_1 \in \Lambda_1$ and $x_2, y_2 \in \Lambda_2$ such that $W^u(x_1) \setminus \Lambda_1$ has a non-empty transverse intersection with $W^s(x_2) \setminus \Lambda_2$ and $W^s(y_1) \setminus \Lambda_1$ has a non-empty transverse intersection with $W^u(y_2) \setminus \Lambda_2$. The closure Λ of the union of basic sets homoclinically related to each other is called a *homoclinic set* if Λ contains more than a single periodic orbit. When one of these basic sets contains a periodic point p of f, Λ is called the homoclinic set of p. We say that a homoclinic set Λ has a *homoclinic tangency* if Λ contains a basic set with a homoclinic tangency. Thirty years after Newhouse's original works, new results were presented by himself. In fact, he showed in [13, theorem 1.4] that there is a residual subset R of $\mathrm{Diff}^r(M)$ such that, if $f \in R$ and $\Lambda(f)$ is a homoclinic set for f which contains a homoclinic tangency and has an associated dissipative saddle point, then $\Lambda(f)$ does not carry an SRB measure. Here, *SRB measure* means an f-invariant Borel probability measure which is ergodic, has a compact support and has absolutely continuous conditional measures on unstable manifolds. Moreover, Newhouse gave the conjecture in [13]:

- for each parameter b, there is a residual set of parameters a such that $f_{a,b}$ has no SRB measure.

On the other hand, Benedicks and Young [2] showed that

- for almost every positive b near 0, there is a positive Lebesgue measure set A_b of a-parameters such that $f_{a,b}$ has an SRB measure supported by the homoclinic set of a fixed point of $f_{a,b}$ if $a \in A_b$.

Following the situation of admitting versus non-admitting of SRB measures, we will show that these two conditions coexist for many parameter values in the original Hénon family. For convenience in our arguments, we adopt the following topologically conjugated formula of the Hénon map $f_{a,b}$:

$$\varphi_{a,b}(x, y) = (y, a - bx + y^2)$$

which is obtained from the classical formula (1.1) by the reparametrization $(a, b) \mapsto (-a, -b)$ and the coordinate change $(x, y) \mapsto (-ab^{-1}y, -ax)$. Note that $p_{a,b} = (y_{a,b}, y_{a,b}) \in \mathbb{R}^2$ with $y_{a,b} = (1 + b + \sqrt{(1 + b)^2 - 4a})/2$ is a fixed point of $\varphi_{a,b}$.

Now we state our main results.

Theorem A. *There exists an open interval I containing 0 such that, for any $b \in I \setminus \{0\}$, there is a positive integer w and a closed interval J_b in the a-parameter space satisfying the following.*

(i) For any $a \in J_b$, $\varphi_a := \varphi^w_{a,b}$ has continuations of two basic sets $\Lambda^{\rm out}_a$ and $\Lambda^{\rm in}_a$ with $p_{a,b} \in \Lambda^{\rm out}_a$ such that there exist persistent quadratic tangencies of $W^u(\Lambda^{\rm out}_a)$ and $W^s(\Lambda^{\rm in}_a)$ which unfold generically with respect to the one-parameter family $\{\varphi_a\}_{a\in J_b}$.

(ii) There is a dense subset J' of J_b such that, for any $\hat{a} \in J'$, $W^u(p_{\hat{a},b})$ and $W^s(p_{\hat{a},b})$ have a quadratic tangency $q_{\hat{a}}$ which unfolds generically with respect to $\{\varphi_a\}_{a\in J_b}$.

See section 2.2 for the definition of persistent quadratic tangency unfolding generically. A more detailed version of theorem A(i) and (ii) is stated as theorems 5.1 and 5.2, respectively. Together with the results of [2, 10, 13, 14, 16, 21], the two theorems also imply the following.

Theorem B. *For any $b \in I \setminus \{0\}$, the interval J_b given in theorem A contains subsets $A^{(1)}_b$, $A^{(2)}_b$, $A^{(3)}_b$ satisfying the following conditions.*

(i) $A^{(1)}_b$ is open dense in J_b and, for any $a \in A^{(1)}_b$, $\varphi_{a,b}$ does not have an SRB measure supported by the homoclinic set of $p_{a,b}$.

(ii) $A^{(2)}_b$ is a residual subset of J_b with $A^{(2)}_b \subset A^{(1)}_b$ and, for any $a \in A^{(2)}_b$, $\varphi_{a,b}$ has infinitely many sinks.

(iii) $A^{(3)}_b$ has Lebesgue measure positive everywhere in J_b and, for any $a \in A^{(3)}_b$, $\varphi_{a,b}$ has an SRB measure supported by an Hénon-like strange attractor.

Note that the Hénon-like strange attractor given in theorem B(iii) is a small invariant set which arises from renormalization near the tangency $q_{\hat{a}}$ of theorem A, while the homoclinic set in theorem B(i) is a large invariant set.

We say that a subset A of an interval J has *Lebesgue measure positive everywhere* if, for any non-empty open subset U of J, $A \cap U$ has positive Lebesgue measure. From the definition, we know that such a set A is dense in J. Also, an invariant set Ω of $\varphi_{a,b}$ is called a *strange attractor* if (a) there exists a saddle point $p \in \Omega$ such that the unstable manifold $W^u(p)$ has dimension 1 and $\mathrm{Cl}(W^u(p)) = \Omega$, (b) there exists an open neighbourhood U of Ω such that $\{f^n(U)\}_{n=1}^{\infty}$ is a decreasing sequence with $\Omega = \bigcap_{n=1}^{\infty} f^n(U)$ and (c) there exists a point $z_0 \in \Omega$ whose positive orbit is dense in Ω and a non-zero vector $v_0 \in T_{z_0}(\mathbb{R}^2)$ with $\|\mathrm{d}\varphi^n_{z_0}(v_0)\| \geqslant \mathrm{e}^{cn}\|v_0\|$ for any integer $n \geqslant 0$ and some constant $c > 0$.

Conditions (i) and (iii) of theorem B imply that the intersection $\tilde{A}_b = A^{(1)}_b \cap A^{(3)}_b$ is a dense subset of J_b satisfying the following conditions.

Corollary C. *For any $b \in I \setminus \{0\}$, there exists a subset $\tilde{A}_b$ of J_b which has Lebesgue measure positive everywhere in J_b and such that, for any $a \in \tilde{A}_b$, $\varphi_{a,b}$ does not have an SRB measure supported by the homoclinic set of $p_{a,b}$ but has an SRB measure supported by a strange attractor.*

We finish introduction by outlining the proofs of theorems A and B. Note that a difficulty in our proof is that we need to find desired diffeomorphisms in the *fixed two*-parameter family $\{\varphi_{a,b}\}$ but *not* a neighbourhood of the family in the infinite dimensional space $\mathrm{Diff}^\infty(\mathbb{R}^2)$. See section 3. Our key mechanism for overcoming it is the *double renormalization* for the two-parameter family. Most of our efforts is devoted to detecting such renormalizations in section 4.

Using the implicit function theorem, we will define a smooth function $h : I \to \mathbb{R}$ such that, for any $b \in I$, $\varphi_{h(b),b}$ has a homoclinic quadratic tangency $q_{h(b),b}$ near the point $(-2, 2) \in \mathbb{R}^2$ unfolding generically with respect to the a-parameter family $\{\varphi_{a,b(\mathrm{fixed})}\}$ (proposition 3.2). Then, one can renormalize $\{\varphi_a\}$ with $\varphi_a := \varphi^w_{a,b}$ in a neighbourhood of $q_{h(b),b}$ as in [14], where

$w > 0$ is the even integer given in section 3. Then, by the thickness lemma [11, 14], there exists a closed interval J_b such that the one-parameter family $\{\varphi_a\}_{a\in J_b}$ has persistent heteroclinic quadratic tangencies q_a ($a \in J_b$), and moreover accompanying lemma (lemma 2.2) implies that all these tangencies unfold generically with respect to φ_a (theorem 5.1). Using these results, we also show that J_b contains a dense subset J' such that, for any $\hat{a} \in J'$, $\varphi_{\hat{a}}$ has a homoclinic tangency $q_{\hat{a}}$ associated with the fixed point $p_{\hat{a}} := p_{\hat{a},b}$ which also unfolds generically (theorem 5.2). Obviously, theorems 5.1 and 5.2 imply theorem A.

One can renormalize $\{\varphi_a\}$ again near the tangency $q_{\hat{a}}$ given in theorem A(ii) for any $\hat{a} \in J'$. Applying then standard arguments of Robinson [16] and Newhouse [13] to our situation, we have an open dense subset $A_b^{(1)}$ of J_b such that φ_a has no SRB measure supported by the homoclinic set of $p_{a,b}$ for any $a \in A_b^{(1)}$, and a residual subset $A_b^{(2)}$ of J_b with $A_b^{(2)} \subset A_b^{(1)}$ such that φ_a has infinitely many sinks for any $a \in A_b^{(2)}$. Moreover, applying the results of Wang–Young [21] to the renormalized maps, we have a dense subset $A_b^{(3)}$ of J_b with Lebesgue measure positive everywhere and such that φ_a has a strange attractor supporting an SRB measure if $a \in A_b^{(3)}$. These results prove theorem B.

2. Preliminaries

First of all, we will briefly review some notation and definitions needed in later sections. Throughout this section, we suppose that $\{\psi_t\}_{t\in J}$ is a one-parameter family in $\mathrm{Diff}^r(\mathbb{R}^2)$ with $r \geqslant 3$ such that the parameter space J is an interval. A family $\{A_t\}_{t\in J}$ of ψ_t-invariant subsets of $\mathbb{R}^2$ is called a *t-continuation* (or shortly *continuation*) if, for any $t \in J$ and some $t_0 \in J$, there exist homeomorphisms $h_t : A_{t_0} \to A_t$ depending on t continuously such that h_{t_0} is the identity of A_{t_0} and $h_t \circ \psi_{t_0}|_{A_{t_0}} = \psi_t|_{A_t} \circ h_t$.

2.1. Thickness of Cantor sets

We recall the definition of thickness given in Newhouse [12] and Palis–Takens [14] for a Cantor set K contained in an interval I. A *gap* of K is a connected component of $I \setminus K$ which does not contain a boundary point of I. Let G be a gap and p a boundary point of G. A closed interval $B \subset I$ is called the *bridge* at p if B is the maximal interval with $G \cap B = \{p\}$ such that B does not intersect any gap whose length is at least that of G. The *thickness* of K at p is defined by $\tau(K, p) = \mathrm{Length}(B)/\mathrm{Length}(G)$. The *thickness* $\tau(K)$ of K is the infimum over these $\tau(K, p)$ for all boundary points p of gaps of K. Let K_1, K_2 be two Cantor sets in I with thickness τ_1 and τ_2, respectively. Then, gap lemma (see [12, 14]) shows that, if $\tau_1 \cdot \tau_2 > 1$, then either K_1 is contained in a gap of K_2, or K_2 is contained in a gap of K_1, or $K_1 \cap K_2 \neq \emptyset$. The thickness of a Cantor subset K of a C^1 curve α in $\mathbb{R}^2$ is defined similarly by supposing that α is parametrized by arc length.

2.2. Persistent quadratic tangencies

A *basic set* of ψ_t is a non-trivial compact transitive hyperbolic invariant set of ψ_t with a dense subset of periodic orbits. Suppose that there exist continuations $\{\Lambda_{1,t}\}_{t\in J}$, $\{\Lambda_{2,t}\}_{t\in J}$ of basic sets or saddle fixed points of ψ_t such that $W^s(\Lambda_{1,t_0})$ and $W^u(\Lambda_{2,t_0})$ have a *quadratic tangency* q_{t_0} for a $t_0 \in J$. That is, one can choose a coordinate (x, y) on a neighbourhood O of q_{t_0} with $q_{t_0} = (0, 0)$ and such that

$$L^s_{t_0} \cap O = \{(x, y) : y = 0\} \qquad \text{and} \qquad L^u_{t_0} \cap O = \{(x, y) : y = ax^2\}$$

for some constant $a \neq 0$, where L_t^s, L_t^u are arcs in $W^s(\Lambda_{1,t})$ and $W^u(\Lambda_{2,t})$, respectively, which depend on t continuously. The tangency is called *homoclinic* (respectively *heteroclinic*) if $\Lambda_{1,t_0} = \Lambda_{2,t_0}$ (respectively $\Lambda_{1,t_0} \neq \Lambda_{2,t_0}$). One can choose these L_t^s, L_t^u so that they vary C^{r-2} with respect to t, for example see [15, propositions 1 and 2] and references therein. Note that the assumption $r \geqslant 3$ continues to be used.

Definition 2.1. *The quadratic tangency q_{t_0} unfolds generically with respect to $\{\psi_t\}_{t\in J}$ if there exist local coordinates on O C^3 depending on t and a C^1 function b on J satisfying the following conditions.*

- *$L_t^s \cap O$ is given by $y = 0$ and $L_t^u \cap O$ by $y = ax^2 + b(t)$ for any t near t_0.*
- *$b(t_0) = 0$ and $\frac{\mathrm{d}b}{\mathrm{d}t}(t_0) \neq 0$.*

The family $\{\psi_t\}_{t\in J}$ is said to have *persistent quadratic tangencies unfolding generically* if, for any $t_1 \in J$, $W^s(\Lambda_{1,t_1})$ and $W^u(\Lambda_{2,t_1})$ have a quadratic tangency q_{t_1} unfolding generically with respect to $\{\psi_t\}$.

2.3. Compatible foliations

Let $\mathcal{F}$ be a foliation consisting of smooth curves in the plane. A smooth curve σ in the plane is said to *cross $\mathcal{F}$ exactly* if each leaf of $\mathcal{F}$ intersects σ transversely in a single point and any point of σ is passed through by a leaf of $\mathcal{F}$.

Suppose that $\{\Lambda_t\}$ is a continuation of non-trivial basic sets of ψ_t and $\{p_t\}$ is a continuation of saddle fixed points in Λ_t. Let $\{I_t\}$ be a set of curves in $W^s_{\mathrm{loc}}(p_t)$ which are shortest among curves in $W^s_{\mathrm{loc}}(p_t)$ containing $\Lambda_t \cap W^s_{\mathrm{loc}}(p_t)$ and depends on t continuously. According to lemma 4.1 in [7] based on results in [4], there exists a t-parameter family of foliations $\mathcal{F}_t^u$ in $\mathbb{R}^2$ satisfying the following conditions. Such foliations are said to be *compatible with* $W^u_{\mathrm{loc}}(\Lambda_t)$.

(i) Each leaf of $W^u_{\mathrm{loc}}(\Lambda_t)$ is a leaf of $\mathcal{F}_t^u$.
(ii) I_t crosses $\mathcal{F}_t^u$ exactly.
(iii) Leaves of $\mathcal{F}_t^u$ are C^3 curves such that themselves, their directions and their curvatures vary C^1 with respect to any transverse direction and t.

Similarly, there exist foliations $\mathcal{F}_t^s$ *compatible with* $W^s_{\mathrm{loc}}(\Lambda_t)$. A leaf of $\mathcal{F}_t^{u/s}$ is said to be a Λ_t*-leaf* if the leaf is contained in $W^{u/s}(\Lambda_t)$.

2.4. Accompanying lemma

We still work with the notation and situation as in the previous subsections. Accompanying lemma given below is used to show that some quadratic tangencies q_t unfold generically.

Suppose that there exists a continuation of saddle fixed points $\hat{p}_t$ of ψ_t other than p_t such that $W^s(\hat{p}_t) \setminus \{\hat{p}_t\}$ has a subarc crossing $\mathcal{F}_t^{u(k_0)} := \psi_t^{k_0}(\mathcal{F}_t^u)$ exactly for some integer $k_0 \geqslant 0$. Let σ be an oriented short segment in $\mathbb{R}^2$ meeting $W^u(\hat{p}_t) \setminus \{\hat{p}_t\}$ almost orthogonally in a single point of $\mathrm{Int}(\sigma)$, which is denoted by c_t. The inclination lemma implies that $W^u_{\mathrm{loc}}(\hat{p}_t)$ is contained in a small neighbourhood of $\mathcal{F}_t^{u(k_0+j)}$ for all sufficiently large integer $j > 0$, see figure 1. In particular, σ contains an arc crossing $\mathcal{F}_t^{u(k_0+j)}$ exactly.

Consider an orientation-preserving arc-length parametrization $\alpha : [v_0, v_1] \to \sigma$ independent of t. Let $v : J \to [v_0, v_1]$ be a C^1 function such that $\alpha(v(t))$ is contained in a Λ_t-leaf of $\mathcal{F}_t^{u(k_0+j)}$, and let $c : J \to [v_0, v_1]$ be a C^1 function satisfying $\alpha(c(t)) = c_t$.

The following lemma is given in [8, lemma 4.1] and the proof is in [8, appendix A].

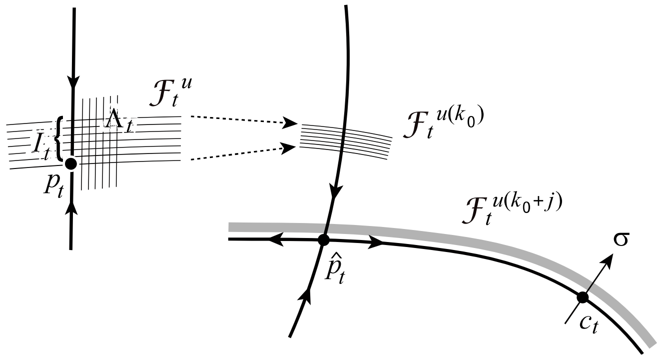

Figure 1. Inclining and accompanying of unstable leaves.

Lemma 2.2 (Accompanying lemma). *For any* $\delta > 0$ *and* $t_0 \in \mathrm{Int}(J)$, *there exist an integer* $j_0 > 0$ *and a number* $\varepsilon > 0$ *such that any* C^1 *function* v *as above satisfies*

$$\left|\frac{\mathrm{d}v}{\mathrm{d}t}(t) - \frac{\mathrm{d}c}{\mathrm{d}t}(t_0)\right| < \delta \tag{2.1}$$

if $j \geqslant j_0$ *and* $|t - t_0| < \varepsilon$.

3. Continuations of homoclinic tangencies

As we have stated at the end of section 1, arguments in [14, section 6.3] which were used to detect diffeomorphisms admitting homoclinic or heteroclinic tangencies in a small neighbourhood of Hénon maps in $\mathrm{Diff}^\infty(\mathbb{R}^2)$ cannot be applied directly to the *fixed* two-parameter family of original Hénon maps:

$$\varphi_{a,b}(x, y) = (y, a - bx + y^2).$$

In this section, we will consider a C^∞ function $h : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that the b-parameter family $\{\varphi_{h(b),b}\}$ admits a b-continuation of homoclinic quadratic tangencies $q_{h(b),b}$ each of which unfolds generically with respect to the a-parameter family $\{\varphi_{a,b(\mathrm{fixed})}\}$. Finally, we will obtain essential results for the a-parameter family.

For any element (a, b) of a small neighbourhood of $(-2, 0)$ in the parameter space, $\varphi_{a,b}$ has the two fixed points $p^\pm_{a,b}$ with

$$p^\pm_{a,b} = (y^\pm_{a,b}, y^\pm_{a,b}), \qquad \text{where } y^\pm_{a,b} = \frac{1 + b \pm \sqrt{(1+b)^2 - 4a}}{2}. \tag{3.1}$$

For short, we set $p^+_{a,b} = p_{a,b}$ and $y^+_{a,b} = y_{a,b}$. Then, the eigenvalues of the differential $(D\varphi_{a,b})_{p_{a,b}}$ at $p_{a,b}$ are

$$\lambda_{a,b} = y_{a,b} - \sqrt{y^2_{a,b} - b}, \qquad \sigma_{a,b} = y_{a,b} + \sqrt{y^2_{a,b} - b}. \tag{3.2}$$

Thus, for any $(a, b) \approx (-2, 0)$ with $b \neq 0$, the eigenvalues satisfy

$$0 < |\lambda_{a,b}| < 1 < \sigma_{a,b} \qquad \text{and} \qquad |\lambda_{a,b}|\sigma_{a,b} < 1. \tag{3.3}$$

A fixed point satisfying condition (3.3) is called a *dissipative saddle* fixed point.

When $b = 0$, $\varphi_{a,0}$ is not a diffeomorphism. Even in this case, one can define the stable and unstable manifolds associated with $p_{a,0}$ in a usual manner. The stable manifold $W^s(p_{a,0})$ of $\varphi_{a,0}$ is the horizontal line $y = y_{a,0}$ in $\mathbb{R}^2$ passing through $p_{a,0}$. Hence, $W^s(p_{a,0})$ contains

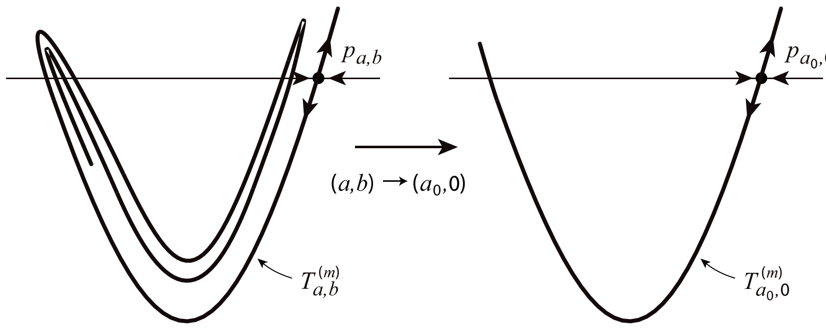

Figure 2. Unstable/stable curves for original Hénon diffeomorphisms and endomorphisms.

the horizontal segment $S_{a,0} = \{(x, y_{a,0}); |x| \leqslant 5/2\}$. By the stable manifold theorem (see, e.g., [17, chapter 5, theorem 10.1]), for any $(a, b) \approx (-2, 0)$, there exists an almost horizontal segment $S_{a,b} \subset W^s(p_{a,b})$ containing $p_{a,b}$ which C^∞ depends on (a, b) and such that one of the end point of $S_{a,b}$ is in the vertical line $x = -5/2$ and the other in $x = 5/2$. In particular, each $S_{a,b}$ is represented as the graph of a C^∞ function $\eta_{a,b}$ of x C^∞ depending on (a, b), that is,

$$S_{a,b} = \{(x, \eta_{a,b}(x)); |x| \leqslant 5/2\}.$$

Since the family $\{\eta_{a,b}\}$ C^∞ converges to the constant function $\eta_{a_0,0}$ uniformly as $(a, b) \to (a_0, 0)$,

$$\begin{aligned} &\lim_{(a,b)\to(a_0,0)} \max\left\{\left|\frac{\mathrm{d}\eta_{a,b}}{\mathrm{d}x}(x)\right|; -5/2 \leqslant x \leqslant 5/2\right\} = 0, \\ &\lim_{(a,b)\to(a_0,0)} \max\left\{\left|\frac{\mathrm{d}^2\eta_{a,b}}{\mathrm{d}x^2}(x)\right|; -5/2 \leqslant x \leqslant 5/2\right\} = 0. \end{aligned} \tag{3.4}$$

From the definition, the unstable manifold $W^u(p_{a,0})$ consists of the points $q \in \mathbb{R}^2$ which admits a sequence $\{q_n\}_{n=0}^\infty$ in $\mathbb{R}^2$ with $q_0 = q$, $q_n \in \varphi_{a,0}^{-1}(q_{n-1})$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} q_n = p_{a,0}$. In particular, $W^u(p_{a,0})$ is contained in the parabolic curve $\mathrm{Im}(\varphi_{a,0}) = \{(x, x^2 + a); -\infty < x < \infty\}$. Then, it is not hard to show that

$$W^u(p_{a,0}) = \{(x, x^2 + a); a \leqslant x < \infty\}$$

for any $a \approx -2$. Again by the stable manifold theorem, for any $(a, b) \approx (-2, 0)$ (possibly $b = 0$), there exist short curves $T_{a,b}$ in $W^u_{\mathrm{loc}}(p_{a,b})$ with $\mathrm{Int}(T_{a,b}) \ni p_{a,b}$ and varying C^∞ with respect to (a, b). Thus, for any integer $m > 0$, $T^{(m)}_{a,b} = \varphi^m_{a,b}(T_{a,b})$ C^∞ converges to $T^{(m)}_{a_0,0} = \varphi^m_{a_0,0}(T_{a_0,0})$ as $(a, b) \to (a_0, 0)$. Intuitively, the curve $T^{(m)}_{a_0,0}$ is obtained by 'folding' $T^{(m)}_{a,b}$ when m is sufficiently large, see figure 2. Let a_0 be any number sufficiently close to -2. For any $(a, b) \approx (a_0, 0)$, take a curve $U_{a,b}$ in $W^u(p_{a,b})$ with $p_{a,b}$ as one of its end points and C^∞ converging to $U_{a_0,0} = \{(x, x^2 + a_0); a_0 \leqslant x \leqslant y_{a_0,0}\} \subset W^u(p_{a_0,0})$ injectively as $(a, b) \to (a_0, 0)$. Set

$$U^1_{a,b} = \varphi_{a,b}(U_{a,b}) \setminus U_{a,b}, \qquad U^2_{a,b} = \varphi_{a,b}(U^1_{a,b}) \quad \text{and} \quad U^3_{a,b} = \varphi_{a,b}(U^2_{a,b}).$$

Fix a sufficiently small $\delta > 0$, let $l_{a,b}$ be the curve in $U^1_{a,b}$ such that its end points are contained in the vertical lines $x = \pm\delta$. The curve $l_{a,b}$ is represented by the graph of a C^∞ function $y = \zeta_{a,b}(t)$ $(-\delta \leqslant t \leqslant \delta)$ C^∞ depending on (a, b) and such that $\{\zeta_{a,b}\}$ uniformly C^∞ converges to the function $\zeta_{a,0}$ with $\zeta_{a,0}(t) = t^2 + a$ as $b \to 0$. Note that $l^1_{a,b} = \varphi_{a,b}(l_{a,b})$ has a maximal point of $U^2_{a,b}$ contained in a small neighbourhood $V(-2, 2)$ of $(-2, 2)$ in $\mathbb{R}^2$, see figure 3.

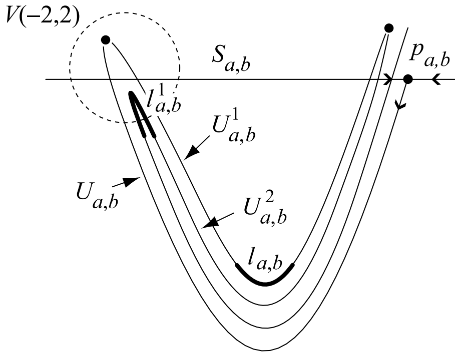

Figure 3. The case of $b > 0$.

Consider C^∞ diffeomorphisms $\Psi_{a,b}$ on $\mathbb{R}^2$ with $\Psi_{a,b}(x, y) = (x, y - \eta_{a,b}(x))$ if $|x| \leqslant 5/2$. Then, for any $(a, b) \approx (-2, 0)$, the image $\Psi_{a,b}(S_{a,b})$ is equal to the horizontal segment $[-5/2, 5/2] \times \{0\}$ in $\mathbb{R}^2$. In particular, any quadratic tangency of $\Psi_{a,b}(S_{a,b})$ and a curve l in $\mathbb{R}^2$ is either a maximal or minimal point of l. Let $\theta_{a,b} : [-\delta, \delta] \to \mathbb{R}$ be the C^∞ function such that $\theta_{a,b}(t)$ is the y-entry of the coordinate of $\Psi_{a,b} \circ \varphi_{a,b}(t, \zeta_{a,b}(t)) \in \mathbb{R}^2$, that is,

$$\theta_{a,b}(t) = a - bt + \zeta_{a,b}(t)^2 - \eta_{a,b}(\zeta_{a,b}(t)).$$

Note that $\theta_{a,b}$ C^∞ depends on (a, b). By (3.4), both $|\mathrm{d}\eta_{a,b}(x)/\mathrm{d}x|$, $|\mathrm{d}^2\eta_{a,b}(x)/\mathrm{d}x^2|$ $(-5/2 \leqslant x \leqslant 5/2)$ are sufficiently small. Since moreover $\mathrm{d}\zeta_{a,b}(t)/\mathrm{d}t \to 2t$ and $\mathrm{d}^2\zeta_{a,b}/\mathrm{d}t^2(t) \to 2$ as $b \to 0$, $\mathrm{d}^2\theta_{a,b}(t)/\mathrm{d}t^2 \approx -8 \neq 0$ for any $(a, b) \approx (-2, 0)$ and $t \approx 0$. Hence, there exists a unique point $t_{a,b} \in (-\delta, \delta)$ at which $\theta_{a,b}$ has the maximal value $\theta_{a,b}(t_{a,b})$. Here, we set

$$H(a, b) := \theta_{a,b}(t_{a,b}) \qquad \text{and} \qquad q_{a,b} := \varphi_{a,b}(t_{a,b}, \zeta_{a,b}(t_{a,b})) \in l^1_{a,b}.$$

The point $q_{a,b}$ is a candidate of the quadratic tangency of $S_{a,b}$ and $W^u(p_{a,b})$ for suitable pairs of (a, b).

From the definition, we know that H is a C^∞ function of $(a, b) \approx (-2, 0)$. Since $S_{a,0}$ is in the horizontal line $y = y_{a,0}$,

$$H(a, 0) = a^2 + a - y_{a,0} = a^2 + a - \frac{1 + \sqrt{1 - 4a}}{2}.$$

Thus, we have

$$H(-2, 0) = 0, \qquad \frac{\partial H}{\partial a}(a, b) \approx -\frac{8}{3} \neq 0$$

for any $(a, b) \approx (-2, 0)$. By the implicit function theorem, there exists a C^∞ function $h : I_\varepsilon = (-\varepsilon, \varepsilon) \to \mathbb{R}$ for a small $\varepsilon > 0$ satisfying

$$h(0) = -2, \qquad H(h(b), b) = 0, \qquad \frac{\mathrm{d}h}{\mathrm{d}b} = -\frac{\partial H/\partial b}{\partial H/\partial a}. \tag{3.5}$$

For each $b \in I_\varepsilon$, since

$$\theta_{h(b),b}(t_{h(b),b}) = 0, \qquad \frac{\mathrm{d}\theta_{h(b),b}}{\mathrm{d}t}(t_{h(b),b}) = 0 \quad \text{and} \quad \frac{\mathrm{d}^2\theta_{h(b),b}}{\mathrm{d}t^2}(t_{h(b),b}) \approx -8 \neq 0,$$

$q_{h(b),b}$ is a quadratic tangency of $W^u(p_{h(b),b})$ and $S_{h(b),b} \subset W^s(p_{h(b),b})$.

Remark 3.1. We note that $S_{h(b),b}$ is almost horizontal, but not in general strictly horizontal when $b \neq 0$. Thus, the slope of $S_{h(b),b}$ at the tangency $q_{h(b),b}$ is not necessarily zero, and hence

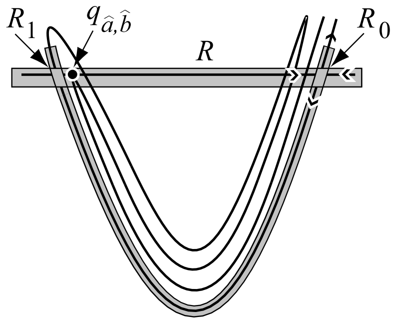

Figure 4. The case of $\hat{b} > 0$.

$q_{h(b),b}$ may not be a maximal point of $l^1_{h(b),b}$. On the other hand, $\Psi_{h(b),b}(q_{h(b),b})$ is a unique maximal point of $\Psi_{h(b),b}(l^1_{h(b),b})$ tangent to the horizontal segment $\Psi_{h(b),b}(S_{h(b),b})$. This is a technical reason for introducing the coordinate change by $\Psi_{a,b}$ to define the function H.

With the notation as above, we will prove the following proposition.

Proposition 3.2. *If $\varepsilon > 0$ is sufficiently small, then for any $b \in I_\varepsilon$, $W^u(p_{h(b),b})$ and $W^s(p_{h(b),b})$ have a transverse intersection and the quadratic tangency $q_{h(b),b}$ unfolds generically with respect to the a-parameter family $\{\varphi_{a,b(\text{fixed})}\}$.*

Proof. Suppose first that b is any positive number sufficiently close to 0. Then, since $l^1_{h(b),b}$ is tangent to $S_{h(b),b} \cap V(-2, 2)$ at $q_{h(b),b}$, $U_{h(b),b} \cup U^1_{h(b),b}$ meets $S_{h(b),b} \cap V(-2, 2)$ transversely at two points, see figure 3. On the other hand, for any $b < 0$ sufficiently close to 0, $U_{h(b),b} \cup U^1_{h(b),b}$ is disjoint from $S_{h(b),b} \cap V(-2, 2)$. But, in this case, $U^3_{h(b),b}$ meets $S_{h(b),b} \cap V(-2, 2)$ transversely at two points. This shows the former assertion.

From the fact that $\partial H(a, b)/\partial a|_{a=h(b)} \neq 0$, we know that the tangency $q_{h(b),b}$ unfolds generically with respect to the a-parameter family $\{\varphi_{a,b(\text{fixed})}\}$. This completes the proof. $\square$

We denote the graph of h in the ab-space by $\mathcal{H}$, that is, $\mathcal{H} = \{(h(b), b);\ b \in I_\varepsilon\}$. For any $(\hat{a}, \hat{b}) \in \mathcal{H}$ with $\hat{b} \neq 0$, one can take a sufficiently thin rectangle $R \subset \mathbb{R}^2$ with Int $R \supset S_{\hat{a},\hat{b}}$ and an even integer $w > 0$ so that $R \cap \varphi^w_{\hat{a},\hat{b}}(R)$ has curvilinear rectangle components R_0, R_1 such that Int R_0 contains the saddle fixed point $p_{\hat{a},\hat{b}}$, and Int R_1 contains a homoclinic transverse point associated with $p_{\hat{a},\hat{b}}$ in $V(-2, 2)$, but $R_0 \cup R_1$ is disjoint from the homoclinic tangency $q_{\hat{a},\hat{b}}$ given in proposition 3.2, see figure 4. One can take such an R so that the intersection

$$\Lambda^{\text{out}}_{\hat{a},\hat{b}} := \bigcap_{n=-\infty}^{\infty} \varphi^{wn}_{\hat{a},\hat{b}}(R_0 \cup R_1)$$

is a horseshoe basic set of $\varphi^w_{\hat{a},\hat{b}}$. Note that, since w is an even integer, $\varphi^w_{\hat{a},\hat{b}}$ is an orientation-preserving diffeomorphism even when $\hat{b} < 0$. Here, the superscript 'out' implicitly suggests that $\Lambda^{\text{out}}_{\hat{a},\hat{b}}$ is outside a small neighbourhood of the homoclinic tangency $q_{\hat{a},\hat{b}}$. Then, for any (a, b) sufficiently close to $(\hat{a}, \hat{b})$ (and hence in particular $b \neq 0$), there exists a continuation of basic sets $\Lambda^{\text{out}}_{a,b}$ of $\varphi^w_{a,b}$ based at $\Lambda^{\text{out}}_{\hat{a},\hat{b}}$.

Lemma 3.3. *For any $(\hat{a}, \hat{b}) \in \mathcal{H}$ with $\hat{b} \neq 0$, there exists an open neighbourhood $\mathcal{O} = \mathcal{O}(\hat{a}, \hat{b})$ of $(\hat{a}, \hat{b})$ in the ab-space and a constant $c = c(\hat{a}, \hat{b}) > 0$ such that, for any $(a, b) \in \mathcal{O}$, the thickness $\tau(\Lambda^{\text{out}}_{a,b} \cap W^s(p_{a,b}))$ is greater than c.*

Proof. Since $\Lambda^{\mathrm{out}}_{\hat{a},\hat{b}} \cap W^s(p_{\hat{a},\hat{b}})$ is a dynamically defined Cantor set, $\tau(\Lambda^{\mathrm{out}}_{\hat{a},\hat{b}} \cap W^s(p_{\hat{a},\hat{b}}))$ is positive, see [14, p 80, proposition 7]. Since moreover $\tau(\Lambda^{\mathrm{out}}_{a,b} \cap W^s(p_{a,b}))$ is continuous on (a, b), see [14, p 85, theorem 2], one can have $c > 0$ satisfying our desired property if the neighbourhood $\mathcal{O}$ is taken sufficiently small. □

4. Double renormalization near quadratic tangencies

In this section, we will work with the notation as in section 3 and reform renormalizations near a homoclinic tangency suitable to the proof of our main theorem.

Take an element $b \in I \setminus \{0\}$ arbitrarily and fix throughout this section. For simplicity, we set

$$\varphi_a = \varphi^w_{a,b} \qquad \text{and} \qquad p_a = p_{a,b}, \tag{4.1}$$

where $w > 0$ is the even integer given in the paragraph preceding lemma 3.3.

Lemma 4.1. *There exists a closed interval $J_{b,1}$ arbitrarily close to $h(b)$ satisfying the following conditions.*

(i) There exist positive integers n_0, m such that, for any $a \in J_{b,1}$ and any $n \geqslant n_0$, φ_a has a basic set $\Lambda^m_{a,n}$ containing a saddle periodic point $Q^m_{a,n}$, and in addition, the thickness of $\Lambda^m_{a,n} \cap W^u(Q^m_{a,n})$ is greater than $2/c$, where $c = c(h(b), b)$ is the constant given in lemma 3.3.

(ii) For any $a \in J_{b,1}$, the thickness of $\Lambda^{\mathrm{out}}_{a,b} \cap W^s(p_{a,b})$ is greater than c.

Proof.

(i) By proposition 3.2, $\varphi_{h(b)}$ has the homoclinic quadratic tangency $q_{h(b),b}$ in $V(-2, 2)$ associated with the dissipative saddle fixed point $p_{h(b)}$ which unfolds generically with respect to the one-parameter family $\{\varphi_a\}_{a\in J_{b,0}}$. We may suppose

- $p_a = (0, 0)$.
- When $a = h(b)$, both the points $r = (1, 0)$ and $r' = (0, 1) \in U$ belong to the orbit of the homoclinic tangency $q_{h(b),b}$ and satisfy $\varphi^N_{h(b)}(r') = r$ for some integer $N > 0$.

By the work of Romero [18, theorem D] based on [1] (see also [5, 9]), without the hypothesis of smooth linearization as in [14, 19, 20], we get the following renormalization near the homoclinic tangency $q_{h(b),b}$: for any sufficiently large integer $n > 0$, one can obtain a C^∞ reparametrization Θ_n on $\mathbb{R}$ and an a-dependent C^2 coordinate change Φ_n on $\mathbb{R}^2$ satisfying the following:

- $\mathrm{d}\Theta_n(\bar{a})/\mathrm{d}\bar{a} > 0$.
- $\Theta_n(\bar{a})$ (respectively $\Phi_n(\bar{x}, \bar{y})$) converges as $n \to \infty$ locally uniformly to the map with constant value $h(b) \in \mathbb{R}$ (respectively $r \in \mathbb{R}^2$).
- For any $\bar{a} \in \mathbb{R}$, the diffeomorphisms $\psi_{\bar{a},n}$ on $\mathbb{R}^2$, defined by
$$(\bar{x}, \bar{y}) \mapsto \psi_{\bar{a},n}(\bar{x}, \bar{y}) := \Phi_n^{-1} \circ \varphi^{N+n}_{\Theta_n(\bar{a})} \circ \Phi_n(\bar{x}, \bar{y}),$$
C^2 converge as $n \to \infty$ locally uniformly to the endomorphism $\psi_{\bar{a}}$ with
$$\psi_{\bar{a}}(\bar{x}, \bar{y}) = (\bar{y}, \bar{y}^2 + \bar{a}).$$

Furthermore, by [14, p 124, proposition], for each integer $m \geqslant 3$, we have a small closed interval $\bar{J}$ with Int $\bar{J} \ni -2$ and an integer $n_0 > 0$ satisfying following conditions.

- For any $\bar{a} \in \bar{J}$ and any integer $n \geqslant n_0$, $\psi_{\bar{a},n}$ has a saddle fixed point $P_{\bar{a},n}$ in a small neighbourhood of $(2, 2)$ and a basic set $\Lambda^m_{\bar{a},n}$ containing a saddle m-periodic point $Q^m_{\bar{a},n}$.

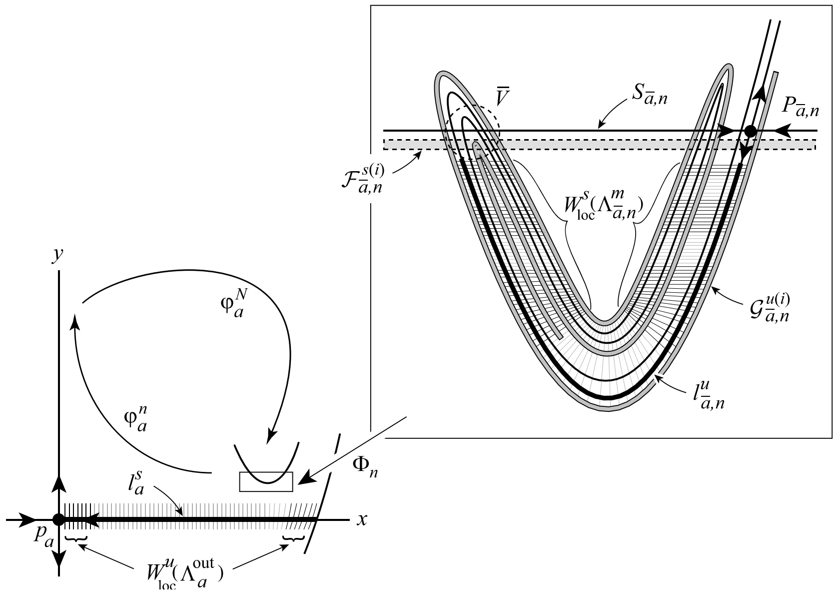

Figure 5. In the left (respectively right)-hand side figure, the short segments meeting l_a^s (respectively $l^u_{\bar{a},n}$) transversely are leaves of $\mathcal{G}^u_a$ (respectively $\mathcal{F}^s_{\bar{a},n}$).

- $\Theta_n(\bar{J}) \subset J_{b,0}$.
- For any $\bar{a} \in \bar{J}$ and any integer $n \geqslant n_0$, the thickness $\tau(\Lambda^m_{\bar{a},n} \cap W^u(Q^m_{\bar{a},n}))$ of the Cantor set $\Lambda^m_{\bar{a},n} \cap W^u(Q^m_{\bar{a},n})$ is greater than an arbitrarily given constant if m is sufficiently large.

So, one can take such $\bar{J}$, n_0 and m so that

$$\tau(\Lambda^m_{\bar{a},n} \cap W^u(Q^m_{\bar{a},n})) \geqslant \frac{2}{c}$$

for any $\bar{a} \in \bar{J}$ and $n \geqslant n_0$. Then, the proof of (i) is completed by letting $J_{b,1} = \Theta_n(\bar{J})$, $\Lambda^m_{a,n} = \Phi_n(\Lambda^m_{\bar{a},n})$ and $Q^m_{a,n} = \Phi_n(Q^m_{\bar{a},n})$.

(ii) For the proof, it suffices to retake the integer n_0 in (i) so that $J_{b,1} = \Theta_n(\bar{J})$ satisfies $J_{b,1} \times \{b\} \in \mathcal{O}(h(b), b)$ for any $n \geqslant n_0$, where $\mathcal{O}(h(b), b)$ is the open subset of the ab-space given in lemma 3.3. □

Convention 4.2. From now on, we will suppose that x, y, a and $\bar{x}, \bar{y}, \bar{a}$ are related by $a = \Theta_n(\bar{a})$ and $(x, y) = \Phi_n(\bar{x}, \bar{y})$ whenever the n is selected. Moreover, for any parametrized subset $Y_{\bar{a}}$ of the $\bar{x}\bar{y}$-space (respectively Z_a of the xy-space), we will denote the image $\Phi_n(Y_{\bar{a}})$ (respectively $\Phi_n^{-1}(Z_a)$) in the xy-space (respectively the $\bar{x}\bar{y}$-space) again by $Y_{\bar{a}}$ (respectively Z_a).

For any sufficiently large integer $n > 0$, one can define a foliation $\mathcal{F}^s_{\bar{a},n}$ in the $\bar{x}\bar{y}$-space compatible with $W^s_{\text{loc}}(\Lambda^m_{\bar{a},n})$ such that there exists an arc $l^u_{\bar{a},n}$ in $W^u(P_{\bar{a},n})$ crossing $\mathcal{F}^s_{\bar{a},n}$ exactly as shown in figure 5. Let $S_{\bar{a},n}$ be the curve in $W^s(P_{\bar{a},n})$ containing $P_{\bar{a},n}$ and such that one of the end point of $S_{a,b}$ is in the vertical line $\bar{x} = -5/2$ and the other in $\bar{x} = 5/2$. For any sufficiently large integer $i > 0$, by the inclination lemma (see for example [17, chapter 5, theorem 11.1]),

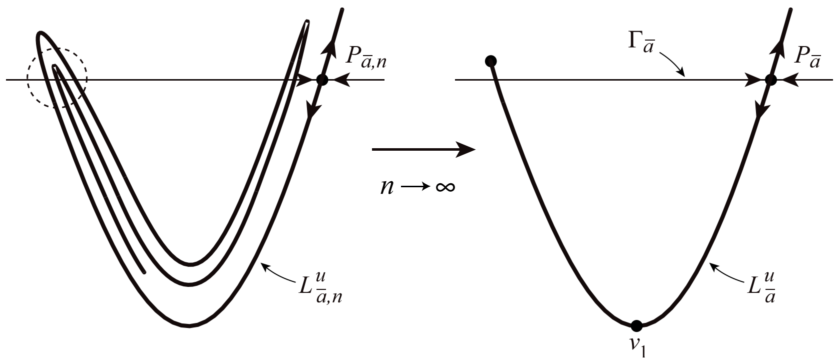

Figure 6. C^2 convergence of unstable curves $L^u_{\bar a,n}$.

one can have foliations $\mathcal{F}^{s(i)}_{\bar a,n}$ obtained by shortening the leaves of $\psi^{-i}_{\bar a,n}(\mathcal{F}^s_{\bar a,n})$ so that all leaves of $\mathcal{F}^{s(i)}_{\bar a,n}$ are well approximated by $S_{\bar a,n}$, see in figure 5.

Since $(a, b) \in \mathcal{O}$ for any $a \in J_{b,1}$, we have the basic set $\Lambda^{\text{out}}_a := \Lambda^{\text{out}}_{a,b}$ given in lemma 3.3. Let $\mathcal{G}^u_a$ be a foliation compatible with $W^u_{\text{loc}}(\Lambda^{\text{out}}_a)$ and l^s_a the segment in $W^s(p_a)$ crossing $\mathcal{G}^u_a$ exactly as shown in figure 5. By [14, p 125, proposition 1], there exists a compact arc $\sigma^s_{\bar a,n}$ in $W^s(P_{\bar a,n})$ containing $P_{\bar a,n}$ and converges in the xy-space to an arc in $W^s(p_a)$ which contains at least one fundamental domain as $n \to \infty$. Moreover, remark 1 in [14, p 129] shows that, for all sufficiently large n and some $j > 0$, $\varphi_a^{-(n+j)}(\sigma^s_{\bar a,n})$ meets $\mathcal{G}^u_a$ non-trivially and transversely. From this fact together with the inclination lemma, for any sufficiently large integer $i > 0$, one can have a foliation $\mathcal{G}^{u(i)}_{\bar a,n}$ in the $\bar x\bar y$-space as in figure 5 obtained by shortening the leaves of the Φ_n^{-1}-image of $\varphi^i_a(\mathcal{G}^u_a)$ so that all leaves of $\mathcal{G}^{u(i)}_{\bar a,n}$ are well approximated by a single curve $L^u_{\bar a,n}$ in $W^u(P_{\bar a,n})$ passing through $P_{\bar a,n}$ and with three maximal points as illustrated in figure 6. Here, we take the sequence $\{L^u_{\bar a,n}\}_n$ so that it C^2 converges onto the graph $L^u_{\bar a}$ of $\bar y = \bar x^2 + \bar a$ with $\bar a \leqslant \bar x \leqslant \bar x_1(\bar a) + \alpha$ for some $\alpha > 0$ as $n \to \infty$.

Recall that a leaf of $\mathcal{F}^{s(i)}_{\bar a,n}$ (respectively $\mathcal{G}^{u(i)}_{\bar a,n}$) is said to be a $\Lambda^m_{\bar a,n}$-leaf (respectively Λ^{out}_a-leaf) if the leaf is contained in $W^s(\Lambda^m_{\bar a,n})$ (respectively $W^u(\Lambda^{\text{out}}_a)$). Consider $\Lambda^m_{\bar a,n}$-leaves $f^{(i)}_{\bar a,n}$ of $\mathcal{F}^{s(i)}_{\bar a,n}$ and Λ^{out}_a-leaves $g^{(i)}_{\bar a,n}$ of $\mathcal{G}^{u(i)}_{\bar a,n}$ C^1 depending on $\bar a \in \bar J_{\bar\delta}$, where $\bar\delta > 0$ is taken sufficiently small so that $\bar J_{\bar\delta} = [-2-\bar\delta, -2+\bar\delta]$ is contained in the interval $\bar J$ given in the proof of lemma 4.1. Let $\bar V$ be a small open neighbourhood of $(-2, 2)$ in the $\bar x\bar y$-space and ξ a vertical segment in $\bar V$ meeting $f^{(i)}_{\bar a,n}$, $g^{(i)}_{\bar a,n}$, $S_{\bar a,n}$, $L^u_{\bar a,n}$ almost orthogonally for any $\bar a \in \bar J_{\bar\delta}$, see figure 7. We denote the the $\bar y$-entries of the coordinates of the intersections $f^{(i)}_{\bar a,n} \cap \xi$, $g^{(i)}_{\bar a,n} \cap \xi$, $S_{\bar a,n} \cap \xi$, $L^u_{\bar a,n} \cap \xi$ by $\bar y(f^{(i)}_{\bar a,n})$, $\bar y(g^{(i)}_{\bar a,n})$, $\bar y(S_{\bar a,n})$ and $\bar y(L^u_{\bar a,n})$, respectively.

Lemma 4.3. *With the notation as above, if we take $\bar\delta > 0$, $\bar V$ sufficiently small and n sufficiently large, then there exists an integer $i_0 = i_0(n) > 0$ such that, if $i \geqslant i_0$, the derivatives of $\bar y(f^{(i)}_{\bar a,n})$ and $\bar y(g^{(i)}_{\bar a,n})$ satisfy*

$$\frac{\mathrm{d}\bar y(f^{(i)}_{\bar a,n})}{\mathrm{d}\bar a} - \frac{\mathrm{d}\bar y(g^{(i)}_{\bar a,n})}{\mathrm{d}\bar a} > 2. \tag{4.2}$$

Proof. For any $\bar a \in \bar J_{\bar\delta}$, consider the endomorphism $\psi_{\bar a}(\bar x, \bar y) = (\bar y, \bar y^2 + \bar a)$ given in the proof of lemma 4.1. The point $P_{\bar a} = (\bar x_1, \bar y_1)$ with $\bar x_1(\bar a) = \bar y_1(\bar a) = (1 + \sqrt{1-4\bar a})/2$ is a fixed point of $\psi_{\bar a}$ close to $(2, 2)$. If necessary slightly extending ξ, we may assume that ξ meets the line $\Gamma_{\bar a} : \bar y = (1 + \sqrt{1-4\bar a})/2$ orthogonally in a single point. Since $\mathrm{d}\bar y_1/\mathrm{d}\bar a \to -1/3$ as $\bar a \to -2$

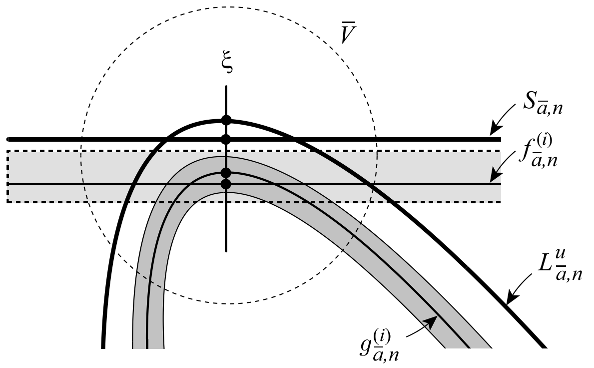

Figure 7. Vertical segment ξ, leaves $f^{(i)}_{\bar{a},n}$, $g^{(i)}_{\bar{a},n}$, stable and unstable curves $S_{\bar{a},n}$, $L^{u}_{\bar{a},n}$ in $\bar{V}$.

and $S_{\bar{a},n}$ C^2 converges to a segment in $\Gamma_{\bar{a}}$ as $n \to \infty$, $\mathrm{d}\bar{y}(S_{\bar{a},n})/\mathrm{d}\bar{a}$ is well approximated by $-1/3$ for all sufficiently large n.

The curve $L^{u}_{\bar{a}}$ has the point $v_1 = (0, \bar{a})$ as a unique minimal point. The $\psi_{\bar{a}}$-image $(\bar{x}_2, \bar{y}_2) = (\bar{a}, \bar{a}^2 + \bar{a})$ of v_1 is the left side end point of $L^{u}_{\bar{a}}$. Since $\mathrm{d}\bar{y}_2/\mathrm{d}\bar{a} = 2\bar{a} + 1 \to -3$ as $\bar{a} \to -2$, $\mathrm{d}\bar{y}(L^{u}_{\bar{a},n})/\mathrm{d}\bar{a}$ is well approximated by -3 for all sufficiently large n and any $\bar{a} \in \bar{J}_{\bar{\delta}}$ if we take $\bar{\delta} > 0$ sufficiently small.

Now, we apply accompanying lemma (lemma 2.2) to the ordered families $(\Lambda^{\mathrm{out}}_{a}, p_a, \mathcal{G}^{u}_{a}, P_{\bar{a},n}, \mathcal{G}^{u(i)}_{\bar{a},n}, \xi)$ and $(\Lambda^{m}_{\bar{a},n}, Q^{m}_{\bar{a},n}, \mathcal{F}^{u}_{\bar{a},n}, P_{\bar{a},n}, \mathcal{F}^{s(i)}_{\bar{a},n}, \xi)$ each of which corresponds to $(\Lambda_t, p_t, \mathcal{F}^{u}_{t}, \hat{p}_t, \mathcal{F}^{u(k_0+j)}_{t}, \sigma)$ in section 2.4 (see also figure 1). Then, there exists an integer $i_0 = i_0(n) > 0$ such that, for any $i \geqslant i_0$,

$$\left|\frac{\mathrm{d}\bar{y}(S_{\bar{a},n})}{\mathrm{d}\bar{a}} - \frac{\mathrm{d}\bar{y}(f^{(i)}_{\bar{a},n})}{\mathrm{d}\bar{a}}\right| < \frac{1}{10} \qquad \text{and} \qquad \left|\frac{\mathrm{d}\bar{y}(L^{u}_{\bar{a},n})}{\mathrm{d}\bar{a}} - \frac{\mathrm{d}\bar{y}(g^{(i)}_{\bar{a},n})}{\mathrm{d}\bar{a}}\right| < \frac{1}{10}.$$

This implies our desired inequality (4.2). □

Here, we consider the case when $f^{(i)}_{\bar{a},n}$ and $g^{(i)}_{\bar{a},n}$ have a quadratic tangency in ξ. Then, lemma 4.3 shows that the tangency unfolds generically with respect to $\psi_{\bar{a},n}$ and hence to φ_a in the sense of definition 2.1.

5. Proof of theorems A and B

In this section, we will work with the notation as in section 4 and convention 4.2. Theorems 5.1 and 5.2 below imply, respectively, the assertions (i) and (ii) of theorem A.

Theorem 5.1. *For any $b \in I \setminus \{0\}$, there exists a closed subinterval J_b of $J_{b,1}$ such that the one-parameter family $\{\varphi_a\}_{a \in J_b}$ with $\varphi_a = \varphi^{w}_{a,b}$ has generically unfolding persistent quadratic tangencies of $W^u(\Lambda^{\mathrm{out}}_{a})$ and $W^s(\Lambda^{m}_{\bar{a},n})$.*

Note that $\Lambda^{m}_{\bar{a},n}$ corresponds to the basic set $\Lambda^{\mathrm{in}}_{a}$ in the statement of theorem A.

Proof. We will work with $\bar{\delta}$, $\bar{V}$, n and i satisfying the conclusion of lemma 4.3. By the intermediate value theorem, there exists an $\bar{a} \in \bar{J}_{\bar{\delta}}$ such that the locally highest leaf of $\mathcal{G}^{u(i)}_{\bar{a},n}$ in $\bar{V}$ and that of $\mathcal{F}^{s(i)}_{\bar{a},n}$ has a tangency in $\bar{V}$, see figure 8(i). Then, for any $\bar{a}_1 \in J_{\bar{\delta}}$ sufficiently close to $\bar{a}$, there exists an almost vertical C^1 arc ρ in $\bar{V}$ containing subarcs ρ^u and ρ^s which, respectively, cross $\mathcal{G}^{u(i)}_{\bar{a},n}$ and $\mathcal{F}^{s(i)}_{\bar{a},n}$ exactly, and such that each point of $\rho \cap \mathcal{G}^{u(i)}_{\bar{a},n} \cap \mathcal{F}^{s(i)}_{\bar{a},n}$ is a

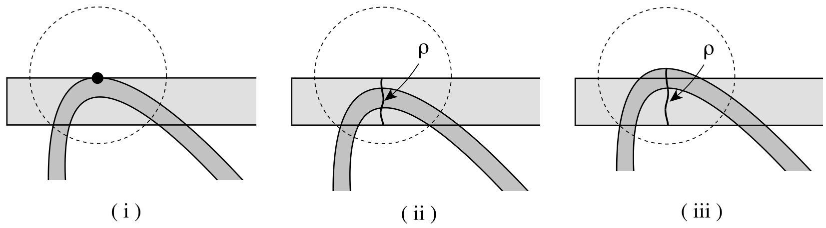

Figure 8. (ii) The case of $\bar{a}_1 > \bar{a}$. (iii) The case of $\bar{a}_1 < \bar{a}$.

tangency of leaves in $\mathcal{G}^{u(i)}_{\bar{a},n}$ and $\mathcal{F}^{s(i)}_{\bar{a},n}$, see figures 8(ii) and (iii). Let $\pi^u : \varphi^i_{a_1}(\mathcal{G}^u_{a_1}) \to \rho^u \subset \rho$ and $\pi^s : \psi^{-i}_{\bar{a}_1,n}(\mathcal{F}^s_{\bar{a}_1,n}) \to \rho^s \subset \rho$ be the C^1 projections along their leaves. Then, the compositions $\eta^u = \pi^u \circ (\varphi^i_{a_1}|l^s_{a_1}) : l^s_{a_1} \to \rho$ and $\eta^s = \pi^s \circ (\psi^{-i}_{\bar{a}_1,n}|l^u_{\bar{a}_1,n}) : l^u_{\bar{a}_1,n} \to \rho$ are C^1 embeddings onto ρ^u and ρ^s, respectively. Note that each point of $\eta^u(W^u_{\mathrm{loc}}(\Lambda^{\mathrm{out}}_{a_1}) \cap l^s_{a_1})$ (respectively, $\eta^s(W^s_{\mathrm{loc}}(\Lambda^m_{\bar{a}_1,n}) \cap l^u_{\bar{a}_1,n})$) is in a $\Lambda^{\mathrm{out}}_{a_1}$-leaf (respectively, $\Lambda^m_{\bar{a}_1,n}$-leaf).

By lemma 4.1 (ii), if necessary replacing n and hence i by greater integers, one can suppose that

$$\tau(\Lambda^m_{\bar{a}_1,n} \cap l^u_{\bar{a}_1,n}) > \frac{1}{c}.$$

This fact together with lemma 4.1(i) shows that

$$\tau(\Lambda^{\mathrm{out}}_{a_1} \cap l^s_{a_1}) \cdot \tau(\Lambda^m_{\bar{a}_1,n} \cap l^u_{\bar{a}_1,n}) > 1.$$

Thus, there exists a closed subinterval $\bar{J}_b$ of $\bar{J}_{\bar{\delta}}$ such that the last inequality holds if $\bar{a}_1 \in \bar{J}_b$. By invoking thickness lemma [11, 14], for any $\bar{a}_1 \in \bar{J}_b$, we have a $\Lambda^{\mathrm{out}}_{a_1}$-leaf in $\mathcal{G}^{u(i)}_{\bar{a}_1,n}$ and a $\Lambda^m_{\bar{a}_1,n}$-leaf in $\mathcal{F}^{s(i)}_{\bar{a}_1,n}$ admitting a quadratic tangency $q_{\bar{a}_1}$ in $\bar{V}$. By considering a vertical segment ξ in $\bar{V}$ passing through $q_{\bar{a}_1}$ and using lemma 4.3, one can show that $q_{\bar{a}_1}$ unfolds generically with respect to $\{\psi_{\bar{a},n}\}$ and hence to $\{\varphi_a\}$. Since $\bar{J}_b \subset \bar{J}_{\bar{\delta}} \subset \bar{J}$, it follows that $J_b = \Theta_n(\bar{J}_b)$ is our desired subinterval of $J_{b,1} = \Theta_n(\bar{J})$. □

Theorem 5.1 presents persistent heteroclinic tangencies associated with two basic sets. The following theorem presents homoclinic tangencies associated with the fixed point p_a of φ_a and so we conclude the assertion (ii) of theorem A.

Theorem 5.2. *With the notation as above, there exists a dense subset J' of J_b such that, for any $\hat{a} \in J'$, $W^u(p_{\hat{a}})$ and $W^s(p_{\hat{a}})$ have a quadratic tangency $q_{\hat{a}}$ which unfolds generically with respect to $\{\varphi_a\}_{a \in J_b}$.*

Proof. For any fixed $a_1 \in J_b$, there exists a heteroclinic quadratic tangency of a $\Lambda^{\mathrm{out}}_{a_1}$-leaf of $\mathcal{G}^{u(i)}_{\bar{a}_1,n}$ and a $\Lambda^m_{\bar{a}_1,n}$-leaf of $\mathcal{F}^{s(i)}_{\bar{a}_1,n}$ in $\bar{V}$ unfolding generically with respect to φ_a. By lemma 4.3, there exists an $a_2 \in J_b$ arbitrarily close to a_1 such that $W^u(\Lambda^{\mathrm{out}}_{a_2})$ and $W^s(\Lambda^m_{\bar{a}_2,n})$ have a transverse intersection z_1 in $\bar{V}$, see figure 9.

Here, we will show that $W^s(\Lambda^{\mathrm{out}}_{a_2})$ and $W^u(\Lambda^m_{\bar{a}_2,n})$ also have a transverse intersection. Let $\mathcal{G}^s_{a_2}$ be a foliation compatible with $W^s_{\mathrm{loc}}(\Lambda^{\mathrm{out}}_{a_2})$ such that any leaves of $\mathcal{G}^s_{a_2}$ and $\mathcal{G}^u_{a_2}$ meet transversely at a single point, see figure 10 (and also figure 5). Again by [14, p 125, proposition 1, p 129, remark 1] and the inclination lemma, for any sufficiently large integer $i > 0$, one can have a foliation $\mathcal{G}^{s(i)}_{\bar{a}_2,n}$ obtained by shortening the leaves of $\varphi^{-i}_{a_2}(\mathcal{G}^s_{a_2})$ so that all leaves of $\mathcal{G}^{s(i)}_{\bar{a}_2,n}$ are well approximated by the arc $S_{\bar{a}_2,n}$ given in section 4. Then, as shown in figure 9, we have a transverse intersection point z_2 of a $\Lambda^{\mathrm{out}}_{a_2}$-leaf of $\mathcal{G}^{s(i)}_{\bar{a}_2,n}$ and a leaf of

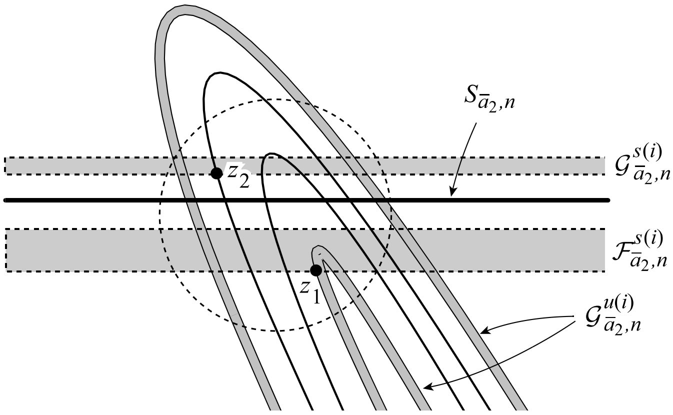

Figure 9. A transverse intersection z_1 between $W^u(\Lambda^{\text{out}}_{\bar{a}_2})$ and $W^s(\Lambda^m_{\bar{a}_2,n})$; a transverse intersection z_2 between a $\Lambda^{\text{out}}_{\bar{a}_2}$-leaf of $\mathcal{G}^{s(i)}_{\bar{a}_2,n}$ and a leaf of $W^u(\Lambda^m_{\bar{a}_2,n})$.

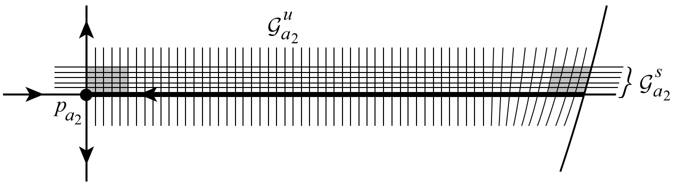

Figure 10. The union of the shaded regions contains $\Lambda^{\text{out}}_{a_2}$.

$W^u(\Lambda^m_{\bar{a}_2,n})$. Applying [12, lemma 8] to the cycle $\{\Lambda^{\text{out}}_{a_2}, z_1, \Lambda^m_{\bar{a}_2,n}, z_2\}$, one can define a basic set Λ_{a_2} with $\Lambda_{a_2} \supset \Lambda^{\text{out}}_{a_2} \cup \Lambda^m_{\bar{a}_2,n}$.

Since a_2 is an element of J_b, by theorem 5.1, $W^u(\Lambda^{\text{out}}_{a_2})$ and $W^s(\Lambda^m_{\bar{a}_2,n})$ have a *heteroclinic* quadratic tangency q_{a_2} unfolding generically. Since the basic sets $\Lambda^{\text{out}}_{a_2}$, $\Lambda^m_{\bar{a}_2,n}$ have dense subsets consisting of saddle periodic points, there exists $a'_2 \in J_b$ arbitrarily close to a_2 and satisfying the following two conditions.

- There exist leaves $l^u_{a'_2}$ of $W^u(\Lambda^{\text{out}}_{a'_2})$ and $l^s_{\bar{a}'_2}$ of $W^s(\Lambda^m_{\bar{a}'_2,n})$ which have a quadratic tangency $q_{a'_2}$ unfolding generically and, moreover, pass through saddle periodic points $\hat{p}^u_{a'_2} \in \Lambda^{\text{out}}_{a_2}$ and $\hat{p}^s_{a'_2} \in \Lambda^m_{\bar{a}_2,n}$, respectively.
- There is a basic set $\Lambda_{a'_2}$ of $\varphi_{a'_2}$ which belongs to a continuation of basic sets based at Λ_{a_2}.

Since $\Lambda_{a'_2}$ is a basic set containing $\Lambda^{\text{out}}_{a'_2} \cup \Lambda^m_{\bar{a}'_2,n}$, both $W^u(p_{a'_2})$ and $W^s(p_{a'_2})$ pass through an arbitrarily small neighbourhood U of $q_{a'_2}$. Since $q_{a'_2}$ is a tangency unfolding generically, there exists $a_3 \in J_b$ arbitrarily close to a'_2 such that $W^u(p_{a_3})$ and $W^s(p_{a_3})$ have a *homoclinic* quadratic tangency q_{a_3} in U. By invoking accompanying lemma, one can show that q_{a_3} is also a tangency unfolding generically. Here, we note that accompanying lemma works only for leaves sufficiently close to a leaf passing through saddle periodic points. In our case, the leaf $l^u_{a_3}$ (respectively $l^s_{a_3}$) passes through the saddle periodic points $\hat{p}^u_{a_3}$ (respectively $\hat{p}^s_{a_3}$). This is the reason for replacing the parameter a_2 by a'_2.

Recall that $a_1 \in J_b$ is taken arbitrarily and $a_3 \in J_b$ is arbitrarily close to $a_1 \in J_b$. From this fact, we have a dense subset J' of J_b such that, for any $\hat{a} \in J'$, $W^u(p_{\hat{a}})$ and $W^s(p_{\hat{a}})$ have a quadratic tangency unfolding generically. This completes the proof of theorem 5.2. □

Next, we prove theorem B using the results of theorems 5.1 and 5.2.

Proof of theorem B. For any $\hat{a} \in J'$, one can apply the Palis–Takens renormalization theory to a small neighbourhood of $q_{\hat{a}}$. Then, proposition 3.3 in [16] and lemma 2.2 in [13] imply the existence of an open subinterval $Y_{\hat{a}}$ of J_b arbitrarily close to $\hat{a} \in J'$ and such that, for any $a \in Y_{\hat{a}}$, φ_a has at least one sink whose basin meets $W^u(p_a)$ non-trivially. From the denseness of J' in J_b and the arbitrary closeness of $Y_{\hat{a}}$ to $\hat{a} \in J'$, we have an open dense subset $A_b^{(1)}$ of J_b such that, for any $a \in A_b^{(1)}$, φ_a admits at least one sink r with $W^u(p_a) \cap B_r \neq \emptyset$, where B_r is the basin of r. According to proposition 2.1 in [13], if φ_a had an SRB measure ν supported by the homoclinic set of p_a, then the support supp(ν) would coincide with the closure Cl($W^u(p_a)$). Since ν is a φ_a-invariant probability measure, it follows that $\mathrm{Cl}(W^u(p_a)) \cap B_r \subset \{r\}$. On the other hand, since $W^u(p_a) \cap B_r \neq \emptyset$, the intersection would contain an arc, a contradiction. Thus, for any $a \in A_b^{(1)}$, the homoclinic set of p_a does not support any SRB measure. This proves (i).

By applying arguments as in the proof of theorem E in [16, section 9] repeatedly, one can have open dense subsets Z_n $(n = 1, 2, \ldots)$ of J_b with $Z_1 = A_b^{(1)}$ such that φ_a has at least 2^n sinks for any $a \in Z_n$ associated with the periodic-doubling bifurcation. Then, $A_b^{(2)} = \bigcap_{n \geqslant 1} Z_n$ is a residual subset of J_b such that φ_a has infinitely many sinks if $a \in A_b^{(2)}$. This shows (ii).

According to Wang–Young [21, appendix A.2], for any $\hat{a} \in J'$, there exists a subset $X_{\hat{a}}$ of J_b with positive Lebesgue measure and contained in an arbitrarily small neighbourhood of $\hat{a}$ in J_b and such that, for any $a \in X_{\hat{a}}$, φ_a has a strange attractor with an SRB measure. Again by the density of J' in J_b, we have a subset $A_b^{(3)}$ of J_b satisfying the conditions required in (iii) of theorem B. □

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