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Stationarization of Stochastic Sequences with Wide-sense Stationary Increments or Jumps by Discrete Wavelet Transforms

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ABSTRACT: *Owing to most physical phenomena observed as nonstationary processes and the form of discrete sequences, it becomes realistic to process the nonstationary sequences in the laboratory if there exists a bijective transformation for stationarization. In this work, our study is emphasized on the class of nonstationary one-dimensional random sequences with wide-sense stationary increments (WSSI), wide-sense stationary jumps (WSSJ) and a famous case, the fractional Brownian motion (FBM) process. Also, the concept of linear algebra is applied to process the stationarization concisely. Our goal is to derive a stationarization theorem developed by linear operators such that a nonstationary sequence with WSSI/WSSJ may be stationarized by an easily realizable perfect reconstruction-quadrature mirror filter structure of the discrete wavelet transform. Some examples for FBM processes and nonstationary signals generated by autoregressive integrated moving average models are provided to demonstrate the stationarization.*

I. Introduction

In most physical phenomena, there are strong long-term dependencies involved and the $1/f\alpha$ spectral behaviors observed over a wide range of frequencies (**1**). The application of fractal characteristics extended to the measurement by power spectral statistics are examples such as measurements in geology and geophysics, the spatial distribution of oxygen isotope ratios in sea floor coves, the distribution of stratigraphic hiatuses, interpolating between measured data (**1**), detection of sea-surface targets and image texture analysis. A convenient modeling to deal with these kinds of process has been developed by Mandelbrot and Van Ness (**2**) and is referred to as fractional Brownian motion (FBM). An important problem appearing in the applications of stochastic processes is the estimation of various statistical parameters in terms of real data. Most parameters are expressed as the moment values. However, the calculation of power

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spectrum of a nonstationary process is not defined in stochastic processing. There have been many methods developed to approximate the power spectrum of a nonstationary process, such as time-interval approximation (**3**), Wiener–Khinchin spectrum (**4**) and Wigner–Ville spectrum (**5**). But the functions of these estimations are not mature since the time interval is not long enough.

Recently, the novelty of wavelet transform emerged as a powerful tool for nonstationary signal analysis and has been considered in the literature (**6–10**). Provided with a time-scale decomposition of processes, the properties of wavelet transform can be read off. In a recent correspondence, Flandrin (**11**) proposed that the stationarization of an FBM process may be approximated by discrete wavelet transform (DWT). However, the result from Flandrin is available only for the FBM process corresponding to is just an example of nonstationary process with WSSI. Until 1993, the second-order statistics of a process characterized through its continuous wavelet transform are further approached by Houdré (**12, 13**) in which some strict assumptions are still necessary to be conditioned. Through the application of continuous wavelet transform (CWT), not only the self-similarity is preserved (**14, 15**), but also a nonstationary process with wide-sense stationary increments (WSSI) is further transformed to a wide-sense stationary (WSS) process (**12, 13, 16**) as well.

However, the CWT at least suffers from two drawbacks of redundancy and heavy load of computation, and these two disadvantages possibly affect one another. Therefore, the DWT proposed in (**7, 17, 18, 9**) is advanced to overcome these problems, so that the computational complexity and redundant information can be reduced dramatically. Additionally, multiresolution signal processing applied to implement the DWT will promote the function efficiency with redundancy nearly cleared. Through the multiresolution analysis (**19, 20**), the DWT will be well performed in the structure of subband filter system, called PR-QMF (perfect reconstruction-quadrature mirror filter) (**9**) which is a realizable finite-impulse-response (FIR) filter system.

The concept of linear algebra by linear operators introduced below will be used to show that a nonstationary stochastic sequence with WSSI or wide-sense stationary jumps (WSSJ) may be stationarized by using a PR-QMF structure of DWT. Some examples such as sampled FBM sequence, WSS sequence generated by ARMA model, stochastic sequences with WSSI and WSSJ generated by ARIMA (auto-regressive integrated moving average) models (**16**) are provided to present the stationarization procedures. Here, the ARIMA model (**16**) is characterized by fitting an ARMA model to increments or jumps, in which the signals generated by the ARIMA models using white stationary kernels are nonstationary with WSSI or WSSJ.

In Section 2, we summarize the notation and brief definitions of nonstationary processes with WSSI/WSSJ. The Discrete-time Stationarization Theorem by the DWT based on the PR-QMF structure is derived in Section 3. Some examples are shown in Section 4. The conclusion and some interesting directions for the further research are given in Section 5.

II. Review

In fact, a physical measure device can only observe a signal at a finite resolution, i.e. a discrete sequence of observed data. The properties of WSSI and WSSJ for stochastic sequences are defined as follows:

Let $\zeta \in \Omega$. If $\mathbf{x}[n, \zeta]$ is a random variable for each fixed integer n in an index set $\mathbf{Z}$, then $\mathbf{x}[n, \zeta]$ is a stochastic sequence (**21**), denoted by $\mathbf{x}[n]$ for simplicity.

Definition I (**13**)

A stochastic sequence $\mathbf{x}[n]$ has WSSI if the second moment of the increment $R_x[n_1, m_1; n_2, m_2] \equiv \mathscr{E}\{(\mathbf{x}[n_1+m_1]-\mathbf{x}[n_1])(\mathbf{x}[n_2+m_2]-\mathbf{x}[n_2])^*\}$ depends on n_1 and n_2 only through n_1-n_2, i.e. $R_x[n_1, m_1; n_2, m_2] = R_x[n_1-n_2, m_1; 0, m_2]$, for all $n_1, n_2, m_1, m_2 \in \mathbf{Z}$.

Definition II (**16**)

A stochastic sequence $\mathbf{x}[n]$ has WSSJ if the jump $x[n]+x[n-m]$, for all $n,m \in \mathbf{Z}$, is WSS.

Furthermore, the definition for the discrete wide-sense cyclostationary (WSCS) is directly followed the continuous case (**23**).

Definition III

A process $\mathbf{x}[n]$ is called WSCS with period T if, for every integer m,

$$\mathscr{E}\{\mathbf{x}[n+mT]\} = \mathscr{E}\{\mathbf{x}[n]\}, \quad R_x[n_1+mT, n_2+mT] = R_x[n_1, n_2].$$

A mathematical tool proposed by Mallat (**9**) to adopt the use of wavelet basis applied in the Discrete-time Stationarization Theorem is extracted here.

III. Stationarization by DWT with PR-QMF Structure

Let the scaling and wavelet coefficients of a stochastic sequence $\mathbf{x}[n]$, denoted by a_m, k and d_m, k, respectively, be defined by

$$a_{m,k} = \sum_{i=-\infty}^{\infty} h[i-2k]a_{m-1,i}, \quad d_{m,k} = \sum_{i=-\infty}^{\infty} g[i-2k]a_{m-1,i}, \tag{1}$$

for $m \geqslant 0$, with the reasonable assumption (**22**)

$$a_{0,k} = \int_{-\infty}^{\infty} x(t)\phi(t-k)\,\mathrm{d}t = x[k].$$

Also, the autocorrelation and crosscorrelation functions of a_m, k and d_m, k are defined as

$$R_a[k_1, k_2]_{m_1,m_2} \equiv \mathscr{E}\{a_{m_1,k_1}a^*_{m_2,k_2}\}, \quad R_d[k_1, k_2]_{m_1,m_2} \equiv \mathscr{E}\{d_{m_1,k_1}d^*_{m_2,k_2}\},$$

$$R_{ad}[k_1, k_2]_{m_1,m_2} \equiv \mathscr{E}\{a_{m_1,k_1}d^*_{m_2,k_2}\}. \tag{2}$$

Let $\mathbf{X}$, $\mathbf{a_m}$ and $\mathbf{d_m}$ be the stochastic vectors of $\{\mathbf{x}[n]\}_{n\in\mathbf{Z}}$, $\{a_{m,k}\}_{k\in\mathbf{Z}}$ and $\{d_{m,k}\}_{k\in\mathbf{Z}}$, respectively. Then a_m, k and d_m, k, for all $k \in \mathbf{Z}$ and $m \geq 0$, can be expressed as

$$\mathbf{a_m} = H\mathbf{a_{m-1}} = H^m\mathbf{X}, \tag{3}$$

$$\mathbf{d_m} = G\mathbf{a_{m-1}} = GH^{m-1}\mathbf{X}, \tag{4}$$

where the linear operators H and G can be found as infinite matrices whose entries are defined as

$$[H]_{k,i} \equiv h[i-2k], \quad [G]_{k,i} \equiv g[i-2k], \quad \text{for all } i, k \in \mathbf{Z}, \tag{5}$$

and satisfy

$$[H]_{k,i} = [H]_{k+1,i+2}, \quad [G]_{k,i} = [G]_{k+1,i+2}, \quad \text{for all } k, i \in \mathbf{Z}. \tag{6}$$

In this framework, the reconstruction formula in (**11**) can be written as

$$\mathbf{X} = H^*\mathbf{a}_1 + G^*\mathbf{d}_1. \tag{7}$$

Now, let $\mathbf{R}_\mathbf{X} \equiv \mathscr{E}\{\mathbf{XX}^*\}$ be the correlation matrix of $\mathbf{X}$. Define $\mathbf{R}_{\mathbf{a}(\mathbf{m}_1,\mathbf{m}_2)}$, as the correlation matrix of $\mathbf{a}_{\mathbf{m}_1}$ and $\mathbf{a}_{\mathbf{m}_2}$, and similarly for $\mathbf{R}_{\mathbf{d}(\mathbf{m}_1,\mathbf{m}_2)}$ and $\mathbf{R}_{\mathbf{ad}(\mathbf{m}_1,\mathbf{m}_2)}$. Then these correlation matrices have the forms as in the following lemma used by the PR-QMF structure according to the recursion of the coefficients $a_{m,k}$ and $d_{m,k}$.

Lemma I

With the above definitions of the infinite matrices H and G, and the correlation matrices $\mathbf{R}_X$, $\mathbf{R}_{\mathbf{a}(\mathbf{m}_1,\mathbf{m}_2)}$ and $\mathbf{R}_{\mathbf{d}(\mathbf{m}_1,\mathbf{m}_2)}$ are linear transformations among $\mathbf{R}_\mathbf{X}$, $\mathbf{R}_{\mathbf{a}(\mathbf{m}_1,\mathbf{m}_2)}$ and $\mathbf{R}_{\mathbf{d}(\mathbf{m}_1,\mathbf{m}_2)}$ and recursive algorithms derived as follows:

$$\mathbf{R}_{\mathbf{a}(\mathbf{m}_1,\mathbf{m}_2)} = \mathrm{H}^{m_1-1}\mathbf{R}_{\mathbf{a}(1,1)}(H^*)^{m_2-1} = H^{m_1}\mathbf{R}_\mathbf{X}(H^*)^{m_2}, \tag{8}$$

$$\mathbf{R}_{\mathbf{d}(\mathbf{m}_1,\mathbf{m}_2)} = GH^{m_1-2}\mathbf{R}_{\mathbf{a}(1,1)}(H^*)^{m_2-2}G^* = GH^{m_1-1}\mathbf{R}_\mathbf{X}(H^*)^{m_2-1}G^*, \tag{9}$$

$$\mathbf{R}_{\mathbf{ad}(\mathbf{m}_1,\mathbf{m}_2)} = H^{m_1-1}\mathbf{R}_{\mathbf{a}(1,1)}(H^*)^{m_2-2}G^* = H^{m_1}\mathbf{R}_\mathbf{X}(H^*)^{m_2-1}G^*, \tag{10}$$

provided that the initial condition $\mathbf{R}_{\mathbf{a}(1,1)}$ *is given.*

Proof: The results are obtained directly by recursive manipulation from Eqs (3) and (4) and (**20, 9**). ■

The main result derived from this section is the following stationarization theorem called the Discrete-time Stationarization Theorem (DTST), in which condition (D) is restricted to the signal itself to ensure the existence of the DWT. The notion of 'cross WSS' means that, for any two stochastic sequences $\mathbf{x}[n]$ and $\mathbf{y}[m]$, $R_{xy}[n,m] \equiv E\{\mathbf{x}[n]\mathbf{y}^*[m]\} = R_{xy}[n-m]$.

Theorem I (Discrete-time Stationarization Theorem, DTST)

Let $\mathbf{x}[n]$, $n \in \mathbf{Z}$, be a stochastic sequence with constant mean and autocorrelation function $R_x[n,m] \equiv \mathscr{E}\{\mathbf{x}[n]\mathbf{x}^*[m]\}$. Suppose that $\{2^{m/2}\phi(2^m t - k)\}_{k\in\mathbf{Z}}$ is a compactly supported orthonormal wavelet basis as in (**9**), two finite-impulse response (FIR) filters h and g satisfy the perfect reconstruction condition as in (**9**), $\mathbf{x}[n]$ satisfies the condition

$$\text{(D)}: \quad \frac{R_x[n_1,n_2]}{(1+n_1^2+n_2^2)^{N_R}} \in l^\infty, \quad \text{for some } N_R > 0, N_R \in \mathbf{Z} \text{ and for all } n_1, n_2 \in \mathbf{Z},$$

and the scaling and wavelet coefficients $a_{m,k}$ and $d_{m,k}$ are defined as in Eq (2). Let m_1 and m_2 be positive integers.

(G1): If $\mathbf{x}[n]$ is WSS then $a_{m,k}$ and $d_{m,k}$ have constant means for all m and k, and the entries of correlation matrices of $a_{m,k}$ and $d_{m,k}$ satisfy

$$[\mathbf{R}_{\mathbf{a}(\mathbf{m}_1,\mathbf{m}_2)}]_{k_1,k_2} = [\mathbf{R}_{\mathbf{a}(\mathbf{m}_1,\mathbf{m}_2)}]_{k_1+2^{m_2},k_2+2^{m_1}}, \tag{11}$$

$$[\mathbf{R}_{\mathbf{d}(\mathbf{m}_1,\mathbf{m}_2)}]_{k_1,k_2} = [\mathbf{R}_{\mathbf{d}(\mathbf{m}_1,\mathbf{m}_2)}]_{k_1+2^{m_2},k_2+2^{m_1}}, \tag{12}$$

$$[\mathbf{R}_{\mathbf{ad}(\mathbf{m}_1,\mathbf{m}_2)}]_{k_1,k_2} = [\mathbf{R}_{\mathbf{ad}(\mathbf{m}_1,\mathbf{m}_2)}]_{k_1+2^{m_2},k_2+2^{m_1}}, \tag{13}$$

for all k_1, $k_2 \in \mathbf{Z}$, which means that these entries are function of $k_2 - 2^{m_1 - m_2}, k_1$, for $m_1 \geq m_2$, or $k_1 - 2^{m_2 - m_1}, k_2$, for $m_1 < m_2$.

(G2): If $\mathbf{x}[n]$ has WSSI then the wavelet coefficient $d_{m,k}$ has constant mean for all m and k, and the entries of the correlation matrices of $d_{m,k}$ satisfy

$$[\mathbf{R_{d(m_1, m_2)}}]_{k_1,k_2} = [\mathbf{R_{d(m_1, m_2)}}]_{k_1+2^{m_2}, k_2+2^{m_1}}, \tag{14}$$

for all k_1, $k_2 \in \mathbf{Z}$, i.e. $[\mathbf{R_{d(m_1,m_2)}}]_{k_1,k_2}$ is a function of $k_2 - 2^{m_1 - m_2}, k_1$, for $m \geqslant m_2$, or $k_1 - 2^{m_2 - m_1}, k_2$, for $m_1 < m_2$.

(G3): If $a_{m,k}$ and $d_{m,k}$ have constant means, and the entries of correlation matrices of $a_{m,k}$ and $d_{m,k}$ satisfy Eqs (11)–(13), for all m and k, then $\mathbf{x}[n]$ is WSCS with period 2.

Moreover, as a special case, for any positive integers $m_1 = m_2 = m$, the results become interesting as follows:

(S1): If $\mathbf{x}[n]$ is WSS, then the scaling coefficient $a_{m,k}$ and wavelet coefficient $d_{m,k}$ are WSS and cross WSS, for all $k \in \mathbf{Z}$.

(S2): If $\mathbf{x}[n]$ has WSSI, then the wavelet coefficient $d_{m,k}$ is WSS for all $k \in \mathbf{Z}$.

(S3): If the scaling coefficient $a_{m,k}$ and wavelet coefficient $d_{m,k}$ are WSS and cross WSS for all $k \in \mathbf{Z}$, then $\mathbf{x}[n]$ is WSCS with period 2.

Proof: We begin by proving (G1). Since $\mathbf{x}[n]$ is WSS with constant mean, its correlation matrix $\mathbf{R}_X$ is Hermitian Toeplitz.

(i) Based on Eqs (3) and (4), we have

$$\mathscr{E}_x\{\mathbf{a_m}\} = H^m \mathscr{E}_x\{\mathbf{X}\}, \tag{15}$$

$$\mathscr{E}_x\{\mathbf{d_m}\} = GH^{m-1} \mathscr{E}_x\{\mathbf{X}\}, \tag{16}$$

for all $m, k \in \mathbf{Z}$. They are constant vectors as the mean of $\mathbf{X}$ is a constant vector.

(ii) For any positive integers m_1 and m_2, the autocorrelation of $a_{m,k}$ is shown to be bounded and qualified as Eq (11) for any bounded integers k_1 and k_2 expressed as

$$\begin{aligned}
|R_a[k_1, k_2]_{m_1,m_2}| &\equiv |\mathscr{E}_x\{a_{m_1,k_1} a^*_{m_2,k_2}\}|, \\
&= \Bigg| \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} \cdots \sum_{i_{m_1}=-\infty}^{\infty} \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \cdots \sum_{l_{m_2}=-\infty}^{\infty} \\
&\quad \times (1 + i_{m_1}^2 + l_{m_2}^2)^{N_R} h[i_1 - 2k_1] h[i_2 - 2i_1] \ldots h[i_{m_1} - 2i_{m_1-1}] h[l_1 - 2k_2] \\
&\quad \times h[l_2 - 2l_1] \ldots h[l_{m_2} - 2l_{m_2} - 1] \frac{R_x[i_{m_1}, l_{m_2}]}{(1 + i_{m_1}^2 + l_{m_2}^2)_{N_R}} \Bigg| \\
&= \Bigg| \sum_{i_1=n_l}^{n_h} \sum_{i_2=n_l}^{n_h} \cdots \sum_{i_2 m_1=n_l}^{n_h} \sum_{l_1=n_l}^{n_h} \sum_{l_2=n_l}^{n_h} \cdots \sum_{l_{m_2}=n_l}^{n_h} (1 + (i_{m_1} + 2i_{m_1} - 1 + \cdots + 2^{m_1} k_1)^2 \\
&\quad + (l_{m_2} + 2_{m_2-1} + \cdots + 2^{m_2} k_2)^2)^{N_R} h[i_1] h[i_2] \ldots h[i_{m_1}] h[l_1] h[l_2] \ldots h[l_{m_2}] \\
&\quad \times \frac{R_x[(i_{m_1} + 2i - m_1 - 1 + \cdots + 2^{m_1} k_1), (l_{m_2} + 2l_{m_2} - 1 + \cdots + 2^{m-2} k_2)]}{(1 + (i_{m_1} + 2i_{m_1-1} + \cdots + 2^{m_1} k_1)^2 + (l_{m_2} + 2l_{m_2} - 1 + \cdots + 2^{m_2} k_2)^2)^{N_R}} \Bigg| \\
&\quad < \infty,
\end{aligned} \tag{17}$$

since $R_x[n_1, n_2]/(1+n_1^2+n_2^2)^{N_R} \in l^\infty$, for some $N_R > 0$, and h is an FIR with finite length $n_h - n_l$ and k_1 and k_2 are bounded.

Now, we will prove the result (G1) by using inductive method. Let $m_1 = 1$ and $m_2 = 0$. From Eqs (8) and (6), we have

$$[\mathbf{R}_{\mathbf{a(1,0)}}]_{k_1,k_2} = [H\mathbf{R_X}]_{k_1,k_2} = \sum_i [H]_{k_1,i}[\mathbf{R_X}]_{i,k_2}$$

$$= \sum_i [H]_{k_1+1,i+2}[\mathbf{R_X}]_{i+2,k_2+2} = [\mathbf{R}_{\mathbf{a(1,0)}}]_{k_1+1,k_2+2}. \tag{18}$$

Assume that, for all $k_1, k_2 \in \mathbf{Z}$,

$$[\mathbf{R}_{\mathbf{a(m_1,m_2)}}]_{k_1,k_2} = [\mathbf{R}_{\mathbf{a(m_1,m_2)}}]_{k_1+2^{m_2},k_2+2^{m_1}}, \forall\, k_1, k_2 \in \mathbf{z}, \tag{19}$$

Then, for any positive integers m_1 and m_2, using Eqs (8), (6) and Eq (19), it yields that

$$[\mathbf{R}_{\mathbf{a(m_1+1,m_2+1)}}]_{k_1,k_2} = [H\mathbf{R}_{\mathbf{a(m_1,m_2)}}H^*]_{k_1,k_2} = \sum_i \sum_j [H]_{k_1,i}[\mathbf{R}_{\mathbf{a(m_1,m_2)}}]_{i,j}[H^*]_{j,k^2},$$

$$= \sum_i \sum_j [H]_{k_1+1,i+2}[\mathbf{R}_{\mathbf{a(m_1,m_2)}}]_{i+2^{m_2},j+2^{m_1}}[H^*]_{j+2,k_2+1}$$

$$= \sum_i \sum_j [H]_{k_1+2^{m_2+1},i+2^{m_2+2}}[\mathbf{R}_{\mathbf{a(m_1,m_2)}}]_{i+2^{m_2+2},j+2^{m_1+2}}$$

$$[H^*]_{j+2^{m_1+2},k_2+2^{m_1+1}},$$

$$= [\mathbf{R}_{\mathbf{a(m_1+1,m_2+1)}}]_{k_1+2^{m_2+1},k_2+2^{m_1+1}}. \tag{20}$$

As a result, the proof of Eq (11) is completed. The proofs of Eqs (12) and (13) are similar to the above. From Eqs (11)–(13), the numbers of the elements of these correlation matrices at (k_1, k_2) and $(k_1+2^{m_2}, k_2+2^{m_1})$ are equal, which indicates that the indices of these two elements have the relationship

$$(k_2) - 2^{m_1-m_2}(k_1) = (k_2+2^{m_1}) - 2^{m_1-m_2}(k_1+2^{m_2}), \quad \text{for } m_1 \geqslant m_2,$$

$$(k_1) - 2^{m_2-m_1}(k_2) = (k_1+2^{m_2}) - 2^{m_2-m_1}(k_2+2^{m_1}), \quad \text{for } m_1 < m_2. \tag{21}$$

It means that, for any m_1 and m_2, these entries are functions of $k_2 - 2^{m_1-m_2}k_1$, for $m_1 \geq m_2$ or $k_1 - 2^{m_2-m_1}k_2$, for $m_1 < m_2$.

Now we prove (G2). Suppose that $\mathbf{x}[n]$ has WSSI.

(iii) The mean of $d_{m,k}$ is shown to be constant as in the proof of (G1).

(iv) Also, for any positive integers m_1 and m_2, the autocorrelation of $d_{m,k}$ will be bounded with the same procedures as in the proof of (ii). Moreover, $\sum_i g[i] = 0$ so that we have, for any $u, v \in \mathbf{Z}$,

$$R_d[k_1, k_2]_{m_1,m_2} = \sum_{i_1} \sum_{l_1} g[i_1 - 2k_1]g[l_1 - 2k_2]R_a[i_1, l_1]_{m_1-1,m_2-1}$$

$$= \sum_{i_1} \sum_{i_2} \cdots \sum_{i_{m_1}} \sum_{l_1} \sum_{l_2} \cdots \sum_{l_{m_2}} g[i_1 - 2k_1]h[i_2 - 2i_1] \ldots h[i_{m_1} - 2i_{m_1} - 1]$$

$$\times g[l_1 - 2k_2]h[l_2 - 2l_1] \ldots h[l_{m_2} - 2l_{m_2} - 1](R_x[i_{m_1}, l_{m_2}] - R_x[i_{m_1}, v]$$

$$- R_x[u, l_{m_2}] + R_x[u, v]). \tag{22}$$

Since the process $\mathbf{x}$ is of WSSI, then define $\mathbf{y}_r[n] \equiv \mathbf{x}[n] - \mathbf{x}[n-r]$, for every $r \in \mathbf{Z}$. Therefore, $\mathbf{x}[n]$ has WSSI if the pair $\mathbf{y}_r$ and y_s is stationarily correlated, for every r and s, i.e. the correlation matrix $\mathbf{R}_{\mathbf{Y}_r,\mathbf{Y}_s}$ is Toeplitz, where $\mathbf{Y}_r$ and $\mathbf{Y}_s$ are the infinite vectors $\{\mathbf{y}_r[n]\}_{n\in\mathbf{Z}}$ and $\{\mathbf{y}_s[n]\}_{n\in\mathbf{Z}}$, respectively. So, choose any $u, v \in \mathbf{Z}$ such that $r = i - v, s = l - u$ and the correlation function of $\mathbf{y}_r[i]$ and $\mathbf{y}_s[l]$ satisfies $R_y[i,l] = R_x[i,l] - R_x[i,v] - R_x[u,l] + R_x[u,v]$. Therefore, from Eq (22), for all k_1 and k_2, we have the correlation matrix of d_m,k as follows:

$$\mathbf{R}_{\mathbf{d}(\mathbf{m}_1,\mathbf{m}_2)} = \mathbf{G}\mathbf{H}^{m_1-1}\mathbf{R}_{\mathbf{Y}_r,\mathbf{Y}_s}(H^*)^{m_2-1}G^*. \tag{23}$$

Based on Eq (6), it implies that the entries of GH satisfy, for all $k, i \in \mathbf{Z}$,

$$[GH]_{k,i} = \sum_j [G]_{k,j}[H]_{j,i} = \sum_j [G]_{k+1,j+2}[H]_{j+2,i+4} = [GH]_{k+1,i+4}. \tag{24}$$

By recursive straightforward manipulation, we find that the elements of GH^{m_1-1} have the following equality, for all $k, i \in \mathbf{Z}$,

$$[GH^{m_1-1}]_{k,i} = [GH^{m_1} - 1]_{k+1,i+2^{m_1}}. \tag{25}$$

Therefore, from Eqs (23) and (25), the entry of $\mathbf{R}_{\mathbf{d}(\mathbf{m}_1,\mathbf{m}_2)}$ has the relationship

$$\begin{aligned}
[\mathbf{R}_{\mathbf{d}(\mathbf{m}_1,\mathbf{m}_2)}]_{k_1,k_2} &= [GH^{m_1-1}\mathbf{R}_{\mathbf{Y}_r,\mathbf{Y}_s}(H^*)^{m_2-1}G^*]_{k_1,k_2} \\
&= \sum_i \sum_j [GH^{m_1-1}]_{k_1,i}[\mathbf{R}_{\mathbf{Y}_r,\mathbf{Y}_s}]_{i,j}[(H^*)^{m_2-1}G^*]_{j,k_2} \\
&= \sum_i \sum_j [GH^{m_1-1}]_{k_1+2^{m_2},i+2^{m_1+m_2}}[\mathbf{R}_{\mathbf{Y}_r,\mathbf{Y}_s}]_{i+2^{m_1+m_2},j+2^{m_1+m_2}} \\
&\quad [(H^*)^{m_2-1}G^*]_{j+2^{m_1+m_2},k_2+2^{m_1}} \\
&= [\mathbf{R}_{\mathbf{d}(\mathbf{m}_1,\mathbf{m}_2)}]_{k_1+2^{m_2},k_2+2^{m_1}}, \forall k_1, k_2 \in \mathbf{Z}.
\end{aligned} \tag{26}$$

Hence, the proof is completed.

Next we prove (G3). Suppose that, for all $k_1, k_2 \in \mathbf{Z}$, a_{m_1,k_1} and d_{m_2,k_2} have constant means and the correlation functions satisfy the properties in Eqs (11)–(13). From Eq (7), we obtain

$$\mathscr{E}\{\mathbf{X}\} = H^*\mathscr{E}\{\mathbf{a}_1\} + G^*\mathscr{E}\{\mathbf{d}_1\}. \tag{27}$$

Therefore, $\mathscr{E}\{\mathbf{x}[n]\}$ is also constant. Since $\mathbf{R}_{\mathbf{a}(1,1)}$, $\mathbf{R}_{\mathbf{d}(1,1)}$, $\mathbf{R}_{\mathbf{ad}(1,1)}$ and $\mathbf{R}_{\mathbf{da}(1,1)}$ are Hermitian Toeplitz, the entries at (n_1, n_2) of the correlation matrix of $\mathbf{x}[n]$ are represented as

$$\begin{aligned}
[\mathbf{R}_\mathbf{X}]_{n_1,n_2} &= [H^*\mathbf{R}_{\mathbf{a}(1,1)}H + G^*\mathbf{R}_{\mathbf{d}(1,1)}G + H^*\mathbf{R}_{\mathbf{ad}(1,1)}G + G^*\mathbf{R}_{\mathbf{da}(1,1)}H]_{n_1,n_2} \\
&= \sum_i \sum_j [H^*]_{n_1,i}[\mathbf{R}_{\mathbf{a}(1,1)}]_{i,j}[H]_{j,n_2} + \sum_i \sum_j [G^*]_{n_1,i}[\mathbf{R}_{\mathbf{d}(1,1)}]_{i,j}[G]_{j,n_2} \\
&\quad + \sum_i \sum_j [H^*]_{n_1,i}[\mathbf{R}_{\mathbf{ad}(1,1)}]_{i,j}[G]_{j,n_2} + \sum_i \sum_j [G^*]_{n_1,i}[\mathbf{R}_{\mathbf{da}(1,1)}]_{i,j}[H]_{j,n_2} \\
&= \sum_i \sum_j [H^*]_{n_1+2,i+1}[\mathbf{R}_{\mathbf{a}(1,1)}]_{i+1,j+1}[H]_{j+1,n_2+2} + \sum_i \sum_j [G^*]_{n_1+2,i+1}
\end{aligned}$$

$$[\mathbf{R}_{\mathbf{d}(1,1)}]_{i+1,j+1}[G]_{j+1,n_2+2}+\sum_i\sum_j[H^*]_{n_1+2,i+1}$$

$$[\mathbf{R}_{\mathbf{ad}(1,1)}]_{i+1,j+1}[G]_{j+1,n_2+2}+\sum_i\sum_j[G^*]_{n_1+2,i+1}$$

$$[\mathbf{R}_{\mathbf{da}(1,1)}]_{i+1,j+1}[H]_{j+1,n_2+2}=[\mathbf{R}_{\mathbf{X}}]_{n_1+2,n_2+2}, \tag{28}$$

for all $n_1, n_2 \in \mathbf{Z}$. Hence, $\mathbf{x}[n]$ is WSCS with period 2.

To prove (S1)–(S3), let $m_1 = m_2 = m$. By substituting into the proof of (G1)–(G3), the results follow. ■

Remark I

Second-order processes always satisfy the condition (D) in DTST, and so the stochastic sequence $\mathbf{x}[n]$ in DTST also contains the case of second-order processes.

Remark II

The case of stationary white sequences is contained in (S1) because the autocorrelation function of a stationary white sequence $\sigma^2\delta[n]$ belongs to the metric (sequence) space l^∞.

Theorem II

Suppose that the same conditions hold as in DTST.

($J1$): if $\mathbf{x}[n]$ is of WSSJ then the wavelet coefficient $d_{m,k}$ has constant mean for all m and k, and the entries of the correlation matrices of $d_{m,k}$ satisfies

$$[\mathbf{R}_{\mathbf{d}(\mathbf{m_1},\mathbf{m_2})}]_{k_1,k_2}=[\mathbf{R}_{\mathbf{d}(\mathbf{m_1},\mathbf{m_2})}]_{k_1+2^{m_2},k_2+2^{m_1}}, \tag{29}$$

for all $k_1, k_2 \in \mathbf{Z}$ and any positive integers m_1 and m_2, i.e. $[\mathbf{R}_{\mathbf{d}(\mathbf{m_1},\mathbf{m_2})}]_{k_1,k_2}$ is a function of $k_2-2^{m_1-m_2}k_1$, for $m_1 \geq m_2$, or $k_1-2^{m_2-m_1}k_2$, for $m_1 < m_2$.

Furthermore, as a special case, for any positive integers $m_1 = m_2 = m$, the results become interesting as follows:

($J2$): if $\mathbf{x}[n]$ is of WSSJ then the wavelet coefficient $d_{m,k}$ is WSS for all $k \in \mathbf{Z}$.

Proof: The proof is similar to the case of WSSI in DTST except the following paragraph:

> Since the process $\mathbf{x}$ is of WSSJ, then define $\mathbf{y}_r[n] \equiv \mathbf{x}[n]+\mathbf{x}[n-r]$, for every $r \in \mathbf{Z}$. Therefore, $\mathbf{x}[n]$ has WSSJ if the pair $\mathbf{y}_r$ and $\mathbf{y}_s$ is stationarily correlated for every r and s, i.e. the correlation matrix $\mathbf{R}_{\mathbf{Y_r},\mathbf{Y_s}}$ is Toeplitz, where $\mathbf{Y}_r$ and $\mathbf{Y}_s$ are the infinite vectors $\{\mathbf{y}_r[n]\}_{n\in\mathbf{Z}}$ and $\{\mathbf{y}_s[n]\}_{n\in\mathbf{Z}}$, respectively. So, choose any $u,v \in \mathbf{Z}$ such that $r = i-v$, $s = l-u$ and the correlation function of $\mathbf{y}_r[i]$ and $\mathbf{y}_s[l]$ satisfies $R_y[i,l] = R_x[i,l]+R_x[i,v]+R_x[u,l]+R_x[u,v]$.

Thereafter, the remaining proof is taken the same manipulation as in the proof of (G2) in DTST, and hence the proof is obtained. ■

The physical meaning of (G1) in DTST is that the correlation functions of the scaling and wavelet coefficients for two different dilations 2^{m_1} and 2^{m_2} are functions of the difference of translation, $(k_2)-2^{m_1-m_2}(k_1)\,(=(k_2+2^{m_1})-2^{m_1-m_2}(k_1+2^{m_2}))$, for $m_1 \geq m_2$, or $(k_1)-2^{m_2-m_1}(k_2)\,(=(k_1+2^{m_2})-2^{m_2-m_1}(k_2+2^{m_1}))$, for $m_1 < m_2$. Therefore, under the same resolution (i.e. $m_1 = m_2 = m$), $a_{m,k}$ and $d_{m,k}$ are stationary, and the case (G1) in DTST will become to the case (S1) in DTST. One of usefulness of the stationarity is that the power spectra of $a_{m,k}$ and $d_{m,k}$ are well-defined by the Fourier transform of the correlation functions

$$S_a(e^{j\omega}) = \sum_{\tau=-\infty}^{\infty} R_a[\tau]_{m,m}\,e^{-j\omega\tau}, \quad S_d(e^{j\omega}) = \sum_{\tau=-\infty}^{\infty} R_d[\tau]_{m,m}\,e^{-j\omega\tau}, \tag{30}$$

where $\tau \equiv k_1 - k_2$. Moreover, for the case of $m < 0$, with the definitions

$$a_{m,k} \equiv \sum_{i=-\infty}^{\infty} h[i-2k]a_{m+1,i}, \quad d_{m,k} \equiv \sum_{i=-\infty}^{\infty} g[i-2k]a_{m+1,i}, \quad \text{for } m<0, \tag{31}$$

the results of stationarization are similar to DTST.

IV. Examples

The first three examples will present the stationarization for WSS process, nonstationary processes with WSSI and WSSJ generated by ARMA and ARIMA models (**16**). The DTST for the discrete FBM sequences will be demonstrated in the fourth example. All examples are Monte Carlo runs with 10 iterations.

In the first three examples, the WSS sequence is generated by ARMA model and the nonstationary sequences with WSSI and WSSJ are created by ARIMA models (**16**). These examples show the cases of (S1) and (S2) with respect to DTST, and Theorem II, respectively. The ARMA model is used to describe WSS processes. The representation of an ARMA(K, L) $\{x[n]\}_{n\in\mathbb{Z}}$ is given by (**16**), denoting $\alpha(q)x[n] = \beta(q)\epsilon[n]$, where $\alpha(q)$ is the regression operator of order K, $\beta(q)$ is the moving average operator of order L, and $\{\epsilon[n]\}_{n\in\mathbb{Z}}$ is a zero-mean stationary white noise. Furthermore, the roots of the polynominal $\alpha(q)$ must lie inside the unit circle to ensure the stability of the model. Krim (**16**) modeled the nonstationarity by fitting an ARMA (K,L) model to $\{\Delta_\lambda^D x[n]\}_{n\in\mathbb{Z}}$ which in turn results in an ARIMA (K, D, L) process for $\{x[n]\}_{n\in\mathbb{Z}}$, where $\Delta_\lambda^D x[n] \equiv (1+\lambda q^{-1})^D x[n]$, and $\lambda = 1$ or -1.

4.1. *The WSS case*

Consider a second-order process $\{x[n]\}_{n\in\mathbb{Z}}$ given by

$$(1-2\rho\cos(2\pi v)q^{-1}+\rho^2 q^{-2})x[n] = \epsilon[n], \tag{32}$$

where $\rho = 0.9$, $v = 0.3$, $q^{-1}x[n] \equiv x[n-1]$ and $\{\epsilon[n]\}_{n\in\mathbb{Z}}$ is an i.i.d. $N(0,1)$.

(1) The process $x[n]$ is WSS as follows. The mean of $x[n]$ is zero and the autocorrelation function of $x[n]$ is

$$R_x[n] \equiv \mathscr{E}\{x[n_2+n]x^*[n_2]\} = \begin{cases} \eta_1(0.9\,e^{j2.828})^{n+1} + \eta_2(0.9\,e^{j4.398})^{n+1}, & n \geq 0, \\ \eta_3\left(\dfrac{1}{0.9}e^{j2.828}\right)^{n+1} + \eta_4\left(\dfrac{1}{0.9}e^{j4.398}\right)^{n+1}, & n < 0, \end{cases} \tag{33}$$

where

$$\eta_1 = [(0.9\,e^{j2.828} - 0.9\,e^{j4.398})(1-0.81\,e^{j5.656})(1-j0.81)]^{-1},$$
$$\eta_2 = [(0.9\,e^{j4.398} - 0.9\,e^{j2.828})(1-j0.81)(1-0.81)]^{-1},$$
$$\eta_3 = [((1/0.9)\,e^{j2.828} - 0.9\,e^{j2.828})((1/0.9)\,e^{j2.828} - 0.9\,e^{j4.398})(1+j0.81)]^{-1}$$
$$\eta_4 = [((1/0.9)\,e^{j4.398} - 0.9\,e^{j2.828})((1/0.9)\,e^{j4.398} - 0.9\,e^{j4.398})(1-j0.81)]^{-1}.$$

Therefore, $x[n]$ is certified to be a second-order WSS process.

(2) Then the DWT of $x[n]$ is also WSS as follows. Let $d_{m,k}$ be the wavelet coefficient of $x[n]$ defined by

$$d_{m,k} = \sum_{i1} \cdots \sum_{i_m} g[i_1 - 2k]h[i_2 - 2i_1] \ldots h[i_m - 2i_{m-1}]x[i_m].$$

The mean of $d_{m,k}$ is also zero and the autocorrelation function of $d_{m,k}$, for any $m \in \mathbf{Z}$, is

$$R_d[\tau] = R_d[k_1, k_2] = \sum_{i_1} \cdots \sum_{i_m} \sum_{l_1} \cdots \sum_{l_m} g[i_1]h[i_2 - 2i_1] \ldots h[i_m - 2i_{m-1}]g[l_1]h[l_2 - 2l_1] \ldots h[l_m - 2l_{m-1}]R_x[i_m - l_m + 2^m\tau]. \quad (34)$$

Obviously, $d_{m,k}$ is also a second-order WSS process.

4.2. *The WSSI case*

Consider a nonstationary process with WSSI $\{x[n]\}_{n \in \mathbf{Z}}$ given by **(16)**

$$(1 - 2\rho \cos(2\pi v)q^{-1} + \rho^2 q^{-2})(1 - q^{-1})x[n] = \epsilon[n], \quad (35)$$

where $\rho = 0.9$, $v = 0.3$, $q^{-1}x[n] \equiv x[n-1]$, and $\{\epsilon[n]\}_{n \in \mathbf{Z}}$ is an i.i.d. $N(0, 1)$. The mean of $x[n]$ is zero and the autocorrelation function of $x[n]$ is

$$\begin{aligned} R_x[n_1, n_2] &\equiv \mathscr{E}\{x[n_1]x^*[n_2]\} \\ &= \mathscr{E}\left\{\left[\sum_{r_1}(h_3[n_1 - r_1] + h_4[n_1 - r_1])\epsilon[r_1]\right]\left[\sum_{r_2}(h_3[n_2 - r_2] + h_4[n_2 - r_2])\epsilon[r_2]\right]^*\right\} \quad (36) \\ &= \alpha_4^2 \min(n_1, n_2) + \sum_{r \le \min(n_1, n_2)} (\alpha_4 h_4[n_1 - r] + \alpha_4 h_4^*[n_2 - r] + h_4[n_1 - r]h_4^*[n_2 - r]), \end{aligned}$$

where $h_3[n] = \alpha_4$, for $n \geq 0$, and $h_3[n] = 0$, for $n < 0$, is the impulse response of $\alpha_4/(1 - q^{-1})$, and $h_4[n]$ is the impulse response of

$$\frac{\alpha_5 + \alpha_6 q^{-1}}{1 - 2\rho \cos(2\pi v)q^{-1} + \rho^2 q^{-2}},$$

where α_4, α_5 and α_6 are the residues of the equation $1/(1 - q^{-1})(1 - 2\rho \cos(2\pi v)q^{-1} + \rho^2 q^{-2}))$.

Since the roots of $1 - 2\rho \cos(2\pi v)q^{-1} + \rho^2 q^{-2}$, denoted by β_1 and β_2, are complex conjugates inside the unit circle, we obtain $h_4[r] = \alpha_7\beta_1^r + \alpha_8\beta_2^r$, where α_7 and α_8 are constant. This signal satisfies condition (D) in DTST. Thus, checking the autocorrelation function of the DWT coefficient of $x[n]$, $d_{m,k}$ for any m, we have

$$\begin{aligned} R_d[k_1, k_2]_{m,m} = \sum_{i_1} \cdots \sum_{i_m} \sum_{l_1} \cdots \sum_{l_m} g[i_1]h[i_2 - 2i_1] \ldots h[i_m - 2_{m-1}]g[l_1]h[l_2 - 2l_1] \\ \ldots h[l_m - 2l_{m-1}]\sum_r (h_3[i_m - r] + h_4[i_m - r])(h_3[l_m - r + 2^m(k_1 - k_2)] \\ + h_4[l_m - r + 2^m(k_1 - k_2)])^* = R_d[k_1 - k_2]_{m,m}. \quad (37) \end{aligned}$$

Hence, it is proved that $d_{m,k}$ is WSS. The experimental signal of WSSI $x[n]$ is shown in

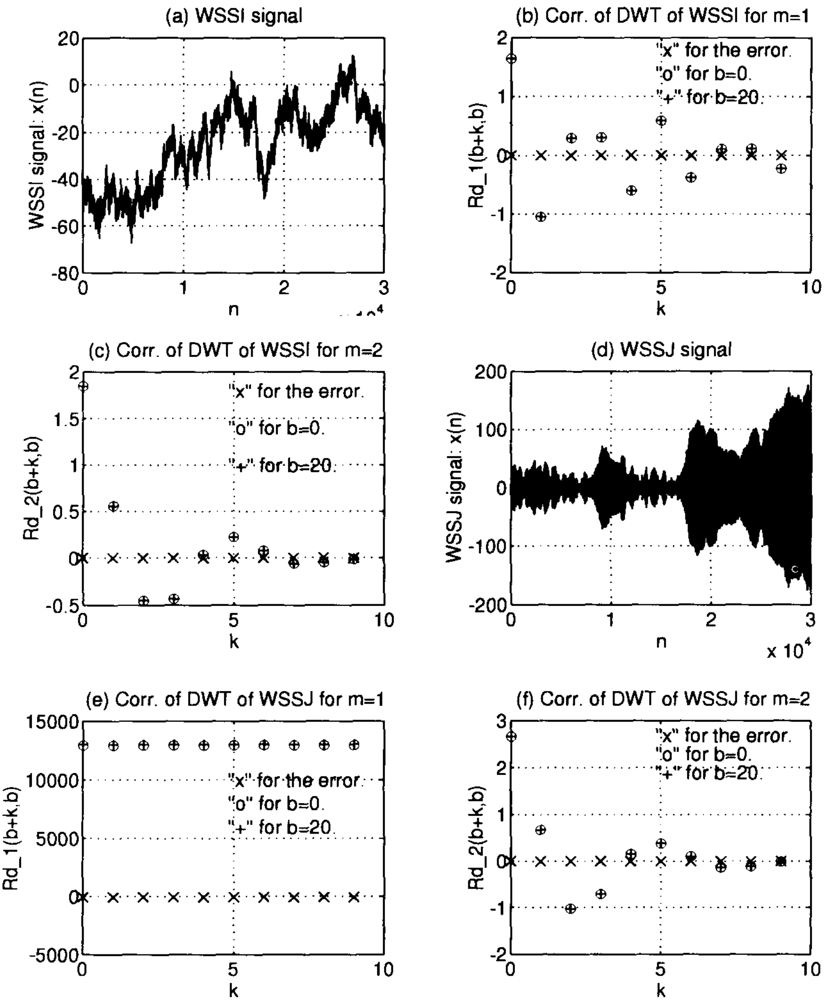

FIG. 1. (a) WSSI signal generated by ARIMA model. (b) Autocorrelation function of the DWT coefficient of the WSSI signal as in (a) for $m = 1$. (c) Autocorrelation function of the DWT coefficient of the WSSI signal as in (a) for $m = 2$, where (a), (b), (c) are the results of the example in Section 4.2. (d) WSSJ signal generated by ARIMA model. (e) Autocorrelation function of the DWT coefficient of the WSSJ signal as in (d) for $m = 1$. (f) Autocorrelation function of the DWT coefficient of the WSSJ signal as in (d) for $m = 2$, where (d), (e), (f) are from the example in Section 4.3.

Fig. 1(a). The autocorrelation function of the wavelet coefficient $R_d[k_1, k_2]$ at two different translation sets and the errors between these two autocorrelations are shown in Fig. 1(b) and (c). Figure 1(b) is for the case of $m = 1$ and Fig. 1(c) is for $m = 2$. In Fig. 1(b) and (c), the autocorrelation functions at the first set $[k_1, k_2] = [0, 0] \sim [9, 0]$ are denoted by o, the autocorrelations at the second set $[k_1, k_2] = [20, 20] \sim [29, 20]$

denoted by $+$, and the errors between these two autocorrelations denoted by x, in which the autocorrelation functions of the DWT of WSSI at different translations are the same with the same increments of translations.

4.3. *The WSSJ case*

Consider a nonstationary process with WSSJ $\{x[n]\}_{n\in\mathbb{Z}}$ given by **(16)**

$$(1-2\rho\cos(2\pi v)q^{-1}+\rho^2q^{-2})(1+q^{-1})x[n] = \epsilon[n], \tag{38}$$

where $\rho = 0.9$, $v = 0.3$, $q^{-1}x[n] \equiv x[n-1]$, and $\{\epsilon[n]\}_{n\in\mathbb{Z}}$ is an i.i.d. $N(0,1)$. The mean of $x[n]$ is zero and the autocorrelation function of $x[n]$ is

$$\begin{aligned} R_x[n_1,n_2] &\equiv \mathscr{E}\{x[n_1]x^*[n_2]\} \\ &= \mathscr{E}\left\{\left[\sum_{r_1}(h_1[n_1-r_1]+h_2[n_1-r_1])\epsilon[r_1]\right]\left[\sum_{r_2}(h_1[n_2-r_2]+h_2[n_2-r_2])\epsilon[r_2]\right]^*\right\} \\ &= (-1)^{n_1+n_2}\alpha_1^2\min(n_1,n_2)+\alpha_1\sum_{r\le\min(n_1,n_2)}(-1)^{n_2}-rh_2[n_1-r]+\alpha_1\sum_{r\le\min(n_1,n_2)}(-1)^{n_1} \\ &\quad -rh_2^*[n_2-r]+\sum_{r\le\min(n_1,n_2)}h_2[n_1-r]h_2^*[n_2-r], \end{aligned}\tag{39}$$

where $h_1[n] = \alpha_1(-1)^n$, for $n \geq 0$, and $h_1[n] = 0$, for $n < 0$, is the impulse response of $\alpha_1/(1+q^{-1})$, $h_2[n]$ is the impulse response of

$$\frac{\alpha_2+\alpha_3q^{-1}}{1-2\rho\cos(2\pi v)q^{-1}+\rho^2q^{-2}},$$

and α_1, α_2 and α_3 are the residues of the equation, $1/(1+q^{-1})(1-2\rho\cos(2\pi v)q^{-1}+\rho^2q^{-2})$. Since the roots of $1-2\rho\cos(2\pi v)q^{-1}+\rho^2q^{-2}$ are inside the unit circle and complex conjugates, $h_2[r] = \alpha_{22}\beta_1^r+\alpha_{33}\beta_2^r$, where α_{22} and α_{33} are constant values, and $\Sigma_{r=-\infty}^{\infty}h_2[r] < \infty$, the $x[n]$ is therefore shown to satisfy condition (D) in DTST also.

Thus, checking the autocorrelation function of the DWT coefficient of $x[n]$, $d_{m,k}$ for any m, it yields that $d_{m,k}$ is WSS, i.e.

$$\begin{aligned} R_d[k_1,k_2]_{m,m} &= \sum_{i_1}\cdots\sum_{i_m}\sum_{l_1}\cdots\sum_{l_m}g[i_1]h[i_2-2i_1]\ldots h[i_m-2i_{m-1}]g[l_1]h[l_2-2l_1] \\ &\quad \ldots h[l_m-2l_{m-1}]\sum_{r_1}\sum_{r_2}(h_1[i_m+2^mk_1-r_1]n+h_2[i_m+2^mk_1-r_1]) \\ &\quad \times(h_1[l_m+2^mk_2-r_2]+h_2[l_m+2^mk_2-r_2])^*\delta_c[r_1-r_2] \\ &= \sum_{i_1}\cdots\sum_{i_m}\sum_{l_1}\cdots\sum_{l_m}g[i_1]h[i_2-2i_1]\ldots h[i_m-2i_{m-1}]g[l_1]h[l_2-2l_1] \\ &\quad \ldots h[l_m-2l_{m-1}]\sum_{r}(h_1[i_m-r]+h_2[i_m-r])(h_1[l_m-r+2^m(k_1-k_2)] \\ &\quad +h_2[l_m-r+2^m(k_1-k_2)])^* = R_d[k_1-k_2]_{m,m}. \end{aligned}\tag{40}$$

The experimental WSSJ sequence is shown in Fig. 1(d). The simulation results are shown in Fig. 1(e) for the case of $m = 1$ and Fig. 1(f) for the case of $m = 2$, where the notation is the same as defined in Fig. 1(b). In Fig. 1(e) and (f), the autocorrelation functions of the DWT at different translations are the same with the same increments of translations.

4.4. *The FBM case*

The FBM process (**11, 24, 25**) is recognized a famous nonstationary stochastic process with WSSI. Since the autocorrelation functions of a sampled FBM sequence satisfy the condition (D) in DTST for $N = 1$, the sampled FBM sequence are used to demonstrate the cases (G2) and (S2) of DTST.

Consider a sampled FBM process (**22**), $B[n] = B_H(n\Delta x)$, $n \in \mathbf{Z}$, where Δx is the sampling period. The autocorrelation of B is denoted by

$$R_B[n_1, n_2] = \frac{\sigma^2}{2}|\Delta x|^{2H}(|n_1|^{2H} + |n_2|^{2H} - |n_1 - n_2|^{2H}), \tag{41}$$

which satisfies condition (D) in DTST for $N_R = 1$. Let $\Delta x = 1$ and choose Haar basis for simplicity, that is $h[0] = h[1] = g[0] = -g[1] = 1/\sqrt{2}$. From (**20, 9**), the wavelet coefficients for $m_1 = 1$ and $m_2 = 1$ is written as $d_{1,n} = \Sigma_{i_1=-\infty}^{\infty} g[i_1 - 2n]a_{0,i_1} = \Sigma_{i_1=-\infty}^{\infty} g[i_1 - 2n]B[i_1]$, where $a_{0,n} = B[n]$. Thus, the autocorrelation functions of the wavelet coefficients are shown as follows:

(G2) case: For $m_1 \neq m_2$, let $m_1 = 2$ and $m_2 = 1$ for simplicity. Substituting Eq (41) into the autocorrelation function of the wavelet coefficients, we have

$$R_d[k_1, k_2]_{2,1} = \frac{\sigma^2}{2}\{|2(2k_1 - k_2) - 1|^{2H} - 2|2(2k_1 - k_2) + 1|^{2H} + |2(2k_1 - k_2) + 3|^{2H}\}. \tag{42}$$

For any positive integers $m_1 \geqslant m_2$, from Eq (41) and (**20, 9**), we obtain that the correlation depends only on $k_2 - 2^{m_1 - m_2}k_1$ as follows:

$$R_d[k_1, k_2]_{m_1,m_2} = \sum_{i'_1}\sum_{i'_2}\cdots\sum_{i'_{m_1}}\sum_{l'_1}\cdots\sum_{l'_{m_2}} g[i'_1]\ldots h[i'_{m_1} - 2i'_{m_1-1}]g[l'_1]\ldots h[l'_{m_2} - 2l'_{m_2-1}]$$

$$\times \frac{\sigma^2}{2}(|i'_m|^{2H} + |l'_m + 2^{m_2}(k_2 - 2^{m_1-m_2}k_1)|^{2H} - |i'_{m_1} - l'_{m_2} - 2^{m_2}(k_2 - 2^{m_1-m_2}k_1)|^{2H}). \tag{43}$$

(S2) case: For $m_1 = m_2 = m = 1$,

$$R_d[k_1, k_2]_{1,1} = \frac{\sigma^2}{2}(|2(k_1 - k_2) - 1|^{2H} + |1 + 2(k_1 - k_2)|^{2H} - 2|2(k_1 - k_2)|^{2H}). \tag{44}$$

For any positive integers $m_1 = m_2 = m$, from Eq (41) and (**20, 9**), we obtain that the correlation depends only upon the difference of translation $k_1 - k_2$. That is,

$$R_d[k_1, k_2]_{m,m} = \sum_{i'_1}\sum_{i'_2}\cdots\sum_{i'_m}\sum_{l'_1}\sum_{l'_2}\cdots\sum_{l'_m} g[i'_1]\ldots h[i'_m - 2i'_{m-1}]g[l'_1]h[l'_2 - 2l'_1]\ldots h[l'_m - 2l'_{m-1}]$$

$$\times \frac{\sigma^2}{2}(|l'_m + 2^m(k_2 - k_1)|^{2H} + |i'_m|^{2H} - |i'_m - l'_m - 2^m(k_2 - k_1)|^{2H}). \tag{45}$$

TABLE 1

Autocorrelation data of the wavelet coefficient $d_{m,k}$, with $m_1 = m_2$ for the sampled FBM sequence

	$[k_1, k_2]$	$H = 0.1$ $\mathscr{E}\{xy^*\}$	$H = 0.5$ $\mathscr{E}\{xy^*\}$	$H = 0.9$ $\mathscr{E}\{xy^*\}$
$R_d[k_1, k_2]_{1,1}$	[0,0]	2.3803e−01	7.0250e−05	2.0389e−08
	[10,10]	2.3804e−01	7.0257e−05	2.0382e−08
	[4,0]	−6.8011e−03	−1.8503e−06	8.0060e−09
	[14,10]	−6.8316e−03	−1.8632e−06	7.9947e−09
$R_d[k_1,k_2]_{2,2}$	[0,0]	1.2292e+00	5.2008e−04	2.6002e−07
	[10,10]	1.2291e+00	5.2008e−04	2.5983e−07
	[14,10]	−2.2313e−02	−7.8963e−06	1.0986e−07
	[14,10]	−2.2489e−02	−8.1351e−06	1.0955e−07
$R_d[k_1,k_2]_{3,3}$	[0,0]	6.5196e+00	4.1752e−03	3.4930e−06
	[10,10]	6.5183e+00	4.1745e−03	3.4879e−06
	[4,0]	−1.5449e−01	−1.0560e−04	1.443e−06
	[14,10]	−1.5257e−01	−1.0572e−04	1.4390e−06

$\mathscr{E}\{xy^*\}$ denotes $\mathscr{E}\{a_{m_1,k_1}a^*_{m_2,k_2}\}$ or $\mathscr{E}\{d_{m_1,k_1}d^*_{m_2,k_2}\}$.

TABLE 2

Autocorrelation data of the wavelet coefficient $d_{m,k}$, with $m_1 \neq m_2$ for the sampled FBM sequence

	$[k_1, k_2]$	$H = 0.1$ $\mathscr{E}\{xy^*\}$	$H = 0.5$ $\mathscr{E}\{xy^*\}$	$H = 0.9$ $\mathscr{E}\{xy^*\}$
$R_d[k_1, k_2]_{2,1}$	[0,0]	5.4444e−01	3.0056e−05	1.7700e−08
	[10,20]	5.4451e−01	3.0087e−05	1.7665e−08
	[10,10]	7.9458e−05	5.4716e−07	7.0506e−09
$R_d[k_1, k_2]_{3,2}$	[0,0]	1.4612e−01	1.2429e−04	1.1829e−07
	[10,20]	1.4634e−01	1.2460e−04	1.1784e−07
	[10,10]	3.2288e−03	2.4195e−06	4.5124e−08
$R_d[k_1, k_2]_{4,3}$	[0,0]	3.6900e−01	4.9992e−04	7.8494e−07
	[10,20]	3.7213e−01	5.0252e−04	7.8092e−07
	[10,10]	2.7953e−02	2.7285e−05	2.9063e−07

The experimental results summarized in Tables 1 and 2 reveal the stationarity of the wavelet coefficient of the FBM sequence for three different parameters $H = 0.1$, $H = 0.5$ and $H = 0.9$. In Table 1 for the case (S2) of DTST, the correlation function at $[k_1, k_2] = [0, 0]$ is equal to the one at $[k_1, k_2] = [10, 10]$ and the correlation function at $[k_1, k_2] = [4, 0]$ is also equal to the one at $[k_1, k_2] = [14, 10]$ for the wavelet coefficient and every resolution $2m$, where $m = 1, 2, 3$. The properties of case (G2) in DTST are shown in Table 2, where the correlation at $[k_1, k_2] = [0, 0]$ is equal to the one at

$[k_1, k_2] = [10, 20]$ and the correlation at $[k_1, k_2] = [0, 0]$ is different from the one at $[k_1, k_2] = [10, 10]$ for distinct resolutions 2^{m_1} and 2^{m_2}, where $m_1 - m_2 = 1$. Therefore, the property $R[k_1, k_2]_{m_1, m_2} = R[k_2 - 2^{m_1 - m_2} k_1]_{m_1, m_2}$ is verified.

V. Conclusion

Physical data are observed in the form of a discrete sequence in practice, and the CWT suffers from redundancy and heavy load of computation. In this work, we have developed the easily realizable FIR PR-QMF structure for the 1-D discrete wavelet transform which can stationarize the random sequences with WSSI/WSSJ. Moreover, we have also shown the stationarization of a sampled FBM signal by the PR-QMF structure of DWT. These results provide a well-defined method for power spectra of those nonstationary stochastic sequences useful for fractal modeling.

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