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Singular Limits of the Klein–Gordon Equation

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Dedicated to Professor Yuh-Jia Lee on his sixtieth birthday.

Abstract

We establish the singular limits, including semiclassical, nonrelativistic and nonrelativistic-semiclassical limits, of the Cauchy problem for the modulated defocusing nonlinear Klein–Gordon equation. For the semiclassical limit, $\hbar \to 0$, we show that the limit wave function of the modulated defocusing cubic nonlinear Klein–Gordon equation solves the relativistic wave map and the associated phase function satisfies a linear relativistic wave equation. The nonrelativistic limit, $c \to \infty$, of the modulated defocusing nonlinear Klein–Gordon equation is the defocusing nonlinear Schrödinger equation. The nonrelativistic-semiclassical limit, $\hbar \to 0, c = \hbar^{-\alpha} \to \infty$ for some $\alpha > 0$, of the modulated defocusing cubic nonlinear Klein–Gordon equation is the classical wave map for the limit wave function and a typical linear wave equation for the associated phase function.

1. Introduction

One of the fundamental partial differential equations is the nonlinear Klein–Gordon equation

$$\frac{\hbar^2}{2mc^2}\partial_t^2\Psi - \frac{\hbar^2}{2m}\Delta\Psi + \frac{mc^2}{2}\Psi + V'(|\Psi|^2)\Psi = 0, \tag{1.1}$$

where m is mass, c is the speed of light and $\hbar$ is the Planck constant. Here $\Psi(x,t)$ is a complex-valued vector field over a spatial domain $\Omega \subset \mathbb{R}^n$, V' is the first derivative of a twice differentiable nonlinear real-valued function over $\mathbb{R}^+$. Thus V' is the potential energy and V is the potential energy density of the fields. Since the Planck constant $\hbar$ has dimension of action $[\hbar] = [\text{energy}] \times [\text{time}] = [\text{action}]$,

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it is easy to check that (1.1) is dimensionally balanced. When the potential energy $V' = 0$, (1.1) is the typical Klein–Gordon equation for a free particle and its dispersion relation has the form $E^2 = p^2c^2 + m^2c^4$, p being the momentum, hence $E = \pm mc^2$ for a resting particle. In quantum field theory the state of a particle with negative energy is interpreted as the state of an antiparticle possessing positive energy, but opposite electric charge.

The Klein–Gordon equation for the complex scalar field is the relativistic version of the Schrödinger equation, which is used to describe spinless particles. It was first considered as a quantum wave equation by Schrödinger in his search for an equation describing de Broglie waves. However, this equation was named after the physicists Oskar Klein and Walter Gordon, who in 1927 proposed that it describes relativistic electrons. Although it turned out that the Dirac equation describes the spinning electron, the Klein–Gordon equation correctly describes the spinless pion. The reader is referred to [26] for a general introduction to nonlinear wave equations and [22] for the physical background. It is not straightforward to identify the nonrelativistic limit, that is, $c \to \infty$, from Equation (1.1). To this end we notice that mc^2t and the Planck constant $\hbar$ have the same dimension of action, $[mc^2t] = [\hbar] = [\text{action}]$, and we may consider the modulated wave function [18]

$$\psi(x,t) = \Psi(x,t)\exp(imc^2t/\hbar), \tag{1.2}$$

where the factor $\exp(imc^2t/\hbar)$ describes the oscillations of the wave function, then ψ satisfies the modulated nonlinear Klein–Gordon equation

$$i\hbar\partial_t\psi - \frac{\hbar^2}{2mc^2}\partial_t^2\psi + \frac{\hbar^2}{2m}\Delta\psi - V'(|\psi|^2)\psi = 0. \tag{1.3}$$

The second term of Equation (1.3) shows the relativistic effect, which is small when considering the light speed c to be large, while the Planck constant $\hbar$ is kept fixed. Thus, in the limit as $c \to \infty$, Equation (1.3) goes over into the defocusing nonlinear Schrödinger equation

$$i\hbar\partial_t\psi + \frac{\hbar^2}{2m}\Delta\psi - V'(|\psi|^2)\psi = 0. \tag{1.4}$$

However, the semiclassical limit, that is $\hbar \to 0$, is not clear from (1.3). One way to tackle this problem is the hydrodynamical formulation, as had been done for the Schrödinger equation [7–9]. In fact, we have discussed in detail the hydrodynamical structure of the modulated Klein–Gordon equation and its relation to the nonlinear Schrödinger equation and to the compressible (relativistic) Euler equations [15]. The most important local conservation laws associated with the modulated Klein–Gordon equation (1.3) are the charge and energy, given respectively by

$$\frac{\partial}{\partial t}\left[|\psi|^2 + \frac{i\hbar}{2mc^2}\left(\overline{\psi}\partial_t\psi - \psi\partial_t\overline{\psi}\right)\right] + \nabla\cdot\left[\frac{i\hbar}{2m}\left(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi\right)\right] = 0, \tag{1.5}$$

$$\frac{\partial}{\partial t}\left[\left(\frac{1}{c^2}|\partial_t\psi|^2 + |\nabla\psi|^2\right) + \frac{2m}{\hbar^2}V\right] - \nabla\cdot\left[\left(\nabla\psi\partial_t\overline{\psi} + \nabla\overline{\psi}\partial_t\psi\right)\right] = 0. \tag{1.6}$$

Examining the charge equation (1.5) we see that although $|\psi|^2$, the Schrödinger part, is positive-definite, the Klein–Gordon part $\frac{i\hbar}{2mc^2}(\overline{\psi}\partial_t\psi - \psi\partial_t\overline{\psi})$ is not. Here we face one of the major difficulties with the Klein–Gordon equation. However, the energy density is positive-definite and can be employed to obtain an estimate of the charge of the Schrödinger part. Thus we introduce the charge-energy inequality to establish singular limits. This is consistent with Einstein's relativity of mass-energy equivalence.

The question of the singular limits of the nonlinear Klein–Gordon equation and related equations has received considerable attention. Quite often, the limiting solution (when it exists) satisfies a completely different nonlinear partial differential equation. The nonrelativistic limit of the modulated nonlinear Klein–Gordon equation is one physical problem involving quantum (dispersion) effects where such a singular limiting process is interesting. In particular, Machihara et al. [18] gave a very complete answer of the Cauchy problem for the modulated Klein–Gordon equation; they proved that any finite energy solution converges to the corresponding solution of the nonlinear Schrödinger equation in the energy space, after infinite oscillation in time is removed. The Strichartz estimate plays a most important role in obtaining the uniform bound in space and time (see also [21,23] and references therein). Furthermore, it was shown by Masmaudi and Nakanishi in [20] that the solutions for the nonlinear Klein–Gordon equation can be described by using a system of two coupled nonlinear Schrödinger equations as the speed of light c tends to infinity. Thus, we may think of the coupled nonlinear Schrödinger equations as the singular limit (nonrelativistic limit) as $c \to \infty$ of the nonlinear Klein–Gordon equation.

The semiclassical limit of the defocusing nonlinear Schrödinger equation (1.3) is very well studied theoretically and numerically. In this limit, the Euler equation for an isentropic compressible flow is formally recovered [7–9,16] through the WKB analysis or Madelung transform (see [3,10,11] for the derivative nonlinear Schrödinger equation). On the other hand, if we forget the hydrodynamical formulation and return to the original nonlinear Schrödinger equation, then the limit wave function is shown to satisfy the wave map equation by Colin and Soyeur [2] for the case when there are no vortices, and by Lin and Xin [16] when there are vortices in two space dimensions. The vortex dynamics is also studied by Colliander and Jerrard in [5]. For long wave-short wave equations we have a similar result for the weak coupling case [14]. However, to the best of our knowledge, the semiclassical limit of the nonlinear Klein–Gordon equation has not been studied theoretically. According to the correspondence principle, the classical or relativistic world should emerge from the quantum world whenever the Planck's constant $\hbar$ is negligible. But the semiclassical limit, that is $\hbar \to 0$, is mathematically singular and is not clear from Equation (1.3) directly. However, as we will discuss in Section 2, the semiclassical limit of the modulated nonlinear Klein–Gordon equation (1.3) can be introduced in the same way as the semiclassical limit of the Schrödinger type equations, and the nonrelativistic-semiclassical limit can be discussed in a similar way.

Let us briefly summarize our results. In Section 2, we investigate the semiclassical limit of the Cauchy problem for the modulated defocusing cubic nonlinear Klein–Gordon Equations (2.1)–(2.2). We prove that any finite charge-energy solution converges to the corresponding solution of the relativistic wave map, and the scattering sound wave is shown to satisfy a linear relativistic wave equation (see Theorem 2.2 below). Unlike the Schrödinger equation, the charge is not positive definite for the Klein–Gordon equation, so we have to introduce a charge-energy inequality obtained by combining the conservation laws of charge and the energy of the nonlinear Klein–Gordon equation. Besides the linear momentum W of the Schrödinger part, we have to introduce one more term Z, defined by (2.16), of the relativistic part. By rewriting the conservation of charge in terms of W and Z we can prove convergence to the relativistic wave map by the compactness argument. SHATAH [24] has proved the existence of global weak solutions of the wave map. For completeness we also prove the nonrelativistic limit of the relativistic wave map in Theorem 2.4.

In Section 3 we employ the same idea used in Section 2 to obtain the nonrelativistic limit of the Cauchy problem for the modulated Klein–Gordon equation for general defocusing nonlinearity $V'(|\psi^c|^2) = |\psi^c|^p$, $p > 0$, and the main result is described in Theorem 3.2, which states that any finite charge-energy solution converges to the corresponding solution of the defocusing nonlinear Schrödinger equation in the energy space. For the sharper Strichartz estimate approach and a more complete result, the reader is referred to [18]. The main difference is that we combine the charge and energy conservation laws together to obtain the charge-energy inequality. Let us remark that in the case of the semiclassical limit, we have $L_t^\infty L_x^2$ as a bound for $\partial_t \psi^\hbar$, but for the non-relativistic limit, we only have $L_t^\infty L_x^2$ as a bound for $\frac{1}{c}\partial_t \psi^c$. Thus we need an extra argument to obtain strong convergence for the non-relativistic limit.

In Section 4, under restrictions similar to those for the semiclassical limit, we study the nonrelativistic-semiclassical limit of the Cauchy problem for the modulated defocusing cubic nonlinear Klein–Gordon equation. We prove that any finite charge-energy solution converges to the corresponding solution of the wave map, and the associated phase function is shown to satisfy a linear wave equation. The main result is stated in Theorem 4.2. Finally, we give a detail proof of Theorem 3.1 in the appendix. The strategy of the proof follows that introduced by Leray in the context of the Navier–Stokes equations, as well as many other existence proofs for weak solutions of other equations.

Notation. In this paper, $L^p(\Omega)$, ($p \geqq 1$) denotes the classical Lebesgue space with norm $\|f\|_p = (\int_\Omega |f|^p \mathrm{d}x)^{1/p}$, the Sobolev space of functions with all its kth partial derivatives in $L^2(\Omega)$ will be denoted by $H^k(\Omega)$, and its dual space is $H^{-k}(\Omega)$. We use $\langle f, g\rangle = \int_\Omega fg\mathrm{d}x$ to denote the standard inner product on the Hilbert space $L^2(\Omega)$. Without loss of generality, the units of length maybe chosen so that $\int_\Omega \mathrm{d}x = 1$. Given any Banach space $\mathbb{X}$ with norm $\|\cdot\|_{\mathbb{X}}$ and $p \geqq 1$, the space of measurable functions $u = u(t)$ from $[0, T]$ into $\mathbb{X}$ such that $\|u\|_{\mathbb{X}} \in L^p([0, T])$ will be denoted $L^p([0, T]; \mathbb{X})$. And $C([0, T]; w\text{-}H^k(\Omega))$ will denote the space of a continuous function from $[0, T]$ into $w\text{-}H^k(\Omega)$. This means that for every $\varphi \in H^{-k}(\Omega)$, the function $\langle \varphi, u(t)\rangle$ is in $C([0, T])$. Finally, we abbreviate "$\leqq C$" to "$\lesssim$", where C is a positive constant depending only on a fixed parameter.

2. Semiclassical limit

The specific problem we will consider in this section is the semiclassical limit of the modulated nonlinear Klein–Gordon equation (1.3) with the potential function given by $V'(|\psi^\hbar|^2) = |\psi^\hbar|^2 - 1$. For the formal analysis of the semiclassical limit from the point of view of hydrodynamics, the reader is referred to [15]. For convenience let us call it the modulated defocusing cubic nonlinear Klein–Gordon equation. After dividing by $\hbar$, we relabel it as

$$i\partial_t\psi^\hbar - \frac{\hbar}{2c^2}\partial_t^2\psi^\hbar + \frac{\hbar}{2}\Delta\psi^\hbar - \left(\frac{|\psi^\hbar|^2 - 1}{\hbar}\right)\psi^\hbar = 0. \tag{2.1}$$

The initial conditions are supplemented by

$$\psi^\hbar(x, 0) = \psi_0^\hbar(x), \quad \partial_t\psi^\hbar(x, 0) = \psi_1^\hbar(x), \quad x \in \Omega. \tag{2.2}$$

The superscript $\hbar$ in the wave function $\psi^\hbar$ indicates the $\hbar$-dependence and the light speed c is assumed to be a fixed number in this section. To avoid complications at the boundary, we concentrate below on the case where $x \in \Omega = \mathbb{T}^n$, the n-dimensional torus. Notice that the fourth term $\frac{|\psi^\hbar|^2-1}{\hbar}$ of (2.1) can serve as the density fluctuation of the sound wave, which is similar to the acoustic wave as discussed in the low Mach number limit of the compressible fluid [1,12,13,17,19]. For this model (2.1)–(2.2) we have the following existence result.

Theorem 2.1. *Let $c, T > 0$ and $0 < \hbar \ll 1$. Given initial data $(\psi_0^\hbar, \psi_1^\hbar) \in H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ and $\frac{|\psi_0^\hbar|^2-1}{\hbar} \in L^2(\mathbb{T}^n)$, there exists a function $\psi^\hbar$ such that*

$$\psi^\hbar \in L^\infty\big([0, T]; H^1(\mathbb{T}^n)\big) \cap C\big([0, T]; L^2(\mathbb{T}^n)\big), \tag{2.3}$$

$$\partial_t\psi^\hbar \in L^\infty\big([0, T]; L^2(\mathbb{T}^n)\big) \cap C\big([0, T]; H^{-1}(\mathbb{T}^n)\big), \tag{2.4}$$

$$\frac{|\psi^\hbar|^2 - 1}{\hbar} \in L^\infty\big([0, T]; L^2(\mathbb{T}^n)\big), \tag{2.5}$$

and satisfies the weak formulation of (2.1) given by

$$\begin{aligned} 0 = i\left\langle\psi^\hbar(\cdot, t_2) - \psi^\hbar(\cdot, t_1), \varphi\right\rangle - \frac{\hbar}{2c^2}\left\langle\partial_t\psi^\hbar(\cdot, t_2) - \partial_t\psi^\hbar(\cdot, t_1), \varphi\right\rangle \\ -\frac{\hbar}{2}\int_{t_1}^{t_2}\left\langle\nabla\psi^\hbar(\cdot, \tau), \nabla\varphi\right\rangle \mathrm{d}\tau - \int_{t_1}^{t_2}\left\langle\left(\frac{|\psi^\hbar|^2 - 1}{\hbar}\right)\psi^\hbar(\cdot, \tau), \varphi\right\rangle \mathrm{d}\tau, \end{aligned} \tag{2.6}$$

for every $[t_1, t_2] \subset [0, T]$ and for all $\varphi \in C_0^\infty(\mathbb{T}^n)$. Moreover, for all $t \in [0, T]$, it satisfies the charge-energy inequality

$$\begin{aligned} &\int_{\mathbb{T}^n} |\psi^\hbar|^2 + \frac{1}{c^2}|\partial_t\psi^\hbar|^2 + |\nabla\psi^\hbar|^2 + \frac{1}{2}\left(\frac{|\psi^\hbar|^2 - 1}{\hbar}\right)^2 \mathrm{d}x \\ &\leqq 2C_1 + \left(1 + \frac{2\hbar^2}{c^2}\right)C_2, \end{aligned} \tag{2.7}$$

where

$$
\begin{aligned}
C_1 &= \int_{\mathbb{T}^n} |\psi_0^\hbar|^2 + \frac{\hbar}{c^2}\frac{i}{2}\left(\psi_1^\hbar\overline{\psi_0^\hbar} - \overline{\psi_1^\hbar}\psi_0^\hbar\right) \mathrm{d}x, \\
C_2 &= \int_{\mathbb{T}^n} \frac{1}{c^2}|\psi_1^\hbar|^2 + |\nabla\psi_0^\hbar|^2 + \frac{1}{2}\left(\frac{|\psi_0^\hbar|^2 - 1}{\hbar}\right)^2 \mathrm{d}x,
\end{aligned} \tag{2.8}
$$

are the initial charge and energy respectively.

The charge consists of the Schrödinger part (positive definite) and the Klein–Gordon part (not positive definite). However, it can be bounded by the energy. We denote by "$\cap$" the intersection of topological spaces equipped with the relative topology induced by the inclusion maps. Since we are concerned with the semi-classical limit in this section, so the proof of this theorem, Theorems 3.1 and 4.1 of the following two sections will be given in the appendix.

We assume that the initial datum satisfy $|\psi_0^\hbar| = 1$ almost everywhere and $(\psi_0^\hbar, \psi_1^\hbar) \to (\psi_0, 0)$ in $H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ as $\hbar \to 0$, thus $|\psi_0| = 1$ almost everywhere. First we deduce from the charge-energy inequality (2.7) that

$$\{\psi^\hbar\}_\hbar \quad \text{is bounded in} \quad L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big), \tag{2.9}$$

$$\{\partial_t\psi^\hbar\}_\hbar \quad \text{is bounded in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big), \tag{2.10}$$

$$\left\{\frac{|\psi^\hbar|^2 - 1}{\hbar}\right\}_\hbar \quad \text{is bounded in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big), \tag{2.11}$$

then the classical compactness argument shows that there exists a subsequence still denoted by $\{\psi^\hbar\}_\hbar$ and a function ψ satisfying

$$\psi \in L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big), \quad \partial_t\psi \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big)$$

such that

$$\psi^\hbar \rightharpoonup \psi \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big), \tag{2.12}$$

$$\partial_t\psi^\hbar \rightharpoonup \partial_t\psi \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big). \tag{2.13}$$

Next, from (2.11), we have

$$|\psi^\hbar|^2 \to 1 \quad \text{almost everywhere and strongly in } L^2(\mathbb{T}^n). \tag{2.14}$$

Note that (2.11) shows only that $\{\frac{|\psi^\hbar|^2-1}{\hbar}\}_\hbar$ is a weakly relative compact set in $L^\infty([0,T]; L^2(\mathbb{T}^n))$. Thus, to overcome the difficulty caused by nonlinearity, that is, the fourth term on the right-hand side of (2.6), we have to prove $\psi^\hbar \to \psi$ strongly in $C([0,T]; L^2(\mathbb{T}^n))$.

Lemma 2.1. *For all* $0 < \hbar \ll 1$, *the sequence* $\{\psi^\hbar\}_\hbar$ *is a relatively compact set in* $C([0,T]; L^2(\mathbb{T}^n))$ *endowed with its strong topology, that is, there exists* $\psi \in C([0,T]; L^2(\mathbb{T}^n))$ *such that*

$$\psi^\hbar \to \psi \quad \textit{strongly in} \quad C\big([0,T]; L^2(\mathbb{T}^n)\big). \tag{2.15}$$

Proof. In this case, compactness requires more than just boundedness because of the strong topology over the time variable t. We appeal to the Arzela–Ascoli theorem, which asserts that $\{\psi^\hbar\}_\hbar$ is a relatively compact set in $C([0,T]; L^2(\mathbb{T}^n))$ if and only if

(1) $\{\psi^\hbar(t)\}_\hbar$ is a relatively compact set in $L^2(\mathbb{T}^n)$ for all $t \geqq 0$;
(2) $\{\psi^\hbar\}_\hbar$ is equicontinuous in $C\big([0,T]; L^2(\mathbb{T}^n)\big)$.

From (2.7) or (2.9) we know that $\{\psi^\hbar(t)\}_\hbar$ is a bounded set in $H^1(\mathbb{T}^n)$ and hence is a relatively compact set in $L^2(\mathbb{T}^n)$ by the Rellich lemma, which states that $H^1(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n)$ is a compact embedding.

In order to establish condition (2), we apply the fundamental theorem of calculus and the uniform bound of $\{\partial_t \psi^\hbar\}_\hbar$ to obtain

$$\|\psi^\hbar(t_2) - \psi^\hbar(t_1)\|_{L^2(\mathbb{T}^n)} \leqq |t_2 - t_1| \|\partial_t \psi^\hbar(s)\|_{L^2(\mathbb{T}^n)} \lesssim |t_2 - t_1|$$

for some $s \in (t_1, t_2)$. This completes the proof of Lemma 2.1. □

The quantity $\frac{|\psi^\hbar(x,t)|^2-1}{\hbar}$ is bounded in $L^\infty([0,T]; L^2(\mathbb{T}^n))$, and hence it converges weakly $*$ to some function $w \in L^\infty([0,T]; L^2(\mathbb{T}^n))$. To find the explicit form of w, we define two functions $W(\psi^\hbar)$ and $Z(\psi^\hbar)$, respectively, by

$$W(\psi^\hbar) = \frac{i}{2}\left(\psi^\hbar \nabla \overline{\psi^\hbar} - \overline{\psi^\hbar} \nabla \psi^\hbar\right), \quad Z(\psi^\hbar) = \frac{i}{2c^2}\left(\overline{\psi^\hbar} \partial_t \psi^\hbar - \psi^\hbar \partial_t \overline{\psi^\hbar}\right). \tag{2.16}$$

We rewrite the conservation of charge (1.5) as

$$\frac{\partial}{\partial t}\left[\frac{|\psi^\hbar|^2 - 1}{\hbar} + Z(\psi^\hbar)\right] + \operatorname{div} W(\psi^\hbar) = 0, \tag{2.17}$$

then, integrating (2.17) with respect to t and using the initial condition $|\psi_0^\hbar|^2 = 1$, we have

$$\frac{|\psi^\hbar(x,t)|^2 - 1}{\hbar} = -Z(\psi^\hbar) + Z(\psi^\hbar(x,0)) - \int_0^t \operatorname{div} W(\psi^\hbar) \mathrm{d}\tau. \tag{2.18}$$

Thus, to obtain the compactness of the sequence $\{\frac{|\psi^\hbar(x,t)|^2-1}{\hbar}\}_\hbar$, we have to treat the compactness of $\{Z(\psi^\hbar)\}_\hbar$ and $\{W(\psi^\hbar)\}_\hbar$ separately. First we have the following lemma.

Lemma 2.2. *Assume the hypothesis of Theorem 2.1, then*

$$\psi^\hbar \partial_t \overline{\psi^\hbar} \rightharpoonup \psi \partial_t \overline{\psi} \tag{2.19}$$

$$\int_0^t \operatorname{div}\left(\psi^\hbar \nabla \overline{\psi^\hbar}\right) d\tau \rightharpoonup \int_0^t \operatorname{div}\left(\psi \nabla \overline{\psi}\right) d\tau \tag{2.20}$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$.

Proof. We observe that $\psi^\hbar \in C([0,T];L^2(\mathbb{T}^n))$ implies $\psi^\hbar \in L^2([0,T]\times\mathbb{T}^n)$ and $\partial_t\psi^\hbar \in L^\infty([0,T];L^2(\mathbb{T}^n))$ implies $\partial_t\psi^\hbar \in L^2([0,T]\times\mathbb{T}^n)$. Also $\psi^\hbar$ converges strongly to ψ in $L^2([0,T]\times\mathbb{T}^n)$ and $\partial_t\psi^\hbar$ converges weakly to $\partial_t\psi$ in $L^2([0,T]\times\mathbb{T}^n)$. Thus for all $\varphi \in C_0^\infty(\mathbb{T}^n)$ we have

$$\lim_{\hbar\to 0}\int_{t_1}^{t_2}\int_{\mathbb{T}^n} \psi^\hbar(x,t)\partial_t\overline{\psi^\hbar}(x,t)\varphi(x)dx\,dt = \int_{t_1}^{t_2}\int_{\mathbb{T}^n} \psi(x,t)\partial_t\overline{\psi}(x,t)\varphi(x)dx\,dt.$$

Similarly $\nabla\psi^\hbar \in L^\infty([0,T];L^2(\mathbb{T}^n))$ implies $\nabla\psi^\hbar \in L^2([0,T]\times\mathbb{T}^n)$ and $\nabla\psi^\hbar$ converges weakly to $\nabla\psi$ in $L^2([0,T]\times\mathbb{T}^n)$, then integration by parts, then by the Fubini theorem and the Lebesgue dominated convergence theorem, we conclude that

$$\begin{aligned}
&-\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\int_0^t \operatorname{div}\left[\psi^\hbar(x,\tau)\nabla\overline{\psi^\hbar}(x,\tau) - \psi(x,\tau)\nabla\overline{\psi}(x,\tau)\right] d\tau\varphi(x)dx\,dt \\
&= \int_{t_1}^{t_2}\int_0^t\int_{\mathbb{T}^n}\left[\psi^\hbar(x,\tau) - \psi(x,\tau)\right]\nabla\overline{\psi^\hbar}(x,\tau)\cdot\nabla\varphi(x)dx\,d\tau\,dt \\
&\quad + \int_{t_1}^{t_2}\int_0^t\int_{\mathbb{T}^n}\left[\nabla\overline{\psi^\hbar}(x,\tau) - \nabla\overline{\psi}(x,\tau)\right]\psi(x,\tau)\cdot\nabla\varphi(x)dx\,d\tau\,dt \to 0
\end{aligned}$$

as $\hbar \to 0$. This completes the proof of Lemma 2.2. □

It follows from Lemma 2.2 that $Z(\psi^\hbar) \rightharpoonup Z(\psi)$, $Z(\psi^\hbar(x,0)) \rightharpoonup 0$ and

$$\int_0^t \operatorname{div} W(\psi^\hbar)d\tau \rightharpoonup \int_0^t \operatorname{div} W(\psi)d\tau$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$, thus

$$\frac{|\psi^\hbar(x,t)|^2 - 1}{\hbar} \rightharpoonup -Z(\psi) - \int_0^t \operatorname{div} W(\psi)d\tau \tag{2.21}$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$, and the limit function w is given explicitly by

$$w = -Z(\psi) - \int_0^t \operatorname{div} W(\psi)d\tau.$$

Passage to the limit ($\hbar \to 0$). The uniform boundness of the sequences $\{\psi^\hbar\}_\hbar$ in $L^\infty\left([0,T];H^1(\mathbb{T}^n)\right)$ and $\{\partial_t\psi^\hbar\}_\hbar$ in $L^\infty\left([0,T];L^2(\mathbb{T}^n)\right)$ imply

$$\frac{\hbar}{2c^2}\left\langle\partial_t\psi^\hbar(\cdot,t_2),\varphi\right\rangle \to 0, \quad \frac{\hbar}{2c^2}\left\langle\partial_t\psi^\hbar(\cdot,t_1),\varphi\right\rangle \to 0, \tag{2.22}$$

$$\frac{\hbar}{2}\int_{t_1}^{t_2}\left\langle\nabla\psi^\hbar(\cdot,\tau),\nabla\varphi\right\rangle d\tau \to 0 \tag{2.23}$$

as $\hbar \to 0$. The strong convergence of $\psi^\hbar$ in $C([0,T]; L^2(\mathbb{T}^n))$ implies

$$\left\langle \psi^\hbar(\cdot, t_2), \varphi \right\rangle \to \langle \psi(\cdot, t_2), \varphi \rangle \ , \quad \left\langle \psi^\hbar(\cdot, t_1), \varphi \right\rangle \to \langle \psi(\cdot, t_1), \varphi \rangle \, . \tag{2.24}$$

The convergence of the nonlinear term follows by combining (2.15) and (2.21), so that for all $t > 0$

$$\left(\frac{|\psi^\hbar|^2 - 1}{\hbar} \right) \psi^\hbar \rightharpoonup - \left[Z(\psi) + \int_0^t \operatorname{div} W(\psi) d\tau \right] \psi \tag{2.25}$$

in $\mathcal{D}'((0,T) \times \mathbb{T}^n)$ and hence

$$\int_{t_1}^{t_2} \left\langle \left(\frac{|\psi^\hbar|^2 - 1}{\hbar} \right) \psi^\hbar(\cdot, \tau), \varphi \right\rangle d\tau$$
$$\to - \int_{t_1}^{t_2} \left\langle \left(Z(\psi) + \int_0^t \operatorname{div} W(\psi) d\tau \right) \psi(\cdot, \tau), \varphi \right\rangle d\tau. \tag{2.26}$$

Putting all the above convergent results into the weak formulation (2.6), the limit wave function ψ satisfies

$$i \partial_t \psi + \left[Z(\psi) + \int_0^t \operatorname{div} W(\psi) d\tau \right] \psi = 0 \tag{2.27}$$

in the sense of distribution. Note $|\psi|^2 = 1$, we have $\overline{\psi} \partial_t \psi + \psi \partial_t \overline{\psi} = 0$ and $\overline{\psi} \nabla \psi + \psi \nabla \overline{\psi} = 0$, hence

$$\frac{1}{2} \left(\overline{\psi} \partial_t \psi - \psi \partial_t \overline{\psi} \right) = \overline{\psi} \partial_t \psi = -\psi \partial_t \overline{\psi}, \quad \frac{1}{2} \left(\overline{\psi} \nabla \psi - \psi \nabla \overline{\psi} \right) = \overline{\psi} \nabla \psi.$$

Differentiating (2.27) with respect to t, we have

$$\partial_t^2 \psi + \left[\frac{1}{c^2} \partial_t (\overline{\psi} \partial_t \psi) - \operatorname{div} (\overline{\psi} \nabla \psi) \right] \psi - \frac{\partial_t \psi}{\psi} \partial_t \psi = 0, \tag{2.28}$$

or

$$\partial_t^2 \psi + \left[\frac{1}{c^2} \left(\overline{\psi} \partial_t^2 \psi + \partial_t \psi \partial_t \overline{\psi} \right) - \left(\overline{\psi} \Delta \psi + \nabla \psi \cdot \nabla \overline{\psi} \right) \right] \psi + |\partial_t \psi|^2 \psi = 0. \tag{2.29}$$

Therefore ψ satisfies the relativistic wave map equation

$$\left(1 + \frac{1}{c^2} \right) \partial_t^2 \psi - \Delta \psi = \left[|\nabla \psi|^2 - \left(1 + \frac{1}{c^2} \right) |\partial_t \psi|^2 \right] \psi,$$
$$|\psi| = 1 \text{ almost everywhere} \tag{2.30}$$

supplemented with the initial conditions

$$\psi(x, 0) = \psi_0(x), \quad \partial_t \psi(x, 0) = 0, \quad x \in \mathbb{T}^n, \quad |\psi_0| = 1 \quad \text{almost everywhere} \tag{2.31}$$

Using the fact $|\psi| = 1$ and writing $\psi = e^{i\theta}$ shows

$$\left(1 + \frac{1}{c^2}\right)\partial_t^2\theta = \Delta\theta\,, \quad \theta(x,0) = \arg\psi_0\,, \quad \partial_t\theta(x,0) = 0, \tag{2.32}$$

that is, θ is a distribution solution of the linear relativistic wave equation. Moreover, $\theta(x,t) \in H^1(\mathbb{T}^n)$ and $\partial_t\theta(x,t) \in L^2(\mathbb{T}^n)$ for all $t \in [0,T]$ implies that θ is the unique weak solution of (2.32) with finite energy. The $\frac{1}{c^2}$ terms in (2.30) and (2.32) show the relativistic effect and formally, letting $c \to \infty$, they reduce to the standard wave map and wave equation, respectively (see [2,6,14,16]).

Theorem 2.2. *Let* $(\psi_0^\hbar, \psi_1^\hbar) \in H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$*,* $|\psi_0^\hbar| = 1$ *almost everywhere and* $(\psi_0^\hbar, \psi_1^\hbar) \to (\psi_0, 0)$ *in* $H^1(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$*,* $|\psi_0| = 1$ *almost everywhere, and let* $\psi^\hbar$ *be the corresponding weak solution of the modulated defocusing cubic nonlinear Klein–Gordon equation* (2.1)–(2.2)*. Then the weak limit* ψ*, satisfying* $|\psi| = 1$ *almost everywhere, solves the relativistic wave map* (2.30)–(2.31)*. Moreover, let* $\psi = e^{i\theta}$*; then the phase function* θ *satisfies the relativistic wave equation* (2.32).

For completeness we also discuss the non-relativistic limit of the relativistic wave map equation (2.30)–(2.31). To indicate the c-dependence of the wave function, we replace ψ by ϕ^c and rewrite (2.30)–(2.31) as

$$\left(1 + \frac{1}{c^2}\right)\partial_t^2\phi^c - \Delta\phi^c = \left[|\nabla\phi^c|^2 - \left(1 + \frac{1}{c^2}\right)|\partial_t\phi^c|^2\right]\phi^c\,, \tag{2.33}$$

$$\phi^c(x,0) = \phi_0^c(x), \quad \partial_t\phi^c(x,0) = 0, \quad x \in \mathbb{T}^n, \tag{2.34}$$

$|\phi^c| = |\phi_0^c| = 1$ almost everywhere. Let $\mathcal{R}e\,\phi^c$ and $\mathcal{I}m\,\phi^c$ denote the real and imaginary parts of ϕ^c, $\phi^c = \mathcal{R}e\,\phi^c + i\mathcal{I}m\,\phi^c$, and $u^c = (\mathcal{R}e\,\phi^c, \mathcal{I}m\,\phi^c)^t$ then (2.33)–(2.34) can be rewritten as

$$\left(1 + \frac{1}{c^2}\right)\partial_t^2 u^c - \Delta u^c = \left[|\nabla u^c|^2 - \left(1 + \frac{1}{c^2}\right)|\partial_t u^c|^2\right]u^c\,, \tag{2.35}$$

$$u^c(x,0) = u_0^c(x), \quad \partial_t u^c(x,0) = 0, \quad x \in \mathbb{T}^n, \tag{2.36}$$

where $u_0^c(x) = (\mathcal{R}e\,\phi_0^c, \mathcal{I}m\,\phi_0^c)^t$ and $|u^c| = |u_0^c| = 1$ almost everywhere. When $c = \infty$, the necessary and sufficient condition for the existence of weak solutions to (2.35)–(2.36) was proved by SHATAH [24] (see also [25]). His result is easily extended to general c by replacing the Riemann metric $\eta = \mathrm{diag}(1, -1, -1, \ldots, -1)$ by $\eta_c = \mathrm{diag}(1 + 1/c^2, -1, -1, \ldots, -1)$ and $\partial^\alpha = \eta^{\alpha\beta}\partial_\beta$ by $\widetilde{\partial}^\alpha = \eta_c^{\alpha\beta}\partial_\beta$.

Lemma 2.3. [24] *If* $|u^c| = 1$ *almost everywhere and satisfies* $\nabla u^c \in L^\infty([0,T]; L^2(\mathbb{T}^n))$*,* $\partial_t u^c \in L^\infty([0,T]; L^2(\mathbb{T}^n))$*, then* u^c *is a weak solution of* (2.35)–(2.36) *if and only if* $\partial_\alpha(\widetilde{\partial}^\alpha u^c \wedge u^c) = 0$*, where* $\wedge$ *denotes the wedge product.*

By Lemma 2.3, we have the existence of weak solutions of the wave map equation.

Theorem 2.3. [24] *Given initial data* $u_0^c \in H^1(\mathbb{T}^n)$ *and* $|u_0^c| = 1$*, there exists a function* u^c*,* $|u^c| = 1$ *almost everywhere, such that*

$$\nabla u^c \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big), \quad \partial_t u^c \in L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big) \tag{2.37}$$

and satisfies the wave map equation

$$\left(1+\frac{1}{c^2}\right)\partial_t^2 u^c - \Delta u^c = \left[|\nabla u^c|^2 - \left(1+\frac{1}{c^2}\right)|\partial_t u^c|^2\right] u^c \tag{2.38}$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$*. Moreover, for all* $t \in [0,T]$*, it satisfies the energy relation*

$$\int_{\mathbb{T}^n}\left(1+\frac{1}{c^2}\right)|\partial_t u^c|^2 + |\nabla u^c|^2 \mathrm{d}x \leqq \int_{\mathbb{T}^n} |\nabla u_0^c|^2 \mathrm{d}x. \tag{2.39}$$

As before, we assume $\phi_0^c \to \phi_0$ strongly in $H^1(\mathbb{T}^n)$ and $|\phi_0| = 1$ almost everywhere, equivalently if $u_0 = (\mathcal{R}e\,\phi_0, \mathcal{I}m\,\phi_0)^t$, $|u_0| = 1$ almost everywhere, then $u_0^c \to u_0$ in $H^1(\mathbb{T}^n)$. We deduce from the energy relation (2.39) and $|u^c| = 1$ almost everywhere that

$$\{u^c\}_c \quad \text{is bounded in} \quad L^\infty([0,T]; H^1(\mathbb{T}^n)), \tag{2.40}$$

$$\{\partial_t u^c\}_c \quad \text{is bounded in} \quad L^\infty([0,T]; L^2(\mathbb{T}^n)). \tag{2.41}$$

By the classical compactness argument and the diagonalization process, there exists a subsequence, still denoted by $\{u^c\}_c$, satisfying $u \in L^\infty([0,T]; H^1(\mathbb{T}^n))$ and $\partial_t u \in L^\infty([0,T]; L^2(\mathbb{T}^n))$ such that

$$u^c \rightharpoonup u \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big), \tag{2.42}$$

$$\partial_t u^c \rightharpoonup \partial_t u \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big). \tag{2.43}$$

Using the same argument as Lemma 2.1, we deduce from (2.40)–(2.41) that

$$u^c \to u \quad \text{strongly in } C\big([0,T]; L^2(\mathbb{T}^n)\big). \tag{2.44}$$

Combining (2.44) and $|u^c| = 1$ almost everywhere, we derive $|u| = 1$ almost everywhere. Moreover, using (2.42)–(2.44), we have

$$\partial_\alpha u^c \wedge u^c \to \partial_\alpha u \wedge u \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^n). \tag{2.45}$$

Note that u^c satisfies $\partial_\alpha(\widetilde{\partial}^\alpha u^c \wedge u^c) = 0$ in the sense of distribution;

$$\left(1+\frac{1}{c^2}\right)\big\langle \partial_t u^c \wedge u^c(t_2,\cdot) - \partial_t u^c \wedge u^c(t_1,\cdot), \varphi\big\rangle + \sum_{i=1}^n \int_{t_1}^{t_2} \big\langle \partial_i u^c \wedge u^c(\cdot,\tau), \partial_i \varphi\big\rangle \mathrm{d}\tau = 0 \tag{2.46}$$

for every $[t_1, t_2] \subset [0, T]$ and for all $\varphi \in C_0^\infty(\mathbb{T}^n)$. Letting $c \to \infty$ in (2.46) and using (2.45), we have shown that u satisfies $\partial_\alpha(\partial^\alpha u \wedge u) = 0$ in the sense of distribution, and by Lemma 2.3 it solves the wave map equation

$$\partial_t^2 u - \Delta u = \left(|\nabla u|^2 - |\partial_t u|^2\right) u \quad \text{in } \mathcal{D}'((0, T) \times \mathbb{T}^n). \tag{2.47}$$

Denote $u = (\alpha, \beta)^t$ and $\phi = \alpha + i\beta$, then we have $\nabla\phi^c \to \nabla\phi$ weakly $*$ in $L^\infty([0, T]; L^2(\mathbb{T}^n))$, $\phi^c \to \phi$ strongly in $L^\infty([0, T]; L^2(\mathbb{T}^n))$ and $\partial_t\phi^c \to \partial_t\phi$ weakly $*$ in $L^\infty([0, T]; L^2(\mathbb{T}^n))$. Moreover, ϕ satisfies the wave map equation

$$\partial_t^2 \phi - \Delta\phi = \left(|\nabla\phi|^2 - |\partial_t\phi|^2\right)\phi, \quad (t, x) \in [0, T] \times \mathbb{T}^n, \tag{2.48}$$

$$\phi(x, 0) = \phi_0(x), \quad \partial_t\phi(x, 0) = 0, \quad x \in \mathbb{T}^n, \tag{2.49}$$

in the sense of distribution and $|\phi| = |\phi_0| = 1$ almost everywhere.

Theorem 2.4. *Let $\phi_0^c, \phi_0 \in H^1(\mathbb{T}^n)$, $|\phi_0^c| = |\phi_0| = 1$ almost everywhere and $\phi_0^c \to \phi_0$ in $H^1(\mathbb{T}^n)$. Let ϕ^c be the corresponding weak solution of the relativistic wave map (2.33)–(2.34). Then the weak limit ϕ of $\{\phi^c\}_c$ satisfies $|\phi| = 1$ almost everywhere and solves the wave map (2.48)–(2.49).*

3. Nonrelativistic limit

This section is devoted to the non-relativistic limit of the modulated nonlinear Klein–Gordon equation with the potential function given by $V'(|\psi^c|^2) = |\psi^c|^p$, $p > 0$,

$$i\hbar\partial_t\psi^c - \frac{\hbar^2}{2c^2}\partial_t^2\psi^c + \frac{\hbar^2}{2}\Delta\psi^c - |\psi^c|^p\psi^c = 0. \tag{3.1}$$

As usual, we supplement the system (3.1) with initial conditions

$$\psi^c(x, 0) = \psi_0^c(x), \quad \partial_t\psi^c(x, 0) = \psi_1^c(x), \quad x \in \mathbb{T}^n. \tag{3.2}$$

Here the Planck's constant $\hbar$ is a fixed positive number and the superscript c in the wave function ψ^c indicates c-dependence. Similarly to the semiclassical limit discussed in the previous section, we discuss only the periodic domain $\mathbb{T}^n$ and state the existence theorem of (3.1)–(3.2) first, leaving the proof to the appendix.

Theorem 3.1. *Let $p, \hbar, T > 0$ and $c \gg 1$. Given initial data (ψ_0^c, ψ_1^c) in $H^1 \cap L^{p+2}(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$, there exists a function ψ^c such that*

$$\psi^c \in L^\infty\big([0, T]; H^1(\mathbb{T}^n)\big) \cap C\big([0, T]; L^2(\mathbb{T}^n)\big), \tag{3.3}$$

$$\partial_t\psi^c \in L^\infty\big([0, T]; L^2(\mathbb{T}^n)\big) \cap C\big([0, T]; H^{-1}(\mathbb{T}^n)\big), \tag{3.4}$$

$$\psi^c \in L^\infty\big([0, T]; L^{p+2}(\mathbb{T}^n)\big), \tag{3.5}$$

and satisfies the weak formulation of (3.1) *given by*

$$0 = -\frac{\hbar^2}{2c^2}\left\langle \partial_t \psi^c(\cdot, t_2) - \partial_t \psi^c(\cdot, t_1), \varphi \right\rangle + i\hbar \left\langle \psi^c(\cdot, t_2) - \psi^c(\cdot, t_1), \varphi \right\rangle$$
$$-\frac{\hbar^2}{2}\int_{t_1}^{t_2} \left\langle \nabla \psi^c(\cdot, \tau), \nabla \varphi \right\rangle \mathrm{d}\tau - \int_{t_1}^{t_2} \left\langle |\psi^c|^p \psi^c(\cdot, \tau), \varphi \right\rangle \mathrm{d}\tau, \tag{3.6}$$

for every $[t_1, t_2] \subset [0, T]$ *and for all* $\varphi \in C_0^\infty(\mathbb{T}^n)$. *Moreover,* ψ^c *satisfies the charge-energy inequality*

$$\int_{\mathbb{T}^n} |\psi^c|^2 + \frac{\hbar^2}{2c^2}|\partial_t \psi^c|^2 + \frac{\hbar^2}{2}|\nabla \psi^c|^2 + \frac{|\psi^c|^{p+2}}{p+2}\mathrm{d}x \leqq 2C_1 + \left(1 + \frac{2}{c^2}\right) C_2, \tag{3.7}$$

where C_1 *and* C_2 *are the initial charge and energy given respectively by*

$$C_1 = \int_{\mathbb{T}^n} |\psi_0^c|^2 + \frac{\hbar}{c^2}\frac{i}{2}\left(\psi_1^c \overline{\psi_0^c} - \overline{\psi_1^c}\psi_0^c\right) \mathrm{d}x\,,$$
$$C_2 = \int_{\mathbb{T}^n} \frac{\hbar^2}{2c^2}|\psi_1^c|^2 + \frac{\hbar^2}{2}|\nabla \psi_0^c|^2 + \frac{1}{p+2}|\psi_0^c|^{p+2}\mathrm{d}x. \tag{3.8}$$

To study the nonrelativistic limit, we will assume that the initial condition (ψ_0^c, ψ_1^c) converges strongly in $H^1(\mathbb{T}^n) \cap L^{p+2}(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ to $(\psi_0, 0)$ when the light speed c tends to ∞. We deduce from the charge-energy inequality (3.7) that

$$\{\psi^c\}_c \quad \text{is bounded in} \quad L^\infty\big([0, T]; H^1(\mathbb{T}^n)\big)\,, \tag{3.9}$$

$$\left\{\frac{1}{c}\partial_t \psi^c\right\}_c \quad \text{is bounded in} \quad L^\infty\big([0, T]; L^2(\mathbb{T}^n)\big)\,, \tag{3.10}$$

$$\{\psi^c\}_c \quad \text{is bounded in} \quad L^\infty\big([0, T]; L^{p+2}(\mathbb{T}^n)\big). \tag{3.11}$$

In the case of the semiclassical limit, we have $L_t^\infty L_x^2$ bound for $\partial_t \psi^\hbar$, but for the non-relativistic limit, we have only $L_t^\infty L_x^2$ as a bound for $\frac{1}{c}\partial_t \psi^c$, so we need further argument to show $\psi^c \to \psi$ in $C([0, T]; L^2(\mathbb{T}^n))$.

Lemma 3.1. *For all* $c \gg 1$, *the sequence* $\{\psi^c\}_c$ *is a relatively compact set in* $C([0, T]; w\text{-}H^1(\mathbb{T}^n))$, *thus there exists* $\psi \in C([0, T]; w\text{-}H^1(\mathbb{T}^n))$ *such that*

$$\psi^c \to \psi \quad \text{in} \quad C\big([0, T]; w\text{-}H^1(\mathbb{T}^n)\big) \quad \text{as} \quad c \to \infty.$$

Furthermore, $\{\psi^c\}_c$ *is a relatively compact set in* $C([0, T]; L^2(\mathbb{T}^n))$ *endowed with its strong topology and*

$$\psi^c \to \psi \quad \text{in} \quad C\big([0, T]; L^2(\mathbb{T}^n)\big) \quad \text{as } c \to \infty.$$

Proof. As discussed in the previous section, we appeal to the Arzela–Ascoli theorem, which states that the sequence $\{\psi^c\}_c$ is a relatively compact set in $C([0, T]; w\text{-}H^1(\mathbb{T}^n))$ if and only if

(1) $\{\psi^c(t)\}$ is a relatively compact set in $w\text{-}H^1(\mathbb{T}^n)$ for all $t \geqq 0$;
(2) $\{\psi^c\}$ is equicontinuous in $C\big([0,T]; w\text{-}H^1(\mathbb{T}^n)\big)$, that is, for every $\varphi \in H^{-1}(\mathbb{T}^n)$ the sequence $\{\langle\psi^c, \varphi\rangle\}_c$ is equicontinuous in the space $C([0,T])$.

Since $\{\psi^c(t)\}_c$ is uniformly bounded in $H^1(\mathbb{T}^n)$, thus $\{\psi^c(t)\}_c$ is a relatively compact set in $w\text{-}H^1(\mathbb{T}^n)$ for every $t > 0$. In order to establish condition (2), let $A \subset C_c^\infty(\mathbb{T}^n)$ be an enumerable set which is dense in H^{-1}; then for any $\rho \in A$, we have

$$\begin{aligned} i\hbar\big\langle\psi^c(\cdot,t_2) - \psi^c(\cdot,t_1), \rho\big\rangle &= \frac{\hbar^2}{2c^2}\big\langle\partial_t\psi^c(\cdot,t_2) - \partial_t\psi^c(\cdot,t_1), \rho\big\rangle \\ &\quad + \frac{\hbar^2}{2}\int_{t_1}^{t_2}\big\langle\nabla\psi^c(\cdot,\tau), \nabla\rho\big\rangle\,\mathrm{d}\tau \\ &\quad + \int_{t_1}^{t_2}\big\langle|\psi^c|^p\psi^c(\cdot,\tau), \rho\big\rangle\,\mathrm{d}\tau, \end{aligned}$$

hence

$$|\langle\psi^c(\cdot,t_2) - \psi^c(\cdot,t_1), \rho\rangle| \lesssim c^{-1}\|\rho\|_{L^2(\mathbb{T}^n)} + |t_2 - t_1|\left(\|\rho\|_{H^1(\mathbb{T}^n)} + \|\rho\|_{L^\infty(\mathbb{T}^n)}\right).$$

Thus for any $\epsilon > 0$, we can choose $\delta = \epsilon$ such that if $|t_2 - t_1| < \delta$ and $c^{-1} < \epsilon$, then

$$|\langle\psi^c(\cdot,t_2) - \psi^c(\cdot,t_1), \rho\rangle| \lesssim \epsilon.$$

Moreover, by the density argument we can prove

$$|\langle\psi^c(\cdot,t_2) - \psi^c(\cdot,t_1), \varphi\rangle| \lesssim \epsilon, \tag{3.12}$$

for all $\varphi \in H^{-1}(\mathbb{T}^n)$. Thus $\{\psi^c\}_c$ is equicontinuous in $C\big([0,T]; w\text{-}H^1(\mathbb{T}^n)\big)$ for c larger. The second statement follows immediately by the Rellich lemma, which states that $H^1(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n)$ compactly, that is, $w\text{-}H^1(\mathbb{T}^n) \hookrightarrow L^2(\mathbb{T}^n)$ continuously. This completes the proof of Lemma 3.1. □

In order to overcome difficulties caused by nonlinearity, we need the following lemma.

Lemma 3.2. *Assume the hypothesis of Theorem 3.1. Let ψ^c be a sequence of weak solution to (3.1)–(3.2) then there exists $\psi \in L^\infty([0,T]; L^{p+1}(\mathbb{T}^n))$ such that*

$$\psi^c \to \psi \quad \text{in } L^\infty\big([0,T]; L^{p+1}(\mathbb{T}^n)\big).$$

Proof. The proof is divided into two cases. First, for $0 < p \leqq 1$, since $L^2(\mathbb{T}^n) \subset L^{p+1}(\mathbb{T}^n)$ for bounded measure $|\mathbb{T}^n| < \infty$, the strong convergence in the space $L^\infty([0,T]; L^2(\mathbb{T}^n))$ also implies the strong convergence in $L^\infty([0,T]; L^{p+1}(\mathbb{T}^n))$. Second, $p > 1$, the strong convergence in the space $L^\infty([0,T]; L^2(\mathbb{T}^n))$ and the weakly $*$ convergence in $L^\infty([0,T]; L^{p+2}(\mathbb{T}^n))$ combined with an interpolation argument yields the result. Indeed, $\psi^c \rightharpoonup \psi$ weakly

$*$ in $L^\infty([0,T];L^{p+2}(\mathbb{T}^n))$, the sequence $\{\psi^c-\psi\}_c$ is a norm bounded set in $L^\infty([0,T];L^{p+2}(\mathbb{T}^n))$, there exists a constant $K>0$ such that

$$\limsup_{c\to\infty}\|\psi^c-\psi\|^{p+2}_{L^\infty([0,T];L^{p+2}(\mathbb{T}^n))}=K<\infty. \tag{3.13}$$

Next, let $\eta>0$ be arbitrary, and choose $\delta<\eta/K$, the Young inequality gives

$$\begin{aligned}|\psi^c-\psi|^{p+1}&=|\psi^c-\psi|^{p+1-2/p}|\psi^c-\psi|^{2/p}\\&\leqq\delta|\psi^c-\psi|^{p+2}+C|\psi^c-\psi|^2.\end{aligned}\tag{3.14}$$

Integrating this inequality over $\mathbb{T}^n$, we have

$$\|\psi^c-\psi\|^{p+1}_{L^{p+1}(\mathbb{T}^n)}\leqq\delta\|\psi^c-\psi\|^{p+2}_{L^{p+2}(\mathbb{T}^n)}+C\|\psi^c-\psi\|^2_{L^2(\mathbb{T}^n)}. \tag{3.15}$$

Thus

$$\limsup_{c\to\infty}\|\psi^c-\psi\|^{p+1}_{L^\infty([0,T];L^{p+1}(\mathbb{T}^n))}\leqq K\delta\leqq\eta. \tag{3.16}$$

Because $\eta>0$ is arbitrary, we have $\psi^c\to\psi$ in $L^\infty([0,T];L^{p+1}(\mathbb{T}^n))$. □

Passage to limit ($c\to\infty$). The uniform boundedness of the sequence $\{\frac{1}{c}\partial_t\psi^c\}_c$ in $L^\infty([0,T];L^2(\mathbb{T}^n))$ yields

$$\frac{\hbar^2}{2c^2}\left\langle\partial_t\psi^c(\cdot,t_2)-\partial_t\psi^c(\cdot,t_1),\varphi\right\rangle\to 0. \tag{3.17}$$

The weak $*$ convergence of ψ^c in $L^\infty([0,T];H^1(\mathbb{T}^n))$ and the strong convergence in $C([0,T];L^2(\mathbb{T}^n))$ imply

$$\int_{t_1}^{t_2}\left\langle\nabla\psi^c(\cdot,\tau),\nabla\varphi\right\rangle\mathrm{d}\tau\to\int_{t_1}^{t_2}\langle\nabla\psi(\cdot,\tau),\nabla\varphi\rangle\,\mathrm{d}\tau, \tag{3.18}$$

$$\left\langle\psi^c(\cdot,t_2)-\psi^c(\cdot,t_1),\varphi\right\rangle\to\langle\psi(\cdot,t_2)-\psi(\cdot,t_1),\varphi\rangle\,. \tag{3.19}$$

For the nonlinear term, we rewrite it as

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\left[|\psi^c|^p\psi^c(x,\tau)-|\psi|^p\psi(x,\tau)\right]\varphi(x)\mathrm{d}x\,\mathrm{d}\tau\\&\quad=\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\left[\psi^c(x,\tau)-\psi(x,\tau)\right]|\psi^c|^p(x,\tau)\varphi(x)\mathrm{d}x\,\mathrm{d}\tau\\&\qquad+\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\left[|\psi^c|^p(x,\tau)-|\psi|^p(x,\tau)\right]\psi(x,\tau)\varphi(x)\mathrm{d}x\,\mathrm{d}\tau\equiv I+II\end{aligned}\tag{3.20}$$

for all $\varphi\in C_0^\infty(\mathbb{T}^n)$. We will estimate the integrals I and II separately. First, by Hölder inequality, we have

$$I\leqq\|\psi^c-\psi\|_{L^{p+1}([t_1,t_2]\times\mathbb{T}^n)}\|\varphi\|_{L^\infty(\mathbb{T}^n)}\|\psi^c\|^p_{L^{p+1}([t_1,t_2]\times\mathbb{T}^n)}\to 0, \tag{3.21}$$

thus I tends to 0 as $c\to\infty$ by Lemma 3.2. The estimate of II requires higher integrability. Since $|\psi^c|^p\rightharpoonup|\psi|^p$ weakly in $L^{\frac{p+2}{p}}([0,T]\times\mathbb{T}^n)$ for $T<\infty$ and

ψ is bounded in $L^q([0,T]\times\mathbb{T}^n)$, $1\leqq q\leqq p+2$, hence we can choose $q=\frac{p+2}{2}$ such that

$$II=\int_{t_1}^{t_2}\int_{\mathbb{T}^n}\left[|\psi^c|^p(x,\tau)-|\psi|^p(x,\tau)\right]\psi(x,\tau)\varphi(x)\mathrm{d}x\,\mathrm{d}\tau\to 0. \quad (3.22)$$

Combining the above convergent results into the weak formulation (3.6), as $c\to\infty$, we deduce that ψ is a distribution solution of the defocusing nonlinear Schrödinger equation;

$$i\hbar\partial_t\psi+\frac{\hbar^2}{2}\Delta\psi-|\psi|^p\psi=0, \quad (x,t)\in\mathbb{T}^n\times(0,T), \quad (3.23)$$

$$\psi(x,0)=\psi_0(x), \quad x\in\mathbb{T}^n. \quad (3.24)$$

Theorem 3.2. *Let* $(\psi_0^c,\psi_1^c)\in H^1\cap L^{p+2}(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$, $(\psi_0^c,\psi_1^c)\to(\psi_0,0)$ *in* $H^1\cap L^{p+2}(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$, *and* ψ^c *be the corresponding weak solution of the modulated defocusing nonlinear Klein–Gordon equation* (3.1)–(3.2). *Then the weak limit* ψ *of* $\{\psi^c\}_c$ *solves the defocusing nonlinear Schrödinger equation* (3.23)–(3.24).

4. Nonrelativistic-semiclassical limit

In this section we will consider the nonrelativistic-semiclassical limit of the modulated nonlinear Klein–Gordon equation with the potential function given by $V'(|\psi|^2)=|\psi|^2-1$. In order to avoid carrying out a double limit, the parameters c and $\hbar$ must be related. For simplicity, we take $\hbar=\varepsilon$, $\frac{1}{c}=\varepsilon^\alpha$ for some $\alpha>0$, $0<\varepsilon\ll 1$ and rewrite the modulated defocusing cubic nonlinear Klein–Gordon equation as

$$i\partial_t\psi^\varepsilon-\frac{1}{2}\varepsilon^{1+2\alpha}\partial_t^2\psi^\varepsilon+\frac{\varepsilon}{2}\Delta\psi^\varepsilon-\left(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\right)\psi^\varepsilon=0, \quad (4.1)$$

supplemented with initial conditions

$$\psi^\varepsilon(x,0)=\psi_0^\varepsilon(x), \quad \partial_t\psi^\varepsilon(x,0)=\psi_1^\varepsilon(x), \quad x\in\mathbb{T}^n. \quad (4.2)$$

Here the superscript ε in the wave function ψ^ε indicates the ε-dependence. As discussed in Sections 2 and 3, we discuss only the periodic domain $\mathbb{T}^n$ and state the following existence theorem.

Theorem 4.1. *Given* $(\psi_0^\varepsilon,\psi_1^\varepsilon)\in H^1(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$ *and* $\frac{|\psi_0^\varepsilon|^2-1}{\varepsilon}\in L^2(\mathbb{T}^n)$, *there exists a function* ψ^ε *such that*

$$\psi^\varepsilon\in L^\infty([0,T];H^1(\mathbb{T}^n))\cap C([0,T];L^2(\mathbb{T}^n)), \quad (4.3)$$

$$\partial_t\psi^\varepsilon\in L^\infty([0,T];L^2(\mathbb{T}^n))\cap C([0,T];H^{-1}(\mathbb{T}^n)), \quad (4.4)$$

$$\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\in L^\infty([0,T];L^2(\mathbb{T}^n)), \quad (4.5)$$

and satisfies the weak formulation of (4.1) *given by*

$$0 = -\frac{1}{2}\varepsilon^{1+2\alpha}\left\langle \partial_t \psi^\varepsilon(\cdot, t_2) - \partial_t \psi^\varepsilon(\cdot, t_1), \varphi \right\rangle + i\left\langle \psi^\varepsilon(\cdot, t_2) - \psi^\varepsilon(\cdot, t_1), \varphi \right\rangle$$
$$-\frac{\varepsilon}{2}\int_{t_1}^{t_2}\left\langle \nabla\psi^\varepsilon(\cdot, \tau), \nabla\varphi \right\rangle \mathrm{d}\tau - \int_{t_1}^{t_2}\left\langle \left(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\right)\psi^\varepsilon(\cdot, \tau), \varphi \right\rangle \mathrm{d}\tau, \quad (4.6)$$

for every $[t_1, t_2] \subset [0, T]$ *and for all* $\varphi \in C_0^\infty(\mathbb{T}^n)$. *Moreover, it satisfies the charge-energy inequality*

$$\int_{\mathbb{T}^n} |\psi^\varepsilon|^2 + \varepsilon^{2\alpha}|\partial_t\psi^\varepsilon|^2 + |\nabla\psi^\varepsilon|^2 + \frac{1}{2}\left(\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\right)^2 \mathrm{d}x$$
$$\leqq 2C_1 + (1 + 2\varepsilon^{2+2\alpha})C_2 \quad (4.7)$$

where C_1 *and* C_2 *denote the initial charge and energy given respectively by*

$$C_1 = \int_{\mathbb{T}^n} |\psi_0^\varepsilon|^2 + \varepsilon^{1+2\alpha}\frac{i}{2}\left(\psi_1^\varepsilon\overline{\psi_0^\varepsilon} - \overline{\psi_1^\varepsilon}\psi_0^\varepsilon\right)\mathrm{d}x,$$
$$C_2 = \int_{\mathbb{T}^n} \varepsilon^{2\alpha}|\psi_1^\varepsilon|^2 + |\nabla\psi_0^\varepsilon|^2 + \frac{1}{2}\left(\frac{|\psi_0^\varepsilon|^2 - 1}{\varepsilon}\right)^2 \mathrm{d}x. \quad (4.8)$$

To study the nonrelativistic-semiclassical limit, we still assume $|\psi_0^\varepsilon| = |\psi_0| = 1$ and $(\psi_0^\varepsilon, \psi_1^\varepsilon) \to (\psi_0, 0)$ in $H^1(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ as $\varepsilon \to 0$. It follows immediately from the charge-energy inequality (4.7) that

$$\{\psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0, T]; H^1(\mathbb{T}^n)\big), \quad (4.9)$$
$$\{\varepsilon^\alpha\partial_t\psi^\varepsilon\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0, T]; L^2(\mathbb{T}^n)\big), \quad (4.10)$$
$$\left\{\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon}\right\}_\varepsilon \quad \text{is bounded in} \quad L^\infty\big([0, T]; L^2(\mathbb{T}^n)\big). \quad (4.11)$$

We deduce from (4.11) that

$$|\psi^\varepsilon|^2 \to 1 \quad \text{almost everywhere and strongly in } L^2(\mathbb{T}^n)$$

as ε tends to 0. As discussed above (4.11) shows only that the quantity $\{\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\}_\varepsilon$ is a weakly relative compact set in $L^\infty([0, T]; L^2(\mathbb{T}^n))$, then (up to a subsequence) the sequence $\{\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\}_\varepsilon$ converges weakly $*$ to some function w in $L^\infty([0, T]; L^2(\mathbb{T}^n))$. In order to find w explicitly, we rewrite the conservation of charge as

$$\frac{|\psi^\varepsilon|^2 - 1}{\varepsilon} = -Z(\psi^\varepsilon) + Z(\psi^\varepsilon(x, 0)) - \int_0^t \operatorname{div} W(\psi^\varepsilon)\mathrm{d}\tau, \quad (4.12)$$

where $Z(\psi^\varepsilon)$ and $W(\psi^\varepsilon)$ are defined similarly to (2.16). We deduce from (4.9) and (4.10) that $Z(\psi^\varepsilon) \rightharpoonup 0$ in $\mathcal{D}'((0, T) \times \mathbb{T}^n)$, and the same discussion as Lemma 2.2, we can prove

$$\int_0^t \text{div } W(\psi^{\hbar})\text{d}\tau \rightharpoonup \int_0^t \text{div } W(\psi)\text{d}\tau$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$, hence

$$\frac{|\psi^\varepsilon|^2-1}{\varepsilon} \rightharpoonup -\int_0^t \text{div } W(\psi)\text{d}\tau \tag{4.13}$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. Therefore

$$\frac{|\psi^\varepsilon|^2-1}{\varepsilon} \rightharpoonup -\int_0^t \text{div } W(\psi)\text{d}\tau \tag{4.14}$$

weakly $*$ in $L^\infty([0,T];L^2(\mathbb{T}^n))$, and thus

$$\left(\frac{|\psi^\varepsilon|^2-1}{\varepsilon}\right)\psi^\varepsilon \rightharpoonup -\psi\int_0^t \text{div } W(\psi)\text{d}\tau \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^n). \tag{4.15}$$

By combining the above convergent results, one can pass to the limit in each term of (4.6) and conclude that the limit ψ satisfies $|\psi|=1$ almost everywhere and

$$i\partial_t\psi + \left(\int_0^t \text{div } W(\psi)\text{d}\tau\right)\psi = 0 \tag{4.16}$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^n)$. Similar to the discussion in the case of the semiclassical limit, using $|\psi|=|\psi_0|=1$ almost everywhere, we can prove that ψ satisfies the wave map equation

$$\partial_t^2\psi - \Delta\psi = \left(|\nabla\psi|^2 - |\partial_t\psi|^2\right)\psi\,, \quad |\psi|=1 \quad \text{almost everywhere} \tag{4.17}$$

$$\psi(x,0)=\psi_0(x), \quad \partial_t\psi(x,0)=0, \quad x\in\mathbb{T}^n. \tag{4.18}$$

Using the fact $|\psi|=|\psi_0|=1$ again and writing $\psi=e^{i\theta}$ shows

$$\partial_t^2\theta = \Delta\theta, \quad \theta(x,0)=\arg\psi_0, \quad \partial_t\theta(x,0)=0. \tag{4.19}$$

Theorem 4.2. *Let* $(\psi_0^\varepsilon,\psi_1^\varepsilon)\in H^1(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$, $|\psi_0^\varepsilon|=1$, *and* $(\psi_0^\varepsilon,\psi_1^\varepsilon)\to(\psi_0,0)$ *in* $H^1(\mathbb{T}^n)\oplus L^2(\mathbb{T}^n)$, $|\psi_0|=1$, *and let* ψ^ε *be the corresponding weak solution of the modulated cubic nonlinear Klein–Gordon equation* (4.1)–(4.2)*. Then the weak limit* ψ *satisfies* $|\psi|=1$ *almost everywhere and solves the wave map* (4.17)–(4.18)*. Moreover, let* $\psi=e^{i\theta}$*; then the phase function* θ *satisfies the wave equation* (4.19).

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Appendix: Proof of Theorem 3.1

The goal of this appendix is a short and direct proof of Theorem 3.1. (The proof of Theorem 2.1 or Theorem 4.1 proceeds along the same lines with modification.) We employ the Fourier–Galerkin method to construct a sequence of approximation solutions, and use the compactness argument to prove the existence of weak solutions; this technique was applied to complex Ginzburg–Landau equation by DOERING ET AL. in [4]. The light speed c and Planck's constant $\hbar$ are assumed to be fixed numbers (or both equal to 1 after proper rescaling) and the proof is decomposed into four steps.

Step 1. *Construction of approximation solutions* ψ^δ *by the Fourier–Galerkin method.* Let P_δ denote the L^2 orthogonal projection onto the span of all Fourier modes of wave vector ξ with $|\xi| \leqq 1/\delta$. Define $\psi_0^\delta = P_\delta\psi_0$, $\psi_1^\delta = P_\delta\psi_1$ and let $\psi^\delta = \psi^\delta(t)$ be the unique solution of the ODE

$$-\frac{\hbar^2}{2c^2}\partial_t^2\psi^\delta + i\hbar\partial_t\psi^\delta + \frac{\hbar^2}{2}\Delta\psi^\delta - P_\delta(|\psi^\delta|^p\psi^\delta) = 0, \tag{5.1}$$

with initial conditions

$$\psi^\delta(x,0) = \psi_0^\delta(x), \quad \partial_t\psi^\delta(x,0) = \psi_1^\delta(x), \quad x \in \mathbb{T}^n. \tag{5.2}$$

The regularized initial data are chosen such that $(\psi_0^\delta, \psi_1^\delta) \to (\psi_0, \psi_1)$ in $H^1 \cap L^{p+2}(\mathbb{T}^n) \oplus L^2(\mathbb{T}^n)$ as δ tends to zero. These solutions will satisfy the regularized version of the weak formulation

$$\begin{aligned}0 = &-\frac{\hbar^2}{2c^2}\left\langle\partial_t\psi^\delta(\cdot,t_2) - \partial_t\psi^\delta(\cdot,t_1), \varphi\right\rangle + i\hbar\left\langle\psi^\delta(\cdot,t_2) - \psi^\delta(\cdot,t_1), \varphi\right\rangle\\ &-\frac{\hbar^2}{2}\int_{t_1}^{t_2}\left\langle\nabla\psi^\delta(\cdot,\tau), \nabla\varphi\right\rangle \mathrm{d}\tau - \int_{t_1}^{t_2}\left\langle|\psi^\delta|^p\psi^\delta(\cdot,\tau), \varphi\right\rangle \mathrm{d}\tau\end{aligned}$$

for every $[t_1, t_2] \subset [0, \infty)$ and for all $\varphi \in C_0^\infty(\mathbb{T}^n)$. Furthermore, the approximate solution $\psi^\delta \equiv P_\delta\psi$ will converge to ψ in C^∞ as δ tends to zero and satisfies the conservation laws of charge and energy given, respectively, by

$$\int_{\mathbb{T}^n} |\psi^\delta|^2 + \frac{\hbar}{c^2}\frac{i}{2}\left(\overline{\psi^\delta}\partial_t\psi^\delta - \psi^\delta\partial_t\overline{\psi^\delta}\right)\mathrm{d}x = C_1^\delta, \tag{5.3}$$

$$\int_{\mathbb{T}^n} \frac{\hbar^2}{2c^2}|\partial_t\psi^\delta|^2 + \frac{\hbar^2}{2}|\nabla\psi^\delta|^2 + \frac{1}{p+2}|\psi^\delta|^{p+2}\mathrm{d}x = C_2^\delta. \tag{5.4}$$

Here C_1^δ and C_2^δ denote the initial charge and initial energy, respectively. By Young's inequality and uniform boundedness of the charge and energy, we derive

$$\begin{aligned}\int_{\mathbb{T}^n} |\psi^\delta|^2\mathrm{d}x &\leqq \frac{\hbar}{c^2}\int_{\mathbb{T}^n} |\partial_t\psi^\delta||\psi^\delta|\mathrm{d}x + C_1^\delta\\ &\leqq \frac{1}{2}\int_{\mathbb{T}^n} |\psi^\delta|^2 + \frac{\hbar^2}{c^4}|\partial_t\psi^\delta|^2\mathrm{d}x + C_1^\delta\\ &\leqq \frac{1}{2}\int_{\mathbb{T}^n} |\psi^\delta|^2\mathrm{d}x + \frac{1}{c^2}C_2^\delta + C_1^\delta,\end{aligned}$$

that is,

$$\int_{\mathbb{T}^n} |\psi^\delta|^2 \mathrm{d}x \leqq 2C_1^\delta + \frac{2}{c^2} C_2^\delta. \tag{5.5}$$

Adding (5.4) and (5.5) together, we have shown that the approximate solution ψ^δ satisfies the charge-energy inequality

$$\int_{\mathbb{T}^n} |\psi^\delta|^2 + \frac{\hbar^2}{2c^2}|\partial_t \psi^\delta|^2 + \frac{\hbar^2}{2}|\nabla \psi^\delta|^2 + \frac{|\psi^\delta|^{p+2}}{p+2}\mathrm{d}x \leqq 2C_1^\delta + \left(1 + \frac{2}{c^2}\right) C_2^\delta. \tag{5.6}$$

Step 2. *Show that* $\{\psi^\delta\}$ *is a relatively compact set in* $C([0,T]; L^2(\mathbb{T}^n)) \cap L^\infty([0,T]; L^{p+1}(\mathbb{T}^n))$ *and* $\{\partial_t \psi^\delta\}$ *is relatively compact in* $C([0,T]; H^{-1}(\mathbb{T}^n))$. We deduce from the charge-energy bound (5.6) that

$$\{\psi^\delta\}_\delta \quad \text{is bounded in} \quad L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big), \tag{5.7}$$

$$\{\partial_t \psi^\delta\}_\delta \quad \text{is bounded in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big), \tag{5.8}$$

$$\{\psi^\delta\}_\delta \quad \text{is bounded in} \quad L^\infty\big([0,T]; L^{p+2}(\mathbb{T}^n)\big). \tag{5.9}$$

It follows from (5.7)–(5.9) and the classical compactness argument that there exists a subsequence of $\{\psi^\delta\}_\delta$, which we still denote by $\{\psi^\delta\}_\delta$, and $\psi \in L^\infty([0,T]; H^1(\mathbb{T}^n))$, $\partial_t \psi \in L^\infty([0,T]; L^2(\mathbb{T}^n))$ such that

$$\psi^\delta \rightharpoonup \psi \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T]; H^1(\mathbb{T}^n)\big), \tag{5.10}$$

$$\partial_t \psi^\delta \rightharpoonup \partial_t \psi \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T]; L^2(\mathbb{T}^n)\big), \tag{5.11}$$

$$\psi^\delta \rightharpoonup \psi \quad \text{weakly} * \text{in} \quad L^\infty\big([0,T]; L^{p+2}(\mathbb{T}^n)\big). \tag{5.12}$$

Using the same technique discussed in Lemma 2.1 and Lemma 3.2, we can apply the Arzela–Ascoli theorem and interpolation theorem to conclude

$$\psi^\delta \to \psi \quad \text{in} \quad C\big([0,T]; L^2(\mathbb{T}^n)\big) \cap L^\infty\big([0,T]; L^{p+1}(\mathbb{T}^n)\big).$$

The convergence of $\partial_t \psi^\delta \to \partial_t \psi$ in $C([0,T]; w - L^2(\mathbb{T}^n))$ also follows by the Arzela–Ascoli theorem. First, it is obvious that $\{\partial_t \psi^\delta(t)\}_\delta$ is a relatively compact set in w-$L^2(\mathbb{T}^n)$ for all $t \geqq 0$ by energy bound. To show $\{\partial_t \psi^\delta\}$ is equicontinuous in $C([0,T]; w\text{-}L^2(\mathbb{T}^n))$, let $A \subset C_0^\infty(\mathbb{T}^n)$ be an enumerable set which is dense in $L^2(\mathbb{T}^n)$, then for any $\rho \in A$, we have

$$\begin{aligned}\frac{\hbar^2}{2c^2}\big\langle \partial_t \psi^\delta(\cdot, t_2) - \partial_t \psi^\delta(\cdot, t_1), \rho\big\rangle = i\hbar \int_{t_1}^{t_2} \big\langle \partial_t \psi^\delta(\cdot, \tau), \rho\big\rangle \mathrm{d}\tau \\ -\frac{\hbar^2}{2}\int_{t_1}^{t_2} \big\langle \nabla \psi^\delta(\cdot, \tau), \nabla \rho\big\rangle \mathrm{d}\tau \\ -\int_{t_1}^{t_2} \big\langle |\psi^\delta|^p \psi^\delta(\cdot, \tau), \rho\big\rangle \mathrm{d}\tau,\end{aligned}$$

so we derive the estimate

$$|\langle \partial_t \psi^\delta(\cdot, t_2) - \partial_t \psi^\delta(\cdot, t_1), \rho\rangle| \lesssim |t_2 - t_1| \left(\|\rho\|_{H^1(\mathbb{T}^n)} + \|\rho\|_{L^\infty(\mathbb{T}^n)}\right).$$

The rest follows by density argument, and this proves the equicontinuity of $\{\partial_t \psi^\delta\}$ in $C([0, T]; w\text{-}L^2(\mathbb{T}^n))$, so $\partial_t \psi^\delta \to \partial_t \psi$ in $C([0, T]; w\text{-}L^2(\mathbb{T}^n))$. Indeed, we have the strong convergence $\partial_t \psi^\delta \to \partial_t \psi$ in $C([0, T]; H^{-1}(\mathbb{T}^n))$ by the Rellich lemma, which states that $L^2 \hookrightarrow H^{-1}$ is a compact embedding.

Step 3. *Passage to the limit* ($\delta \to 0$). The weak $*$ convergence of ψ^δ in $L^\infty([0, T]; H^1(\mathbb{T}^n))$, the strong convergence of ψ^δ in $C([0, T]; L^2(\mathbb{T}^n))$ and the strong convergence of $\partial_t \psi^\delta$ in $C([0, T]; H^{-1}(\mathbb{T}^n))$ give the following convergent results:

$$\int_{t_1}^{t_2} \left\langle \nabla \psi^\delta(\cdot, \tau), \nabla \varphi \right\rangle \mathrm{d}\tau \to \int_{t_1}^{t_2} \langle \nabla \psi(\cdot, \tau), \nabla \varphi\rangle \, \mathrm{d}\tau, \tag{5.13}$$

$$\left\langle \psi^\delta(\cdot, t_2) - \psi^\delta(\cdot, t_1), \varphi \right\rangle \to \langle \psi(\cdot, t_2) - \psi(\cdot, t_1), \varphi\rangle , \tag{5.14}$$

$$\left\langle \partial_t \psi^\delta(\cdot, t_2) - \partial_t \psi^\delta(\cdot, t_1), \varphi \right\rangle \to \langle \partial_t \psi(\cdot, t_2) - \partial_t \psi(\cdot, t_1), \varphi\rangle . \tag{5.15}$$

Moreover, the same argument we applied to the non-relativistic limit shows $|\psi^\delta|^p \psi^\delta \to |\psi|^p \psi$ in the sense of distribution, that is,

$$\int_{t_1}^{t_2} \left\langle |\psi^\delta|^p \psi^\delta(\cdot, \tau), \varphi \right\rangle \mathrm{d}\tau \to \int_{t_1}^{t_2} \left\langle |\psi|^p \psi(\cdot, \tau), \varphi \right\rangle \mathrm{d}\tau. \tag{5.16}$$

Therefore ψ satisfies the weak formulation of (3.1).

Step 4. *Proof of the charge-energy inequality.* The strong convergence of ψ^δ in $C([0, T]; L^2(\mathbb{T}^n))$ implies

$$\int_{\mathbb{T}^n} |\psi^\delta|^2 \mathrm{d}x \to \int_{\mathbb{T}^n} |\psi|^2 \mathrm{d}x. \tag{5.17}$$

The weak convergence of ψ^δ in $L^\infty([0, T]; H^1(\mathbb{T}^n)) \cap L^\infty([0, T]; L^{p+2}(\mathbb{T}^n))$, together with the fact that the norm of the weak limit of a sequence is a lower bound for the inferior limit of the norms, yields

$$\int_{\mathbb{T}^n} |\nabla \psi|^2 \mathrm{d}x \leqq \liminf_{\delta \to 0} \int_{\mathbb{T}^n} |\nabla \psi^\delta|^2 \mathrm{d}x, \tag{5.18}$$

$$\int_{\mathbb{T}^n} |\psi|^{p+2} \mathrm{d}x \leqq \liminf_{\delta \to 0} \int_{\mathbb{T}^n} |\psi^\delta|^{p+2} \mathrm{d}x. \tag{5.19}$$

Similarly, the weak convergence of $\partial_t \psi^\delta$ in $L^\infty([0, T]; L^2(\mathbb{T}^n))$ implies

$$\int_{\mathbb{T}^n} |\partial_t \psi|^2 \mathrm{d}x \leqq \liminf_{\delta \to 0} \int_{\mathbb{T}^n} |\partial_t \psi^\delta|^2 \mathrm{d}x. \tag{5.20}$$

By combining (5.6) and the above inequalities, we obtain the charge-energy inequality

$$\int_{\mathbb{T}^n} |\psi|^2 + \frac{\hbar^2}{2c^2} |\partial_t \psi|^2 + \frac{\hbar^2}{2} |\nabla \psi|^2 + \frac{|\psi|^{p+2}}{p+2} \mathrm{d}x \leqq 2C_1 + \left(1 + \frac{2}{c^2}\right) C_2, \tag{5.21}$$

where the two constants

$$C_1 = \int_{\mathbb{T}^n} |\psi_0|^2 + \frac{\hbar}{c^2}\frac{i}{2}\left(\psi_1\overline{\psi_0} - \overline{\psi_1}\psi_0\right) \mathrm{d}x,$$
$$C_2 = \int_{\mathbb{T}^n} \frac{\hbar^2}{2c^2}|\psi_1|^2 + \frac{\hbar^2}{2}|\nabla\psi_0|^2 + \frac{1}{p+2}|\psi_0|^{p+2}\mathrm{d}x, \tag{5.22}$$

represent the initial charge and energy, respectively. This completes the proof of Theorem 3.1. □

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