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The flipping puzzle on a graph[☆]

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ABSTRACT

Let S be a connected graph which contains an induced path of $n-1$ vertices, where n is the order of S. We consider a puzzle on S. A configuration of the puzzle is simply an n-dimensional column vector over $\{0, 1\}$ with coordinates of the vector indexed by the vertex set S. For each configuration u with a coordinate $u_s = 1$, there exists a move that sends u to the new configuration which flips the entries of the coordinates adjacent to s in u. We completely determine if one configuration can move to another in a sequence of finite steps.

1. Introduction

Let S be a simple connected graph with vertex set $S = \{s_1, s_2, \ldots, s_n\}$. By a *flipping puzzle* on S, we mean a set of *configurations* of S and a set of *moves* on the configurations defined below. The configuration of the flipping puzzle is S, together with an assignment of white or black state to each vertex of S. A move applied to a configuration u in the puzzle is to select a vertex s_i which has a black state, and then flip the states of all neighbors of s_i in u. For convenience we use the set F_2^n of column vectors over $F_2 := \{0, 1\}$, coordinates indexed by S, to denote the set of configurations of S. Precisely, for a configuration $u \in F_2^n$, $u_{s_i} = 1$ iff u has a black state in the vertex s_i. Then for a configuration u with $u_{s_i} = 1$ for some $s_i \in S$, we can apply a move to u by changing u into $u + A\widetilde{s_i}$, where $A\widetilde{s_i}$ is the column indexed by s_i in the adjacency matrix A of S. A flipping puzzle is also called a *lit-only* σ-game in [19]. The study of flipping puzzles is related to the representation theory of Coxeter groups [8] and Lie algebras [1,2,4,5,11].

Two configurations in the flipping puzzle on S are said to be *equivalent* if one can be obtained from the other by a sequence of selected moves. Let $\mathscr{P}$ denote the partition of F_2^n according to the above equivalent relation. A general question in solving the flipping puzzle on S is to realize that for a given

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pair of configurations $u, v \in F_2^n$, whether v can be obtained from u by a sequence of selected moves or not. This can be done if $\mathscr{P}$ is completely determined.

In this paper we are mainly concerned about the class of graphs, each of which contains an induced path on $\{s_1, s_2, \ldots, s_{n-1}\}$. This class of graphs includes the simply-laced Dynkin diagrams and simply-laced extended Dynkin diagrams with exceptions $\widetilde{D}_n$ and $\widetilde{E}_6$. In each case of such graphs we determine $\mathscr{P}$.

For $u \in F_2^n$ let

$$w(u) := |\{s_i \in S \mid u_{s_i} = 1\}|$$

denote the Hamming weight of u, and for an orbit $O \in \mathscr{P}$,

$$w(O) := \min\{w(u) \mid u \in O\}$$

is called the *weight* of the orbit O. The number

$$M(S) := \max\{w(O) \mid O \in \mathscr{P}\}$$

is called the *maximum-orbit-weight* of the graph S. A consequence of our result on $\mathscr{P}$ we find $M(S) \leq 2$ and we give a necessary and sufficient condition for $M(S) = 1$. We also determine the cardinality of $\mathscr{P}$. A summary of our results is given in a table of Section 7. Besides these results, a byproduct is Theorem 3.9.

If S is a tree with ℓ leaves, Wang, Wu [19] and Wu, Chang [20] independently prove $M(S) \leq \lceil \ell/2 \rceil$. For each case of Dynkin diagrams and extended Dynkin diagrams, $\mathscr{P}$ is completely determined by Chuah and Hu [4,5]. The study of flipping puzzles is related to a rich research subject called "groups generated by transvections". We will provide this connection in Appendix.

2. Matrices representing the puzzle

Let S be a simple connected graph with n vertices. Let F_2 denote the 2-element finite field with addition identity 0 and multiplication identity 1, and let F_2^n denote the set of n-dimensional column vectors over F_2 indexed by S. We shall embed the graph S in F_2^n canonically. For $s \in S$, let $\widetilde{s}$ denote the characteristic vector of s in F_2^n; that is $\widetilde{s} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^t$, where 1 is in the position corresponding to s. The set $\{\widetilde{s} \mid s \in S\}$ is called the *standard basis* of F_2^n. In this setting, for $T \subseteq S$ the vector

$$\sum_{s\in T} \widetilde{s}$$

represents the configuration with black states in T in the flipping puzzle on S as stated in the introduction. We shall assign each move as an $n \times n$ matrix that acts on F_2^n by left multiplication. Let $\mathrm{Mat}_n(F_2)$ denote the set of $n \times n$ matrices over F_2 with rows and columns indexed by S.

Definition 2.1. For $s \in S$, we associate a matrix $\mathbf{s} \in \mathrm{Mat}_n(F_2)$, denoted by the bold type of s, as

$$\mathbf{s}_{ab} = \begin{cases} 1, & \text{if } a = b, \text{ or } b = s \text{ and } ab \in R; \\ 0, & \text{else,} \end{cases}$$

where $a, b \in S$ and R is the edge set of S. The matrix $\mathbf{s}$ is called the *flipping move* associated with vertex s.

It is easy to check that for $s, b \in S$,

$$\mathbf{s}\widetilde{b} = \begin{cases} \widetilde{b}, & \text{if } b \neq s; \\ \widetilde{b} + \sum\limits_{ab\in R} \widetilde{a} & \text{if } b = s. \end{cases}$$

Hence if a configuration $u \in F_2^n$ with $u_s = 1$ then $\mathbf{s}u$ is the new configuration after the move to select the vertex s. Note that if $u_s = 0$, we have $\mathbf{s}u = u$, so we can view the action of $\mathbf{s}$ on u as a *feigning move* on u which is not originally defined as a move in the flipping puzzle. Note that $\mathbf{s}$ is an involution and hence is invertible for $s \in S$.

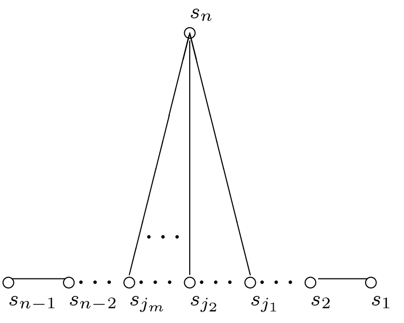

Fig. 1. The graph S.

Definition 2.2. Let $\mathbf{W}$ denote the subgroup of $\mathrm{GL}_n(F_2)$ generated by the set $\{\mathbf{s} \mid s \in S\}$ of flipping moves. $\mathbf{W}$ is called the *flipping group* of S.

The flipping groups of simply-laced Dynkin diagrams are studied in [8]. The flipping group of the line graph of a tree with n vertices is isomorphic to the symmetric group S_n on n elements if $n \geq 3$ [21]. However, we do not need the information of the flipping group $\mathbf{W}$ of S in this paper.

3. The sets Π, Π_0 and Π_1

For the remaining of the paper, the following assumption is assumed.

Assumption 3.1. Let S be a simple connected graph with n vertices $s_1, s_2, \ldots, s_n$, and suppose that the sequence $s_1, s_2, \ldots, s_{n-1}$ is an induced path, among them, $s_{j_1}, s_{j_2}, \ldots, s_{j_m}$ the neighbors of s_n, where $1 \leq j_1 < j_2 < \cdots < j_m \leq n-1$. See Fig. 1.

In the remaining of this paper, we always assume $n \geq 2$ and set

$$\overline{1} = \widetilde{s}_1, \overline{i+1} = \mathbf{s_i s_{i-1}} \cdots \mathbf{s_1} \overline{1} \quad \text{for } 1 \leq i \leq n-1. \tag{3.1}$$

Set

$$\Pi = \{\overline{1}, \overline{2}, \ldots, \overline{n}\}, \tag{3.2}$$

$$\Pi_0 = \{\overline{i} \in \Pi \mid < \overline{i}, \widetilde{s}_n >= 0\}, \tag{3.3}$$

$$\Pi_1 = \Pi - \Pi_0, \tag{3.4}$$

where $\langle , \rangle$ is the dot product of vectors. From (3.1) and the construction,

$$\Pi_0 = \{\overline{i} \mid \overline{i} = \widetilde{s}_{i-1} + \widetilde{s}_i,\ 1 \leq i \leq n-1 \text{ or } \overline{i} = \widetilde{s}_{n-1}\}, \tag{3.5}$$

$$\Pi_1 = \{\overline{i} \mid \overline{i} = \widetilde{s}_{i-1} + \widetilde{s}_i + \widetilde{s}_n,\ 1 \leq i \leq n-1 \text{ or } \overline{i} = \widetilde{s}_{n-1} + \widetilde{s}_n\}, \tag{3.6}$$

where $\widetilde{s}_0 = 0$. Note that $1 \leq |\Pi_0|, |\Pi_1| \leq n-1$ and $|\Pi_0| + |\Pi_1| = n$. Precisely,

$$\Pi_0 = \{\overline{i} \in \Pi \mid i \in (0, j_1] \cup (j_2, j_3] \cup \cdots \cup (j_{2k}, j_{2k+1}]\} \tag{3.7}$$

$$\Pi_1 = \{\overline{i} \in \Pi \mid i \in (j_1, j_2] \cup (j_3, j_4] \cup \cdots \cup (j_{2k-1}, j_{2k}]\} \tag{3.8}$$

where $k = \lceil \frac{m}{2} \rceil$, $j_t := n$ if $t > m$ and $(a, b] = \{x \mid x \in \mathbb{Z}, a < x \leq b\}$. In particular we have the following proposition.

Proposition 3.2.

$$|\Pi_1| = \sum_{k=1}^{\lceil \frac{m}{2} \rceil} j_{2k} - j_{2k-1}. \quad \square$$

From (3.5) and (3.6), we immediately have the following lemma.

Lemma 3.3. *For* $1 \le i \le n-1$,

$$\bar{1}+\bar{2}+\cdots+\bar{i} = \begin{cases} \widetilde{s}_i + \widetilde{s}_n, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is odd;} \\ \widetilde{s}_i, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is even,} \end{cases}$$

and

$$\bar{1}+\bar{2}+\cdots+\bar{n} = \begin{cases} \widetilde{s}_n, & \text{if } |\Pi_1| \text{ is odd;} \\ 0, & \text{if } |\Pi_1| \text{ is even,} \end{cases}$$

where $[\bar{i}] := \{\bar{1}, \bar{2}, \ldots, \bar{i}\}$. □

From Lemma 3.3 and (3.7) we have the following lemma.

Lemma 3.4. $\sum_{\bar{i} \in \Pi_0} \bar{i} = \sum_{k=1}^{m} \widetilde{s}_{j_k}$. □

From (3.1) we have the following lemma.

Lemma 3.5. $\mathbf{s_i}\bar{i} = \overline{i+1}$, $\mathbf{s_i}\overline{i+1} = \bar{i}$ *and* $\mathbf{s_i}$ *fixes other vectors in* $\Pi - \{\bar{i}, \overline{i+1}\}$ *for* $1 \le i \le n-1$. □

From Lemma 3.5, $\mathbf{s_i}$ acts on Π as the transposition $(\bar{i}, \overline{i+1})$ in the symmetric group S_n of Π for $1 \le i \le n-1$. Let $\mathbf{W}$ denote the flipping group of S. By a $\mathbf{W}$-*submodule* of F_2^n we mean a subspace U of F_2^n such that $\mathbf{W}U \subseteq U$.

Corollary 3.6. *The subspace U spanned by the vectors in Π is a* $\mathbf{W}$-*submodule of* F_2^n.

Proof. From Lemma 3.5, U is closed under the action of $\mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_{n-1}}$. Note that for $\bar{i} \in \Pi$ we have

$$\mathbf{s_n}\bar{i} = \begin{cases} \bar{i}, & \text{if } \bar{i} \in \Pi_0; \\ \bar{i} + \sum_{\bar{j} \in \Pi_0} \bar{j}, & \text{if } \bar{i} \in \Pi_1 \end{cases}$$
$$\in U$$

by Lemma 3.4. □

Proposition 3.7. *The subspace U in Corollary 3.6 has the basis*

$$\begin{cases} \Pi, & \text{if } |\Pi_1| \text{ is odd;} \\ \Pi - \{\bar{j}\}, & \text{if } |\Pi_1| \text{ is even} \end{cases}$$

for any $\bar{j} \in \Pi$. *Moreover* $\widetilde{s}_n \notin U$ *if* $|\Pi_1|$ *is even.*

Proof. By Lemma 3.3, $\bar{1}, \bar{2}, \ldots, \overline{n-1}$ are linearly independent and hence U has dimension at least $n-1$. Since $\widetilde{s}_n \notin \operatorname{Span}\{\bar{1}, \bar{2}, \ldots, \overline{n-1}\}$, the proposition follows from the second case of Lemma 3.3. □

Let $\mathbf{W}_P$ denote the subgroup of $\mathbf{W}$ generated by $\mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_{n-1}}$. From Lemma 3.5, Proposition 3.7 and the fact $G\widetilde{s}_n = \widetilde{s}_n$ for $G \in \mathbf{W}_P$, we have the following corollary.

Corollary 3.8. *The subgroup* $\mathbf{W}_P$ *of* $\mathbf{W}$ *is isomorphic to the symmetric group* S_n *on* Π. □

Let S' be another graph satisfying Assumption 3.1, $\mathbf{s'_n}$ be the corresponding matrix in Definition 2.1 and Π', Π'_0, Π'_1 be the corresponding sets of vectors in (3.2)–(3.4). For this moment we suppose $|\Pi_1| = |\Pi'_1|$. Let $f : \Pi \cup \{\widetilde{s}_n\} \to \Pi' \cup \{\widetilde{s}'_n\}$ be a bijection such that $f(\widetilde{s}_n) = \widetilde{s}'_n$ and $f(\Pi_1) = \Pi'_1$. Then

$$\mathbf{s'_n} f(\widetilde{s}_n) = f(\widetilde{s}_n) + \sum_{\bar{j} \in \Pi_0} f(\bar{j})$$

and

$$\mathbf{s}'_{\mathbf{n}} f(\bar{i}) = \begin{cases} f(\bar{i}), & \text{if } \bar{i} \in \Pi_0; \\ f(\bar{i}) + \sum_{\bar{j} \in \Pi_0} f(\bar{j}), & \text{if } \bar{i} \notin \Pi_0 \end{cases}$$

are corresponding to the way that $\mathbf{s_n}$ acts on $\Pi \cup \{\widetilde{s}_n\}$. From Corollary 3.8 and the above arguments we have the following theorem.

Theorem 3.9. $\mathbf{W}$ *is unique up to isomorphism among all the graphs satisfying Assumption 3.1 with a given cardinality* $|\Pi_1|$ *computed from (3.2).* □

The flipping group $\mathbf{W}$ of a simply-laced Dynkin diagram S is isomorphic to the quotient group $W/Z(W)$ of the Coxeter group W of S by its center $Z(W)$ [8], and the study of Coxeter groups W is notoriously interesting. With this in mind, one might expect the flipping groups are very different on different graphs. Theorem 3.9 is surprising since up to isomorphism the number of flipping groups is at most $n-1$, which is much less than the number of graphs satisfying Assumption 3.1.

4. Simple basis Δ of F_2^n

To better describe the orbits in $\mathcal{P}$ later, we need to choose a new basis of F_2^n. Set

$$\Delta := \begin{cases} \Pi, & \text{if } |\Pi_1| \text{ is odd;} \\ \Pi \cup \{\overline{n+1}\} - \{\bar{n}\}, & \text{if } |\Pi_1| \text{ is even,} \end{cases}$$

where $\overline{n+1} := \widetilde{s}_n$. With referring to Proposition 3.7, Δ is a basis of F_2^n. To distinguish from the *standard basis* $\{\widetilde{s}_1, \widetilde{s}_2, \dots, \widetilde{s}_n\}$ of F_2^n, we refer Δ to the *simple basis* of F_2^n. For each vector $u \in F_2^n$, u can be written as a linear combination of elements in Δ, so let $\Delta(u)$ be the subset of Δ such that

$$u = \sum_{\bar{i} \in \Delta(u)} \bar{i},$$

set $sw(u) := |\Delta(u)|$, and we refer $sw(u)$ to be the *simple weight* of u. Note that for $1 \le i \le n-1$, the vector $\bar{1} + \bar{2} + \cdots + \bar{i}$ has simple weight i, but has weight

$$w(\bar{1} + \bar{2} + \cdots + \bar{i}) = \begin{cases} 1, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is even;} \\ 2, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is odd} \end{cases} \tag{4.1}$$

by Lemma 3.3.

The following notation will be used in the sequel. For $V \subseteq F_2^n$ and $T \subseteq \{0, 1, \dots, n\}$,

$$V_T := \{u \in V \mid sw(u) \in T\},$$

and for shortness $V_{t_1,t_2,\dots,t_i} := V_{\{t_1,t_2,\dots,t_i\}}$. Let *odd* be the subset of $\{1, 2, \dots, n\}$ consisting of odd integers.

5. The case $|\Pi_1|$ is odd

In this section we assume $|\Pi_1|$ to be odd and the counter part is treated in the next section. Note that $\Delta = \{\bar{1}, \bar{2}, \dots, \bar{n}\}$ is a basis of $U = F_2^n$ in this case. From Lemma 3.3, for $1 \le i \le n-1$,

$$\widetilde{s}_i = \begin{cases} \bar{1} + \bar{2} + \cdots + \bar{i}, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is even;} \\ \overline{i+1} + \overline{i+2} + \cdots + \bar{n}, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is odd,} \end{cases}$$

and

$$\widetilde{s}_n = \bar{1} + \bar{2} + \cdots + \bar{n}.$$

Hence, for $1 \le i \le n-1$,

$$sw(\widetilde{s_i}) = \begin{cases} i, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is even;} \\ n-i, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is odd,} \end{cases}$$

and $sw(\widetilde{s_n}) = n$. In other words, there exists a vector with simple weight i and weight 1 if and only if $|[\bar{i}] \cap \Pi_1|$ is even, $i = n$ or $|[\overline{n-i}] \cap \Pi_1|$ is odd. Set

$$I := \{i \in [n] \mid |[\bar{i}] \cap \Pi_1| \text{ is even, } i = n \text{ or } |[\overline{n-i}] \cap \Pi_1| \text{ is odd}\},$$

where $[n] := \{1, 2, \ldots, n\}$. Note that $w(U_i) \le 2$ by Lemma 3.3, and

$$w(U_i) = 1 \qquad \text{if and only if} \qquad i \in I \tag{5.1}$$

for $1 \le i \le n$.

Lemma 5.1. *For $u \in F_2^n$, we have*

$$\mathbf{s_n} u = \begin{cases} u, & \text{if } |\Delta(u) \cap \Pi_1| \text{ is even;} \\ u + \sum_{\bar{i} \in \Pi_0} \bar{i}, & \text{else.} \end{cases}$$

In particular,

$$sw(\mathbf{s_n} u) = \begin{cases} sw(u), & \text{if } |\Delta(u) \cap \Pi_1| \text{ is even;} \\ n - |\Pi_1| + 2k - sw(u), & \text{else,} \end{cases}$$

where $k = |\Pi_1 \cap \Delta(u)|$.

Proof. If $|\Delta(u) \cap \Pi_1|$ is even then $\langle u, \widetilde{s_n} \rangle = 0$ and $\mathbf{s_n} u = u$ by construction. If $|\Delta(u) \cap \Pi_1|$ is odd, then

$$\begin{aligned} \mathbf{s_n} u &= u + \sum_{k=1}^{m} \widetilde{s_{j_k}} \\ &= u + \sum_{\bar{i} \in \Pi_0} \bar{i} \end{aligned}$$

by Lemma 3.4, and $sw(\mathbf{s_n} u) = |\Delta(u) \cap \Pi_1| + (|\Pi_0| - |\Delta(u) \cap \Pi_0|) = n - |\Pi_1| + 2k - sw(u)$. □

The following lemma follows from Corollary 3.8 and $\Delta = \Pi$.

Lemma 5.2. *The nontrivial orbits of F_2^n under $\mathbf{W}_P$ are U_i for $1 \le i \le n$.* □

The following theorem solves the flipping puzzle when $3 \le |\Pi_1| \le n-3$.

Theorem 5.3. *Suppose $3 \le |\Pi_1| \le n-3$. Then the nontrivial orbits of F_2^n under $\mathbf{W}$ are $U_{A_1}, U_{A_2}, U_{A_3}, U_{A_4}$, where*

$$A_i := \{j \in [n] \mid j \equiv i, n + |\Pi_1| - i \pmod 4\}.$$

In particular the number of orbits (including the trivial one) of F_2^n under $\mathbf{W}$ is

$$|\mathcal{P}| = \begin{cases} 3, & \text{if } n \text{ is even;} \\ 4, & \text{else,} \end{cases}$$

and the maximum-orbit-weight $M(S)$ of S is

$$M(S) = \begin{cases} 1, & \text{if } A_i \cap I \ne \emptyset \text{ for all } i; \\ 2, & \text{else.} \end{cases}$$

Proof. Fix an integer $1 \le i \le n$. By Lemma 5.2, U_i is contained in an orbit of F_2^n under $\mathbf{W}$. To put two orbits under $\mathbf{W}_P$ to an orbit under $\mathbf{W}$ is only by the action of $\mathbf{s_n}$. Hence U_i and $U_{n-|\Pi_1|+2k-i}$ are in the same orbit by Lemma 5.1, where k runs through possible odd integers $|\Pi_1 \cap \Delta(u)|$ for $u \in U_i$. In fact k is any odd number that satisfies $k \le |\Pi_1|$ and $0 \le i - k \le |\Pi_0|$; equivalently

$$\max\{1, i + |\Pi_1| - n\} \le k \le \min\{|\Pi_1|, i\}. \tag{5.2}$$

Such an odd integer k exists for any $1 \le i \le n$, and note that

$$n - |\Pi_1| + 2k - i \equiv n + |\Pi_1| - i \pmod 4$$

since k and $|\Pi_1|$ are odd integers. To see the orbits as stated in the theorem, it remains to show that U_i and U_{i+4} are in the same orbit under **W** for $1 \le i \le n-4$. Set k to be the least odd integer greater than or equal to $\max\{1, i + |\Pi_1| - n + 2\}$. For this k, (5.2) holds and then U_i and $U_{n-|\Pi_1|+2k-i}$ are in the same orbit. Here we use the assumption $|\Pi_1| \le n-3$ to guarantee the existence of such k. Note that if we use $(n - |\Pi_1| + 2k - i, k+2)$ to replace (i, k) in (5.2), we have

$$\max\{1, 2k - i\} \le k + 2 \le \min\{|\Pi_1|, n - |\Pi_1| + 2k - i\}. \tag{5.3}$$

The above k and the assumption $3 \le |\Pi_1|$ guarantee the Eq. (5.3). Since $n - |\Pi_1| + 2(k+2) - (n - |\Pi_1| + 2k - i) = i + 4$, we have $U_{n-|\Pi_1|+2k-i}$ and U_{i+4} in the same orbit. Putting these together, U_i and U_{i+4} are in the same orbit. The remaining statements of the theorem are obtained from the orbits description immediately and by using (5.1). □

The following theorem does the remaining cases.

Theorem 5.4. *Suppose $|\Pi_1| = 1, n-2$ or $n-1$. Then the nontrivial orbits of F_2^n under* **W** *are*

$$\begin{cases} U_{i,n+1-i}, & \text{if } |\Pi_1| = 1; \\ U_{odd}, U_{2j}, & \text{if } |\Pi_1| = n-2; \\ U_{2i-1,2i}, & \text{if } |\Pi_1| = n-1 \end{cases}$$

for $1 \le i \le \lceil n/2 \rceil$ and $1 \le j \le (n-1)/2$. In particular the number of orbits (including the trivial one) of F_2^n under **W** *is*

$$|\mathcal{P}| = \begin{cases} \lceil (n+2)/2 \rceil, & \text{if } |\Pi_1| = 1; \\ (n+3)/2, & \text{if } |\Pi_1| = n-2; \\ (n+2)/2, & \text{if } |\Pi_1| = n-1, \end{cases}$$

and the maximum-orbit-weight $M(S)$ of S is at most 2. Moreover $M(S) = 1$ if and only if

$$\begin{cases} \{i, n+1-i\} \cap I \ne \emptyset \quad \text{for all } 1 \le i \le \lceil n/2 \rceil, & \text{if } |\Pi_1| = 1; \\ odd \cap I \ne \emptyset \text{ and } U_{2j} \cap I \ne \emptyset \quad \text{for all } 1 \le j \le \lfloor n/2 \rfloor, & \text{if } |\Pi_1| = n-2; \\ \{2i-1, 2i\} \cap I \ne \emptyset \quad \text{for all } 1 \le i \le \lceil n/2 \rceil, & \text{if } |\Pi_1| = n-1. \end{cases}$$

Proof. As the proof in Theorem 5.3, U_i and $U_{n-|\Pi_1|+2k-i}$ are in the same orbit under **W**, where k needs to satisfy (5.2). In the case $|\Pi_1| = 1$, $k = 1$ is the only possible choice and hence U_{n+1-i} is the only orbit under $\mathbf{W}_P$ been put together with U_i to become an orbit under **W**. In the case $|\Pi_1| = n-2$, we have $k = i-2$ or i if i is odd; $k = i-1$ if i is even. In the case $|\Pi_1| = n-1$, we have $k = i$ if i is odd; $k = i-1$ if i is even. In each of the remaining the proof follows similarly. □

Example 5.5. Let S be an odd cycle of length n, i.e. n is odd, $m = 2$, $j_1 = 1$ and $j_2 = n-1$. Then $\Pi_0 = \{\overline{1}, \overline{n}\}$ and $\Pi_1 = \{\overline{2}, \overline{3}, \ldots, \overline{n-1}\}$. Note that $|\Pi_1| = n-2$ is odd, and $I = \{1, 3, \ldots, n\}$. Hence Theorem 5.4 applies. We have

$$\mathcal{P} = \{U_{odd}, U_0, U_2, U_4, \ldots, U_{n-1}\}.$$

In particular, $|\mathcal{P}| = (n+3)/2$, and $M(S) = 2$.

6. The case $|\Pi_1|$ is even

In this section we assume $|\Pi_1|$ to be even. Recall that in this case $\Delta = \Pi \cup \{\overline{n+1}\} - \{\overline{n}\}$ and $\Delta - \{\overline{n+1}\}$ are bases of F_2^n and U respectively. Recall that

$$\overline{1} + \overline{2} + \cdots + \overline{n} = 0. \tag{6.1}$$

Let $\overline{U} := F_2^n - U$, and note that $\overline{U} = \overline{n+1} + U$, $\overline{U}_1 = \{\overline{n+1}\}$ and $U_n = \emptyset$. From Lemma 3.3, for $1 \le i \le n-1$,

$$\widetilde{s}_i = \begin{cases} \bar{1} + \bar{2} + \cdots + \bar{i} \in U, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is even;} \\ \bar{1} + \bar{2} + \cdots + \bar{i} + \overline{n+1} \in \overline{U}, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is odd,} \end{cases}$$

and

$$\widetilde{s}_n = \overline{n+1} \in \overline{U}.$$

Moreover, for $1 \le i \le n-1$,

$$sw(\widetilde{s}_i) = \begin{cases} i, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is even;} \\ i+1, & \text{if } |[\bar{i}] \cap \Pi_1| \text{ is odd,} \end{cases}$$

and $sw(\widetilde{s}_n) = 1$. In other words, there exists a vector in U with simple weight i and weight 1 if and only if $|[\bar{i}] \cap \Pi_1|$ is even; there exists a vector in $\overline{U}$ with simple weight i and weight 1 if and only if $|[\overline{i-1}] \cap \Pi_1|$ is odd or $i = 1$. Set

$$I = \{i \in [n-1] \mid |[\bar{i}] \cap \Pi_1| \text{ is even}\}$$

and

$$J = \{i \in [n] \mid |[\overline{i-1}] \cap \Pi_1| \text{ is odd or } i = 1\}.$$

Note that $w(U_i), w(\overline{U}_j) \le 2$, and

$$\begin{aligned} w(U_i) = 1 \quad &\text{if and only if} \quad i \in I; \\ w(\overline{U}_j) = 1 \quad &\text{if and only if} \quad j \in J \end{aligned} \tag{6.2}$$

for $1 \le i \le n-1$, $1 \le j \le n$.

Lemma 6.1. *For $u \in F_2^n$, let $k = |\Pi_1 \cap \Delta(u)|$. Then the following* (i), (ii) *hold*

(i) *For $u \in U$, we have*

$$\mathbf{s_n} u = \begin{cases} u, & \text{if } |\Delta(u) \cap \Pi_1| \text{ is even;} \\ u + \sum_{\bar{i} \in \Pi_0} \bar{i}, & \text{else.} \end{cases}$$

In particular, the simple weight $sw(\mathbf{s_n} u)$ of $\mathbf{s_n} u$ is

$$\begin{cases} sw(u), & \text{if } |\Delta(u) \cap \Pi_1| \text{ is even;} \\ n - |\Pi_1| + 2k - sw(u), & \text{if } |\Delta(u) \cap \Pi_1| \text{ is odd and } \bar{n} \in \Pi_1; \\ sw(u) + |\Pi_1| - 2k, & \text{else.} \end{cases}$$

(ii) *For $u \in \overline{U}$, we have*

$$\mathbf{s_n} u = \begin{cases} u, & \text{if } |\Delta(u) \cap \Pi_1| \text{ is odd;} \\ u + \sum_{\bar{i} \in \Pi_0} \bar{i}, & \text{else.} \end{cases}$$

In particular, the simple weight $sw(\mathbf{s_n} u)$ of $\mathbf{s_n} u$ is

$$\begin{cases} sw(u), & \text{if } |\Delta(u) \cap \Pi_1| \text{ is odd;} \\ n - |\Pi_1| + 2k + 2 - sw(u), & \text{if } |\Delta(u) \cap \Pi_1| \text{ is even and } \bar{n} \in \Pi_1; \\ sw(u) + |\Pi_1| - 2k, & \text{else.} \end{cases}$$

Proof. The proof is similar to the proof of Lemma 5.1, except that at this time since the choice of simple basis Δ is different, the action of $\mathbf{s_n}$ on a vector is a little different, and we need to use (6.1) to adjust the simple weight of a vector. $\square$

By Corollary 3.6 the orbits of F_2^n under $\mathbf{W}$ (resp. under $\mathbf{W}_P$) are divided into two parts, one in U and the other in $\overline{U}$.

Lemma 6.2. *The nontrivial orbits of F_2^n under $\mathbf{W}_P$ are $\overline{U}_1$, $\overline{U}_{i+1,n+1-i}$ and $U_{i,n-i}$ for $1 \le i \le \lfloor n/2 \rfloor$.* □

Proof. By construction, $\overline{U}_1 = \{\widetilde{s}_n\}$ is an orbit under $\mathbf{W}_P$. By Corollaries 3.6 and 3.8, U_i is contained in an orbit of F_2^n under $\mathbf{W}_P$ and $\overline{U}_i$ is contained in another one for $1 \le i \le n-1$. The Eq. (6.1) and our choice of Δ imply that U_i and U_{n-i} are in the same orbit of F_2^n under $\mathbf{W}_P$; $\overline{U}_{i+1}$ and $\overline{U}_{n+1-i}$ are in another one for $1 \le i \le n-1$. Since no other ways to put these sets together, we have the lemma. □

Theorem 6.3. *Suppose $4 \le |\Pi_1| \le n-3$. Then the nontrivial orbits of F_2^n under $\mathbf{W}$ are $U_{B_1}, U_{B_2}, U_{B_3}, U_{B_4}, \overline{U}_{C_1}, \overline{U}_{C_2}, \overline{U}_{C_3}, \overline{U}_{C_4}$, where*

$$B_i = \{j \in [n-1] \mid j \equiv i, i+|\Pi_1|-2, n-i, n-i+|\Pi_1|-2 \pmod 4\}$$

and

$$C_i = \{j \in [n] \mid j \equiv i, i+|\Pi_1|, n+2-i, n+2-i+|\Pi_1| \pmod 4\}.$$

In particular the number of orbits (including the trivial one) of F_2^n under $\mathbf{W}$ is

$$|\mathcal{P}| = \begin{cases} 6, & \text{if } n \text{ is even;} \\ 4, & \text{else,} \end{cases}$$

and the maximum-orbit-weight $M(S)$ of S is

$$M(S) = \begin{cases} 1, & \text{if } B_i \cap I \ne \emptyset \text{ and } C_i \cap J \ne \emptyset \text{ for all } i; \\ 2, & \text{else.} \end{cases}$$

Proof. Firstly we determine the orbits of U under $\mathbf{W}$. By Lemma 6.2, $U_{i,n-i}$ is contained in an orbit under $\mathbf{W}$ for $1 \le i \le n-1$. We suppose $\overline{n} \in \Pi_0$ and the case $\overline{n} \in \Pi_1$ is left to the reader. In this case U_i and $U_{i+|\Pi_1|-2k}$ are in the same orbit of F_2^n under $\mathbf{W}$ by Lemma 6.1(i), where $1 \le i+|\Pi_1|-2k \le n-1$ and k runs through possible odd integers $|\Pi_1 \cap \Delta(u)|$ for $u \in U_i$. In fact k is any odd number that satisfies $k \le |\Pi_1|-1$ and $0 \le i-k \le |\Pi_0|-1$; equivalently

$$\max\{1, i+|\Pi_1|-n+1\} \le k \le \min\{|\Pi_1|-1, i\}. \tag{6.3}$$

Such an odd k exists for any $1 \le i \le n-3$, and note that

$$i+|\Pi_1|-2k \equiv i+|\Pi_1|-2 \pmod 4.$$

To determine the orbits of U under $\mathbf{W}$ in this case, it remains to show that U_i and U_{i+4} are in the same orbit under $\mathbf{W}$ for $1 \le i \le \lfloor n/2 \rfloor$. Suppose $4 \le |\Pi_1| \le 6$. Set $k=1$ to conclude U_i and U_{i+2} in an orbit if $|\Pi_1| = 4$; U_i and U_{i+4} in an orbit if $|\Pi_1| = 6$. Suppose $|\Pi_1| \ge 8$. Then $n \ge 11$ and $\lfloor n/2 \rfloor \le n-6$. Set k to be the least odd integer greater than or equal to $\max\{1, i+|\Pi_1|-n+3\}$. For this k, (6.3) holds and then U_i and $U_{i+|\Pi_1|-2k}$ are in the same orbit. Here we use the assumption $|\Pi_1| \le n-3$. Note that if we use $(i+|\Pi_1|-2k, |\Pi_1|-k-2)$ to replace (i,k) in (6.3), we have

$$\max\{1, i+2|\Pi_1|-2k-n+1\} \le |\Pi_1|-k-2 \le \min\{|\Pi_1|-1, i+|\Pi_1|-2k\}. \tag{6.4}$$

The above k, the assumption $4 \le |\Pi_1|$ and $i \le n-6$ guarantee the Eq. (6.4). Since $(i+|\Pi_1|-2k)+|\Pi_1|-2(|\Pi_1|-k-2) = i+4$, we have $U_{i+|\Pi_1|-2k}$ and U_{i+4} in the same orbit. Putting these together, U_i and U_{i+4} are in the same orbit. Then the orbits of U under $\mathbf{W}$ are $U_{B_1}, U_{B_2}, U_{B_3}, U_{B_4}$ as in the statement.

Secondly, we determine the orbits of $\overline{U}$ under $\mathbf{W}$. Since the proof is similar to the above case, we only give a sketch. By Lemma 6.2, $\overline{U}_{i,n+2-i}$ is contained in an orbit for $2 \le i \le n$. We suppose $\overline{n} \in \Pi_1$ and leave the case $\overline{n} \in \Pi_0$ to the reader. By Lemma 6.1(ii), we have U_i and $U_{n-|\Pi_1|+2k+2-i}$ in an orbit, where $k = |\Delta(u) \cap \Pi_1|$ is an even number for some $u \in U_i$ and $1 \le i \le n-4$. From the same argument with k been replaced by $k+2$, we find $U_{n-|\Pi_1|+2k+2-i}$ and U_{i+4} in an orbit to finish the proof.

The remaining statements of the theorem are obtained from the orbits description. □

The following theorem determine the nontrivial orbits of F_2^n under $\mathbf{W}$ in the remaining cases.

Theorem 6.4. *Suppose $|\Pi_1| = 2$, $n-2$ or $n-1$. Then with referring to the notation in Theorem 6.3, the nontrivial orbits of F_2^n under $\mathbf{W}$ are*

$$\begin{cases} U_{i,n-i}, \overline{U}_{C_1}, \overline{U}_{C_2}, & \text{if } |\Pi_1| = 2; \\ U_{odd}, U_{2j,n-2j}, \overline{U}_{odd}, \overline{U}_{2t,n+2-2t}, & \text{if } |\Pi_1| = n-2; \\ U_{2j-1,2j,n-2j,n+1-2j}, \overline{U}_{2t-1,2t,n+2-2t,n+3-2t}, & \text{if } |\Pi_1| = n-1, \end{cases}$$

for $1 \le i \le \lfloor n/2 \rfloor$, $1 \le j \le \lceil (n-2)/4 \rceil$ and $1 \le t \le \lceil n/4 \rceil$. In particular the number of orbits (including the trivial one) of F_2^n under W is

$$|\mathcal{P}| = \begin{cases} (n+6)/2, & \text{if } |\Pi_1| = 2 \text{ and } n \text{ is even, or } |\Pi_1| = n-2; \\ (n+3)/2, & \text{if } |\Pi_1| = 2 \text{ and } n \text{ is odd, or } |\Pi_1| = n-1, \end{cases}$$

and the maximum-orbit-weight $M(S)$ of S is at most 2. Moreover $M(S) = 1$ if and only if

$$\{i, n-i\} \cap I \ne \emptyset \quad \text{and} \quad \overline{U}_{C_j} \cap J \ne \emptyset \quad \text{for } 1 \le j \le 2, \quad \text{if } |\Pi_1| = 2;$$

$$\begin{cases} odd \cap I \ne \emptyset, \{2j, n-2j\} \cap I \ne \emptyset \\ \text{for all } 1 \le j \le \lceil (n-2)/4 \rceil, \\ odd \cap J \ne \emptyset, \{2t, n+2-2t\} \cap J \ne \emptyset \\ \text{for all } 1 \le t \le \lceil n/4 \rceil, \end{cases} \quad \text{if } |\Pi_1| = n-2;$$

$$\begin{cases} \{2j-1, 2j, n-2j, n+1-2j\} \cap I \ne \emptyset \\ \text{for all } 1 \le j \le \lceil (n-2)/4 \rceil, \\ \{2t-1, 2t, n+2-2t, n+3-2t\} \cap J \ne \emptyset \\ \text{for all } 1 \le t \le \lceil n/4 \rceil, \end{cases} \quad \text{if } |\Pi_1| = n-1.$$

Proof. The proof is similar to the proof of Theorem 5.4 that follows from the proof of Theorem 5.3. At this time, to determine the orbits of U we check what values of odd k occur in (6.3) in each case of $|\Pi_1| \in \{2, n-2, n-1\}$. To determine the orbits of $\overline{U}$ under $\mathbf{W}$, we do similarly as in the second part of the proof of Theorem 6.3. □

Example 6.5. Let S be an even cycle of length n, i.e. n is even, $m = 2$, $j_1 = 1$ and $j_2 = n-1$. Then $\Pi_0 = \{\overline{1}, \overline{n}\}$ and $\Pi_1 = \{\overline{2}, \overline{3}, \dots, \overline{n-1}\}$. Note that $|\Pi_1| = n-2$ is even and $I = J = \{1, 3, \dots, n-1\}$. Hence Theorem 6.4 applies. We have

$$\mathcal{P} = \{U_{odd}, U_0, U_{2,n-2}, U_{4,n-4}, \dots, U_{2j,n-2j}, \overline{U}_{odd}, \overline{U}_{2,n}, \overline{U}_{4,n-2}, \dots, \overline{U}_{2t,n-2t+2}\},$$

where $j = \lceil (n-2)/4 \rceil$ and $t = \lceil n/4 \rceil$. In particular

$$|\mathcal{P}| = \lceil (n-2)/4 \rceil + \lceil n/4 \rceil + 3 = (n+6)/2,$$

and $M(S) = 2$.

7. Summary

We list the main results as follows. Let S be a connected graph with n vertices $s_1, s_2, \dots, s_n$ that contains an induced path $s_1, s_2, \dots, s_{n-1}$ of $n-1$ vertices, and s_n has neighbors $s_{j_1}, s_{j_2}, \dots, s_{j_m}$ with $1 \le j_1 < j_2 \cdots < j_m \le n-1$. Let $\widetilde{s}_1, \widetilde{s}_2, \dots, \widetilde{s}_n$ denote the characteristic vectors of F_2^n and let $\mathbf{s_1}, \mathbf{s_2}, \dots, \mathbf{s_n}$ denote the flipping moves associated with $s_1, s_2, \dots, s_n$ respectively.

Set

$$\overline{1} = \widetilde{s}_1, \overline{i+1} = \mathbf{s_i s_{i-1}} \cdots \mathbf{s_1} \overline{1} \quad (1 \le i \le n-1), \quad \overline{n+1} := \widetilde{s}_n$$

and consider the following three sets

$$\Pi = \{\overline{1}, \overline{2}, \dots, \overline{n}\},$$
$$\Pi_0 = \{\overline{i} \in \Pi \mid \langle \overline{i}, \widetilde{s}_n \rangle = 0\},$$
$$\Pi_1 = \Pi - \Pi_0.$$

Table 1
The summary.

$\lvert\Pi_1\rvert$	n	Nontrivial $O \in \mathcal{P}$ (might be repeated)	$\lvert\mathcal{P}\rvert$
$3 \le \lvert\Pi_1\rvert \le n-3$, $\lvert\Pi_1\rvert$ is odd	Even	U_{A_j}	3
$3 \le \lvert\Pi_1\rvert \le n-3$, $\lvert\Pi_1\rvert$ is odd	Odd	U_{A_j}	4
$4 \le \lvert\Pi_1\rvert \le n-3$, $\lvert\Pi_1\rvert$ is even	Even	$U_{B_j}, \overline{U}_{C_j}$	6
$4 \le \lvert\Pi_1\rvert \le n-3$, $\lvert\Pi_1\rvert$ is even	Odd	$U_{B_j}, \overline{U}_{C_j}$	4
$\lvert\Pi_1\rvert = 1$		$U_{t,n+1-t}$	$\lceil (n+2)/2 \rceil$
$\lvert\Pi_1\rvert = 2$	Even	$U_{i,n-i}, \overline{U}_{C_1}, \overline{U}_{C_2}$	$(n+6)/2$
$\lvert\Pi_1\rvert = 2$	Odd	$U_{i,n-i}, \overline{U}_{C_1}, \overline{U}_{C_2}$	$(n+3)/2$
$\lvert\Pi_1\rvert = n-2$, $\lvert\Pi_1\rvert$ is odd	Odd	U_{odd}, U_{2i}	$(n+3)/2$
$\lvert\Pi_1\rvert = n-2$, $\lvert\Pi_1\rvert$ is even	Even	$U_{odd}, U_{2h,n-2h}$, $\overline{U}_{odd}, \overline{U}_{2g,n+2-2g}$	$(n+6)/2$
$\lvert\Pi_1\rvert = n-1$, $\lvert\Pi_1\rvert$ is odd	Even	$U_{2t-1,2t}$	$(n+2)/2$
$\lvert\Pi_1\rvert = n-1$, $\lvert\Pi_1\rvert$ is even	Odd	$U_{2h-1,2h,n-2h,,n+1-2h}$, $\overline{U}_{2g-1,2gn+2-2g,n+3-2g}$	$(n+3)/2$

where $1 \le j \le 4$, $1 \le t \le \lceil n/2 \rceil$, $1 \le i \le \lfloor n/2 \rfloor$, $1 \le h \le \lfloor (n-2)/4 \rfloor$, $1 \le g \le \lceil n/4 \rceil$.

By using the graph structure we can compute the following value

$$|\Pi_1| = \sum_{k=1}^{\lceil \frac{m}{2} \rceil} j_{2k} - j_{2k-1}$$

as shown in Proposition 3.2. Let

$$\Delta := \begin{cases} \Pi, & \text{if } |\Pi_1| \text{ is odd;} \\ \Pi \cup \{\overline{n+1}\} - \{\overline{n}\}, & \text{if } |\Pi_1| \text{ is even} \end{cases}$$

be the simple basis of F_2^n as shown in the beginning of Section 4. For a vector $u \in F_2^n$ let $sw(u)$ denote the simple weight of u, i.e. the number nonzero terms in writing u as a linear combination of elements in Δ. Let U be the subspace spanned by the vectors in Π. For $V \subseteq F_2^n$ and $T \subseteq \{0, 1, \ldots, n\}$,

$$V_T := \{u \in V \mid sw(u) \in T\},$$

and for shortness $V_{t_1,t_2,\ldots,t_i} := V_{\{t_1,t_2,\ldots,t_i\}}$. Let *odd* be the subset of $\{1, 2, \ldots, n\}$ consisting of odd integers. Set

$$A_i = \{j \in [n] \mid j \equiv i, n + |\Pi_1| - i \pmod 4\},$$
$$B_i = \{j \in [n-1] \mid j \equiv i, i + |\Pi_1| - 2, n - i, n - i + |\Pi_1| - 2 \pmod 4\},$$
$$C_i = \{j \in [n] \mid j \equiv i, i + |\Pi_1|, n + 2 - i, n + 2 - i + |\Pi_1| \pmod 4\}.$$

Let $\mathcal{P}$ denote the set of orbits of the flipping puzzle on S. Then the set $\mathcal{P}$ and its cardinality $|\mathcal{P}|$ are given in Table 1 according to the different cases of the pair $(|\Pi_1|, n)$ in the first two columns.

Appendix

We are indebted to a referee for the information in this section. Let S be a simple connected graph with n vertices and adjacency matrix A. The adjacency matrix defines an alternating form $\langle , \rangle_A$ on F_2^n by

$$\langle u, v \rangle_A = u^{t} A v$$

and a quadratic form q on F_2^n that satisfies $q(\widetilde{s}) = 1$ and

$$q(u + v) = q(u) + q(v) + \langle u, v \rangle_A$$

for all vertices $s \in S$ and $u, v \in F_2^n$. For a vertex $s \in S$, the associating matrix $\mathbf{s}$ in Definition 2.1 satisfies

$$\mathbf{s}A\mathbf{s}^{\mathrm{t}} = A. \tag{A.1}$$

Hence $\mathbf{s}^{\mathrm{t}}$ is an element of the symplectic group $S(n, F_2)$ [18, p. 69], and therefore the transpose group $\mathbf{W}^{\mathrm{t}}$ of the flipping group $\mathbf{W}$ of S is a subgroup of $S(n, F_2)$. Moreover $\mathbf{W}^{\mathrm{t}}$ preserves q in the sense that $q(\mathbf{w}^{\mathrm{t}}u) = q(u)$ for any $\mathbf{w}^{\mathrm{t}} \in \mathbf{W}^{\mathrm{t}}$ and any $u \in F_2^n$. Note that from Definition 2.1,

$$\mathbf{s}^{\mathrm{t}}u = u + \langle \tilde{s}, u\rangle_A \tilde{s} \tag{A.2}$$

for $s \in S$ and $u \in F_2^n$. Such an $\mathbf{s}^{\mathrm{t}}$ is called a *transvection* in the literature. The study of arbitrary groups generated by transvections was largely instituted by McLaughlin [12,13]. Hamelink's work on Lie algebras led to a question about groups generated by symplectic transvections over F_2 [7]. Hamelink's question was answered by Seidel, as reported and generalized by Shult in his Breukelen lectures [15,17]. Graphical notation is implicit in this earlier work and explicit in that of Brown and Humphries [3,10]. A survey of related work, a brief discussion of Humphries results, and a discussion of the isomorphism types of groups occurring are given by Hall [6]. More recent results are in [14,16].

Let $\mathcal{P}'$ denote the set of orbits under the action of $\mathbf{W}^{\mathrm{t}}$ on F_2^n. Several of the papers discussed above (or referenced therein) also focus on and discuss orbit lengths for $\mathcal{P}'$. As before let $\mathcal{P}$ be the set of orbits under the action of $\mathbf{W}$ on F_2^n (the set of orbits of the flipping puzzle on S). By (A.1) and using $\mathbf{s}^2 = I$, the map

$$O \to AO$$

is a map from $\mathcal{P}'$ into $\mathcal{P}$, where $AO = \{Au \mid u \in O\}$. In particular if A is nonsingular over F_2, this map is a bijection. But when A is singular, the orbit structures can presumably differ. See [9] for more connections between $\mathcal{P}'$ and $\mathcal{P}$.

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