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RESEARCH NOTE

Estimating and testing process accuracy with extension to asymmetric tolerances

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Abstract Pearn et al. (Commun. Stat. Theory Methods, 27(4):985–1000, 1998) introduced the process accuracy index C_a to measure the degree of process centering, the ability to cluster around the center. In this paper, we derive an explicit form of the cumulative distribution function for the estimator $\hat{C}_a$ with the case of symmetric tolerances. Subsequently, the distributional and inferential properties of the estimated process accuracy index C_a are provided. Calculations of the critical values, P-values, and lower confidence bounds are developed for testing process accuracy. Further, a generalization of C_a for the case with asymmetric tolerances is proposed to measure the process accuracy. Based on the results practitioners can easily perform the testing of the process accuracy, and make reliable decisions on whether actions should be taken to improve the process quality. An application is given to illustrate how we test the process accuracy using the actual data collected from the factory.

Keywords Asymmetric tolerances · Critical value · Process accuracy · Process centering

1 Introduction

In recent years, process capability indices (PCIs) have received substantial research attention in quality assurance and statistical literatures as well (see Kotz and Lovelace 1998; Kotz and Johnson 2002; Spiring et al. 2003; Pearn et al. 2004 for more details). Those indices have become popular as unit-less measures on whether a process is capable of reproducing items

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Table 1 C_a values and ranges of μ

C_a value	Range of μ	C_a value	Range of μ
$C_a = 1.00$	$\mu = M$	$0.25 < C_a < 0.50$	$d/2 < \lvert\mu - M\rvert < 3d/4$
$0.75 < C_a < 1.00$	$0 < \lvert\mu - M\rvert < d/4$	$0.00 < C_a < 0.25$	$3d/4 < \lvert\mu - M\rvert < d$
$0.67 < C_a < 0.75$	$d/4 < \lvert\mu - M\rvert < d/3$	$C_a = 0.00$	$\mu = LSL$ or $\mu = USL$
$0.50 < C_a < 0.67$	$d/3 < \lvert\mu - M\rvert < d/2$	$C_a < 0.00$	$\mu < LSL$ or $\mu > USL$

meeting the quality requirement preset by the product designer. Based on analyzing the PCIs, a production department can trace and improve a poor process so that the quality level can be enhanced and the requirements of the customers can be satisfied.

The first, and the original, process capability index was C_p. The C_p index reflects product consistency by considering the overall process variability relative to manufacturing tolerances, which is designed to provide an indirect measure of potential ability to meet requirements (Kane 1986). The C_a index measures the degree of process centering (the ability to cluster around the center), which can be regarded as a process accuracy index (see Pearn et al. 1998). The indices C_p and C_a are defined as the following:

$$C_p = \frac{USL - LSL}{6\sigma}, \quad C_a = 1 - \frac{|\mu - m|}{d},$$

where LSL and USL are the lower and upper specification limit, μ is the process mean, σ is the process standard deviation, $m = (USL + LSL)/2$ is the mid-point between the upper and the lower specification limits, and $d = (USL - LSL)/2$ is the half length of the specification interval. For processes with two-sided specification limits, the percentage of non-conforming items can be calculated as $1 - F(USL) + F(LSL)$, where $F(\cdot)$ is the cumulative distribution function (CDF) of the process characteristic X. On the assumption of normality, non-conforming items (NC) can be expressed in parts per million (PPM) as:

$$\text{NC} = \left[1 - \Phi\left(\frac{USL - \mu}{\sigma}\right) + \Phi\left(\frac{LSL - \mu}{\sigma}\right)\right] \times 10^6$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution. If the process is perfectly centered at the specification range ($\mu = m$), then the percentage of non-conforming items can be expressed as $2\Phi(-3C_p)$. For example, $C_p = 1.00$ corresponds to NC = 2,700 PPM, and $C_p = 1.33$ corresponds to NC = 63 PPM. However, C_p does not refer to the mean of the process, it will not give an exact measure of the percentage of non-conforming items in the general case, i.e., $\mu \neq m$. Therefore, it provides a lower bound on NC with $2\Phi(-3C_p)$. On the other hand, the index C_a provides a quantified measure of ability to cluster around the center, which alerts the user if the process deviates from its midpoint. For example, $C_a = 1$ indicates that the process is perfectly centered ($\mu = m$), $C_a = 0$ indicates that the process mean μ is located at one of the specification limits. Thus, when $0 < C_a < 1$, the process mean is located between the mid-point and one of the specification limits. Obviously, if $C_a < 0$ then it indicates that μ fall outside the specification limits (i.e., $\mu > USL$ or $\mu < LSL$), the process is severely off-center and it needs an immediate troubleshooting. Table 1 displays various C_a values and the corresponding ranges of the departure magnitude of μ.

We remark here that the process capability approach can be used only if the manufacturing process is under statistical control. If the process is out of control in the early stages of process capability analysis, it will be unreliable and meaningless to estimate process capability.

Proper understanding and accurate estimation of the capability index is essential for the company to maintain a capable supplier. The usual practice of judging process capability by evaluating the point estimates of process capability indices, have flaws since there is no assessment of the sampling errors. As the use of the capability indices grows more widespread, users are becoming educated and sensitive to the impact of the estimators and their sampling distributions, learning that capability measures must be reported in confidence intervals or via capability testing (Kushler and Hurley 1992; Vännman and Kotz 1995; Zimmer et al. 2001; Pearn and Yang 2003). Critical values are usually used for making decisions in capability testing with a designated Type I error α, the risk of misjudging an incapable process as a capable one. The P-values present the actual risk of misjudging an incapable process as a capable one. That is, if P-value $< \alpha$ then we reject the null hypothesis, and conclude that the process is capable with actual Type I error α. Similarly, the lower confidence bounds convey the minimum capability of the process which is essential to quality assurance.

Therefore, in order to assess process performance and make decisions in the manufacturing capability testing, the exact cumulative distribution functions (CDF) needs to be derived in advance. This paper is organized as follows. In Sect. 2, we derive the explicit forms of the cumulative distribution functions (CDF) and the probability density function (PDF) for the estimator $\hat{C}_a$ with the case of symmetric tolerances. The distributional and inferential properties of the estimated process accuracy index C_a are provided. The calculations of the critical value, P-value and lower confidence bound are developed for testing process quality. Furthermore, extensions to the case of asymmetric tolerances are discussed in Sect. 3. The decision making rules are presented in Sect. 4. Practitioners can use the proposed results to perform quality testing and determine how well the process can reproduce product items to meet the specified quality requirement. Finally, some concluding remarks are made in Sect. 5.

2 Estimation of accuracy index C_a for symmetric tolerances

A process is said to have symmetric tolerances if the target value T is set to be the midpoint of the specification interval, i.e. $T = m = (USL + LSL)/2$. Most research in quality assurance literature has focus on cases in which the manufacturing tolerances are symmetric. Examples include Chan et al. (1988), Pearn et al. (1992), Vännman and Kotz (1995), Spiring (1997), Zimmer et al. (2001), Vännman and Hubele (2003), and many others. To estimate the accuracy index C_a, Pearn et al. (1998) considered the natural estimator $\hat{C}_a$ as

$$\hat{C}_a = 1 - \frac{|\bar{X} - m|}{d},$$

where $\bar{X} = \sum_{i=1}^{n} x_i/n$ is the conventional estimator of the process mean μ, which may be obtained from a stable process.

To derive the explicit form for the CDF of the estimator $\hat{C}_a$, we first define $D = \sqrt{n}d/\sigma$, $Z' = \sqrt{n}(\bar{X} - m)/\sigma$, and $H = |Z'|$, where Z' is distributed as $N(\sqrt{n}(\mu - m)/\sigma, 1)$ and H is a folded normal distribution. Then, the estimator $\hat{C}_a$ can be rewritten as $\hat{C}_a = 1 - H/D$. Thus,

$$F_{\hat{C}_a}(x) = P\left\{\hat{C}_a \le x\right\} = P\left\{1 - \frac{H}{D} \le x\right\} = 1 - P\left\{H \le D\left(1-x\right)\right\}$$
$$= 1 - \int_0^{D(1-x)} f_H\left(h\right)dh \tag{1}$$

since $H = \left|Z'\right|$ is a folded normal distribution, we have

$$f_H\left(t\right) = \phi\left(t + \xi\sqrt{n}\right) + \phi\left(t - \xi\sqrt{n}\right), \tag{2}$$

where $\xi = (\mu - T)/\sigma$ and $\phi(\cdot)$ is the PDF of the standard normal distribution $N(0, 1)$.

Substituting (2) into (1) gives

$$F_{\hat{C}_a}\left(x\right) = 1 - \int_0^{D(1-x)} \left[\phi\left(t + \xi\sqrt{n}\right) + \phi\left(t - \xi\sqrt{n}\right)\right]dt, \quad \text{for } -\infty \le x \le 1. \tag{3}$$

By differential with respect to x gives the PDF of $\hat{C}_a$ as

$$f_{\hat{C}_a}\left(x\right) = D\left[\phi\left(D\left(1-x\right) + \xi\sqrt{n}\right) + \phi\left(D\left(1-x\right) - \xi\sqrt{n}\right)\right], \quad \text{for } -\infty \le x \le 1. \tag{4}$$

In order to determine whether a given process capability meets the customers' demands and runs under the desired quality condition, the statistical hypothesis testing can be stated as follows:

$H_0 : C_a \le C$ (process is inaccurate),
$H_1 : C_a > C$ (process is accurate).

We will reject the null hypothesis $H_0(C_a \le C)$, when $\hat{C}_a > c_0$ with Type I error $\alpha(c_0)$, the chance of incorrectly concluding an inaccurate process $(C_a \le C)$ as an accurate $(C_a > C)$ process. Based on the CDF of $\hat{C}_a$ expressed in (3), given values of capability requirement C, parameter ξ, sample size n, and α risk, the critical value c_0 can be obtained by solving the Eq. 5 using available numerical methods.

$$\int_0^{b\sqrt{n}(1-c_0)} \left[\phi\left(t + \xi\sqrt{n}\right) + \phi\left(t - \xi\sqrt{n}\right)\right]dt = \alpha, \tag{5}$$

where $b = d/\sigma$.

For users' convenience in applying our proposed procedure, we tabulate the critical values of $\hat{C}_a$ for values of $\alpha = 0.01$ and 0.05 with $n = 10, 25(25)100$ in Table 2 for various values of C. For example, if $C = 0.667$ is the minimum capability requirement, then for $\alpha = 0.01$, with sample size $n = 50$, $\xi = 1.0$ and d/σ we can find $c_0 = 0.777$ from Table 2. Thus, the critical value of $\hat{C}_a$ required for the process capable is $c_0 = 0.777$. That is, if $\hat{C}_a$ is greater than 0.777, we say that the process is capable.

Given $C_a = C$, $b = d/\sigma$ expressed as $b = |\xi|/(1 - C)$, the P-value corresponding to c^*, a specific value of $\hat{C}_a$ calculated from the sample data, is:

$$P\text{-value} = P(\hat{C}_a \ge c^* | C_a = C) = \int_0^{b\sqrt{n}(1-c^*)} \left[\phi\left(t + \xi\sqrt{n}\right) + \phi\left(t - \xi\sqrt{n}\right)\right]dt. \tag{6}$$

Furthermore, based on the CDF of $\hat{C}_a$ expressed in (3), given the sample of size n, the confidence level γ, the estimated value $\hat{C}_a$, and ξ, then the lower confidence bounds C_a^L can be obtained by solving the following Eq. 7

$$\int_0^{b\sqrt{n}\left(1-\hat{C}_a\right)} \left[\phi\left(t + \xi\sqrt{n}\right) + \phi\left(t - \xi\sqrt{n}\right)\right]dt = 1 - \gamma, \tag{7}$$

Table 2 Critical values c_0 for values of $\alpha = 0.01$ and 0.05 and $n = 25, 50,$ and 75

$\lvert\xi\rvert$		0.5			1.0			1.5		
n	$\frac{d}{\sigma}$	C_a	c_0		C_a	c_0		C_a	c_0	
			$\alpha = 0.01$	$\alpha = 0.05$		$\alpha = 0.01$	$\alpha = 0.05$		$\alpha = 0.01$	$\alpha = 0.05$
25	2	0.750	0.973	0.914	0.500	0.733	0.664	0.250	0.483	0.414
	3	0.833	0.982	0.943	0.667	0.822	0.777	0.500	0.655	0.610
	4	0.875	0.987	0.957	0.750	0.866	0.832	0.625	0.741	0.707
	5	0.900	0.989	0.966	0.800	0.893	0.866	0.700	0.793	0.766
50	2	0.750	0.914	0.866	0.500	0.664	0.616	0.250	0.414	0.366
	3	0.833	0.943	0.911	0.667	0.777	0.744	0.500	0.610	0.578
	4	0.875	0.957	0.933	0.750	0.832	0.808	0.625	0.707	0.683
	5	0.900	0.966	0.947	0.800	0.866	0.847	0.700	0.766	0.747
75	2	0.750	0.884	0.845	0.500	0.634	0.595	0.250	0.384	0.345
	3	0.833	0.923	0.896	0.667	0.756	0.730	0.500	0.590	0.563
	4	0.875	0.942	0.922	0.750	0.817	0.797	0.625	0.692	0.672
	5	0.900	0.954	0.938	0.800	0.854	0.838	0.700	0.754	0.738

Table 3 Lower confidence bounds C_a^L of $\hat{C}_a = 0.75$ for $\lvert\xi\rvert =$ 1.0(0.1)2.0 and $n = 20, 30, 50,$ and 70 with $\alpha = 0.05$

$\lvert\xi\rvert$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$n = 20$	0.605	0.625	0.640	0.651	0.661	0.669	0.675	0.681	0.686	0.690	0.694
$n = 30$	0.643	0.656	0.667	0.675	0.682	0.688	0.692	0.696	0.700	0.703	0.706
$n = 50$	0.674	0.683	0.690	0.696	0.700	0.704	0.708	0.710	0.713	0.715	0.717
$n = 70$	0.689	0.696	0.701	0.706	0.709	0.712	0.715	0.717	0.719	0.721	0.723

where $b = \lvert\xi\rvert/(1 - C_a^L)$. Table 3 displays the lower confidence bounds C_a^L of $\hat{C}_a = 0.75$, for various parameter values, with $\alpha = 0.05$, $\lvert\xi\rvert = 1.0(0.1)2.0$, and $n = 20(10)80$.

Therefore, practitioners can use the proposed previous results to perform quality testing and determine if the process can reproduce product items to meet the specified process accuracy requirement.

3 Estimation of accuracy index C_a for asymmetric tolerances

Although cases with symmetric tolerances are common in practical situations, cases with asymmetric tolerances ($T \neq m$) often occur in the manufacturing industry. From the customer's point of view, asymmetric tolerances reflect that deviations from the target are less tolerable in one direction than in the other (see Boyles 1994 and Wu and Tang 1998). Usually they are not related to the shape of the supplier's process distribution. However, asymmetric tolerances can also arise in situations where the tolerances are symmetric to begin with, but the process distribution is skewed or follows a non-normal distribution. Dealing with this, the data have been transformed to achieve approximate normality, as shown by

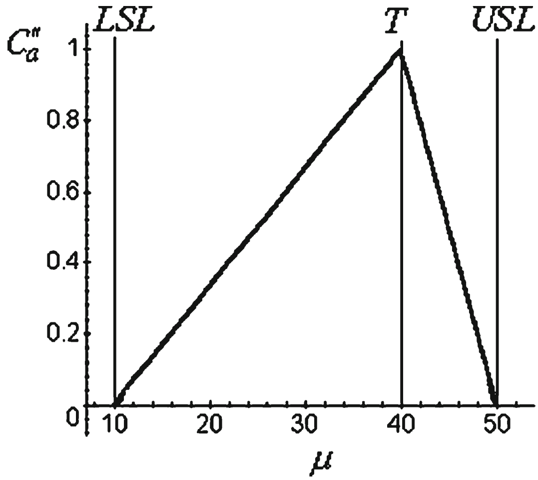

Fig. 1 Plots of C''_a values for processes with $10 \leq \mu \leq 50$ under $(LSL, T, USL) = (10, 40, 50)$

Chou et al. (1998) who have used Johnson curves to transform non-normal process data. Moreover, these indices presented above, are designed to monitor the performance for only normal and near-normal processes with symmetric tolerances, which are shown to be inappropriate for cases with asymmetric tolerances (Boyles 1994; Pearn and Chen 1998). Unfortunately, there has been comparatively little research published on cases with asymmetric tolerances. Exceptions are Boyles (1994), Chen (1998), Pearn and Chen (1998), Jessenberger and Weihs (2000), and Shu and Chen (2005).

To overcome the asymmetric cases ($USL - T \neq T - LSL$), we modify C_a index, denoted here as C''_a:

$$C''_a = 1 - \frac{A^*}{d^*},$$

where $A^* = max\{d^*(\mu - T)/D_u, d^*(T - \mu)/D_l\}$, $d^* = min\{D_u, D_l\}$, $D_u = USL - T$, $D_l = T - LSL$. Obviously, if $T = m$ (symmetric tolerance), then $d^* = D_u = D_l = d$, C''_a reduces to the original index C_a. The factor A^* ensures that the new generalization C''_a obtains its maximal value at $\mu = T$ (process is on-target) regardless of whether the tolerances are symmetric ($T = m$) or asymmetric ($T \neq m$). Figure 1 displays the plots of C''_a for processes with $10 \leq \mu \leq 50$ where $(LSL, T, USL) = (10, 40, 50)$ is an asymmetric tolerance. $C''_a = 0$ can be verified when the process mean is on the specification limit ($\mu = LSL$ or $\mu = USL$). On the other hand, $C''_a > 0$ when $LSL < \mu < USL$. Thus, given two processes E and F with $\mu_E > T$ and $\mu_F < T$, satisfying $(\mu_E - T)/D_u = (T - \mu_F)/D_l$ (i.e., processes E and F have an equal departure ratio), the C''_a values given to processes E and F are the same. For example, consider processes E and F with $\mu_E = 45 > T$ and $\mu_F = 25 < T$. Clearly, the corresponding departure ratios are 1/2 for both processes E and F (i.e., $(45 - 40)/10 = (40 - 25)/30 = 1/2$). We have $C''_a = 0.50$ for both processes E and F. In addition, the index C''_a decreases when mean μ shifts away from target T in either direction. In fact, C''_a decreases faster when μ shifts away from T to the closer specification limit than that to the farther specification limit. This is an advantage since the index would respond faster to the shift towards "the wrong side" of T than towards the middle of the specification interval.

To estimate the generalization C''_a, we can define the natural estimator $\hat{C}''_a$ as:

$$\hat{C}''_a = 1 - \frac{\hat{A}^*}{d^*},$$

where $\hat{A}^* = max\left\{d^*\left(\bar{X} - T\right)/D_u, d^*\left(T - \bar{X}\right)/D_l\right\}$ and $\bar{X} = \sum_{i=1}^{n} x_i/n$. We now define $D^* = n^{1/2}\left(d^*/\sigma\right), Z = n^{1/2}\left(\bar{X} - T\right)/\sigma, Y = \left[max\left\{(d^*/D_u)\,Z\;, -(d^*/D_l)\,Z\right\}\right]^2 \lambda = \delta^2$, and $\delta = n^{1/2}(\mu - T)/\sigma$. Then, the estimator $\hat{C}''_a$ can be rewritten as:

$$\hat{C}''_a = 1 - \frac{\sqrt{Y}}{D^*}.$$

Under the assumption of normality, Z is distributed as the normal distribution $N(\delta, 1)$. We note that the statistic Z^2 follows a non-central chi-square distribution with one degree of freedom and non-centrality parameter $\lambda = \delta^2$. Chen (1998) defined the distribution of Y as a weighted non-central chi-square distribution with one degree of freedom and non-centrality parameter λ under the assumption of normality. The PDF of Y is derived as:

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \left\{ h_j(\lambda)\,\Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} \frac{(-1)^{ij}}{d_i^2} f_{Y_j}\left(\frac{y}{d_i^2}\right)\right\}, \quad y > 0. \tag{8}$$

where $\lambda = \delta^2, \delta = n^{1/2}(\mu - T)/\sigma, h_j(\lambda) = (2\lambda)^{j/2}/(j!), d_1 = d^*/D_l, d_2 = d^*/D_u$ and Y_j is distributed as χ^2_{1+j}, the Chi-square distribution with degree of freedom $1 + j$. Thus, the CDF of $\hat{C}''_a$ can be obtained as follows:

$$\begin{aligned} F_{\hat{C}''_a}(x) &= P\left\{\hat{C}''_a \le x\right\} = P\left\{1 - \frac{\sqrt{Y}}{D^*} \le x\right\} = 1 - P\left(Y \le [D\,(1-x)]^2\right) \\ &= 1 - \int_0^{[D^*(1-x)]^2} f_Y(y)dy, \text{ for } x < 1. \end{aligned} \tag{9}$$

By substituting (8) into (9) gives

$$\begin{aligned} F_{\hat{C}''_a}(x) = 1 - \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} &\left\{ h_j(\lambda)\Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} \frac{(-1)^{ij}}{d_i^2} \right. \\ &\left. \times \int_0^{[D^*(1-x)]^2} f_{Y_j}\left(\frac{y}{d_i^2}\right) dy \right\}, \quad \text{for } x < 1. \end{aligned} \tag{10}$$

Changing the variable with $t = y/d_i^2$, then $dy = d_i^2 dt$ for $i = 1, 2$, the CDF of $\hat{C}''_a$ can be rewritten as

$$F_{\hat{C}''_a}(x) = \begin{cases} 1 - \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} h_j(\lambda)\Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} (-1)^{ij} \int_0^{[D^*(1-x)/d_i]^2} f_{Y_j}(t)dt, & x < 1 \\ 1, & x \ge 1 \end{cases} \tag{11}$$

Taking the derivative of the CDF of $\hat{C}''_a$ in x, the PDF of $\hat{C}''_a$ will be obtained as

$$f_{\hat{C}''_a}(x) = \frac{e^{-\lambda/2}}{\sqrt{\pi}} \sum_{j=0}^{\infty} h_j(\lambda)\Gamma\left(\frac{1+j}{2}\right) \times \sum_{i=1}^{2} (-1)^{ij}(1-x)\left(\frac{D^*}{d_i}\right)^2 f_{Y_j}\left(\left[\frac{(1-x)D^*}{d_i}\right]^2\right), \quad x < 1. \tag{12}$$

If the manufacturing tolerance is symmetric ($T = m$), then $d^* = D_u = D_l = d, d_1 = d_2 = 1$, the CDF of $\hat{C}''_a$ in (10) is reduced to:

$$F_{\hat{C}''_a}(x) = \begin{cases} 1 - \sum_{\ell=0}^{\infty} P_\ell(\lambda) \int_0^{[D(1-x)]^2} f_{Y_{2\ell}}(t)dt, & x < 1 \\ 1, & x \geq 1 \end{cases} \tag{13}$$

and the corresponding PDF is:

$$f_{\hat{C}''_a}(x) = \sum_{\ell=0}^{\infty} P_\ell(\lambda) \frac{D^{2\ell+1}(1-x)^{2\ell}}{\Gamma\left((2\ell+1)/2\right) 2^{(2\ell-1)/2}} e^{-[D(1-x)]^2/2}, \quad x < 1 \tag{14}$$

where $D = n^{1/2}(d/\sigma)$ and $P_\ell(\lambda) = e^{-\lambda/2}(\lambda/2)^\ell/(\ell!)$. It can be remarked that (13) and (14) are equivalent to (3) and (4), respectively.

4 Decision making rule for asymmetric tolerances

To make a decision for the case of asymmetric tolerances, we consider a testing hypothesis with the null hypothesis $C''_a \leq C$ (the process is inaccurate) and the alternative hypothesis $C''_a > C$ (the process accurate). The null hypothesis will be rejected if $\hat{C}''_a > c_0$, where the critical value c_0 is determined by $P\left(\hat{C}''_a > c_0 | C''_a = C\right) = \alpha$. Hence, we can find c_0 by solving Eq. 15

$$P\left(\hat{C}''_a > c_0 | C''_a = C\right) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \left\{ h_j(\lambda)\Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} \frac{(-1)^{ij}}{d_i^2} \times \int_0^{[D^*(1-c_\alpha)]^2} f_{Y_j}\left(\frac{y}{d_i^2}\right) dy \right\} = \alpha. \tag{15}$$

Table 4 displays the critical values c_0 for $d^*/\sigma = 2, 3, 4$ and 5 with sample sizes $n = 25, 50,$ and 75, $|\xi| = 0.5(0.5)1.5$, $\alpha = 0.01$ and 0.05 and $D_l : D_u = 6 : 4$.

Similarly, if the estimated index value is c^*, given the values of C, ξ and sample size n, then the P-value can be calculated as:

$$P\text{-value} = P\left(\hat{C}''_a > c^* | C''_a = C\right) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \left\{ h_j(\lambda)\Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} \frac{(-1)^{ij}}{d_i^2} \int_0^{[D^*(1-c^*)]^2} f_{Y_j}\left(\frac{y}{d_i^2}\right) dy \right\}, \tag{16}$$

Table 4 Critical value c_0 of C_a'' for various parameter values, with $D_l : D_u = 6 : 4$, $\alpha = 0.01$ and 0.05, and $n = 25$, 50, and 75

$\|\xi\|$		0.5			1.0			1.5		
n	$\frac{d^*}{\sigma}$	C_a''	c_0		C_a''	c_0		C_a''	c_0	
			$\alpha = 0.01$	$\alpha = 0.05$		$\alpha = 0.01$	$\alpha = 0.05$		$\alpha = 0.01$	$\alpha = 0.05$
25	2	0.750	0.965	0.909	0.500	0.732	0.664	0.250	0.482	0.414
	3	0.833	0.977	0.939	0.667	0.821	0.776	0.500	0.655	0.609
	4	0.875	0.982	0.954	0.750	0.866	0.832	0.625	0.741	0.707
	5	0.900	0.986	0.963	0.800	0.893	0.865	0.700	0.793	0.765
50	2	0.750	0.914	0.866	0.500	0.665	0.616	0.250	0.414	0.366
	3	0.833	0.942	0.910	0.667	0.776	0.744	0.500	0.609	0.577
	4	0.875	0.957	0.933	0.750	0.832	0.808	0.625	0.707	0.683
	5	0.900	0.965	0.946	0.800	0.866	0.846	0.700	0.765	0.746
75	2	0.750	0.884	0.844	0.500	0.634	0.595	0.250	0.387	0.345
	3	0.833	0.922	0.896	0.667	0.756	0.730	0.500	0.591	0.563
	4	0.875	0.942	0.922	0.750	0.817	0.797	0.625	0.693	0.672
	5	0.900	0.953	0.937	0.800	0.853	0.838	0.700	0.754	0.738

Table 5 Lower Confidence Bounds $C_a^{L}{}''$ for various parameter values, $\hat{C}_a'' = 0.75$, $D_l : D_u = 7 : 3$, $\alpha = 0.01, 0.05$, $n = 20(10)50$, and $|\xi| = 0.7(0.1)1.0$

n	$\|\xi\| = 0.7$		$\|\xi\| = 0.8$		$\|\xi\| = 0.9$		$\|\xi\| = 1.0$	
	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$
20	0.027	0.474	0.286	0.538	0.408	0.578	0.479	0.605
30	0.365	0.563	0.468	0.600	0.527	0.625	0.566	0.643
40	0.474	0.603	0.538	0.630	0.578	0.648	0.605	0.663
50	0.529	0.626	0.576	0.648	0.606	0.663	0.628	0.675

Furthermore, given the true process capability level γ and the estimated value $\hat{C}_a''$ and ξ, the lower confidence bounds $C_a^{L}{}''$ can be obtained by solving the following Eq. 17

$$\frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \left\{ h_j(\lambda)\Gamma\left(\frac{1+j}{2}\right) \sum_{i=1}^{2} \frac{(-1)^{ij}}{d_i^2} \int_0^{[D^*(1-\hat{C}_a'')]^2} f_{Y_j}\left(\frac{y}{d_i^2}\right) dy \right\} = 1-\gamma. \quad (17)$$

Table 5 displays the lower confidence bounds for the asymmetric case $D_l : D_u = 7 : 3$, with $\hat{C}_a'' = 0.75$ $\alpha = 0.01, 0.05$, $n = 20(10)50$, and $|\xi| = 0.6(0.1)1.0$.

4.1 An example of testing the laser marking on IC packages

Integrated circuit (IC) packages are used for encapsulating the chips and connections against environmental, electrical and electromagnetic effects, and for establishing thermal path from the semi-conductor junction to the environment and electrical connection from the chip to the printed circuit board (PCB). In mask marking of IC packages, the marking area is determined by the output power of the laser, the cross-section of the beam and the properties of the material being marked. Marked area of 75 mm^2 at 10 J/pulse with pulse duration of about 0.69 ms has been reported for uncoated plastic packages. For the coated plastic packages (e.g., marker ink or varnish), the marked area can reach 600 mm^2 per pulse with pulse energy ranging from 0.1 J to 0.75 J and a pulse repetition frequency of 30 Hz. For mask marking, a minimum character height of about 0.5 mm has been reported.

The quality of a mark is assessed by its legibility characteristics such as mark contrast, mark with, mark depth, spattering, and micro cracks. The characteristics are usually evaluated using complementary techniques such as optical microscopy, ultrasonics microscopy, electron microscopy, surface roughness measurement, and contrast evaluation devices. The acceptance of level of each of these characteristics generally depends on the manufacturer's requirements. Mark width refers to the width of the line segment that forms a character. With the mask image marking, the mark width in the characters is essentially determined by the mask geometry and the lens imaging quality. It can be as small as a few micro-meters, which can only be read under a microscope. In beam deflected marking, the line width is mainly determined by the focused beam spot size, which varies between 20 and 100 mm.

The IC packages company monitors the process that laser marking on IC packages by measuring the mark width. According to the customer's requirement, this company considers the following normally distributed process with asymmetric tolerances $LSL = 20$ mm, $T = 26.5$ mm and $USL = 32$ mm for the mark width and the $\alpha -$ risk $= 0.05$. We calculate that $D_l = T - LSL = 6.5$, $D_u = USL - T = 5.5$, $d^* = \min\{D_l, D_u\} = 5.5$. The sample of $n = 100$, the sample mean $\bar{X} = 27.35$ and the sample standard deviation $S = 2.0$. We can calculate $\hat{A}^* = \max\left\{d^*\left(\bar{X} - T\right)/D_u, d^*\left(T - \bar{X}\right)/D_l\right\} = 0.85$, $\hat{\xi} = \left(\bar{X} - T\right)/S = 0.425$ and $\hat{C}_a'' = 0.845$. The corresponding P-value is 0.0532 from calculating (20). Because the $\alpha -$ risk $= 0.05$ is small than 0.0532, we do not have sufficient information to conclude that the process meets the present accuracy requirement. The further improvement actions for the accuracy are needed in this laser marking process.

5 Conclusions

The use of indices to measure process capability and communicate information about processes and the specified requirements has become widespread. In this paper, the process accuracy index C_a is proposed to measure the degree of process centering (the ability to cluster around the center). An explicit form of the cumulative distribution function for the estimator $\hat{C}_a$ with the case of symmetric tolerances is derived. We provide the distributional and inferential properties of the estimated process accuracy index C_a. The calculations of critical value, P-value and lower confidence bound are developed for testing process quality. Furthermore, a new generalization of C_a for cases with asymmetric tolerances is proposed to measure the process accuracy. The obtained results are useful for the practitioners in performing the process accuracy testing, and make decisions whether improvement actions should be taken. For illustrative purpose, a real-world application is also given to show how to test process accuracy by using the actual data collected from the factory.

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