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SENSOR AND ACTUATOR PLACEMENT FOR MODAL IDENTIFICATION

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In modal identification, the use of different actuator and sensor locations results in different frequency response functions (FRFs) and consequently affects the modes shown on the FRF plots. This paper presents a method of placing sensors and actuators to obtain reliable measured data for estimating the modal parameters. The method is based on proposed controllability and observability measures for second-order linear systems, which are suitable for identification purposes and have a physical interpretation on the FRF plots. These measures, combined with an objective criterion, provide a method for ranking the effectiveness of alternative actuator and sensor distributions and hence a rational basis for choosing their locations. The method is applied successfully to the selection of actuator and sensor locations for a set of target modes for the structural characterisation of a simulated system. Finally, an iterative procedure for performing modal surveys is proposed.

1. INTRODUCTION

The aim of modal analysis or testing is to estimate and identify the modal parameters (natural frequencies, damping values, or mode shapes) of a mechanical structure. The model identified by modal testing allows one to predict, monitor, and control the real structure. It also allows the analytical dynamic model to be optimised so as to improve the dynamic behaviour [1]. All these applications depend greatly on the accuracy of the identified model. In order to increase the accuracy of the estimated modal parameters from frequency response functions (FRFs), it is necessary to excite the interested modes to a sufficient level by some form of input. The form of input is sometimes determined by test and analysis methods, e.g. sinusoids used in frequency response analysis and step functions and impulses in transient analysis. Although different forms of input influence the response of the system, the system impulse FRFs are fixed and independent of the type of input once the response and the input locations and their directions are determined. Some of the most frequently recurring questions in this regard deal with the advance determination of the number of excitations, their spatial locations, and the number of measurements and their locations, so that the quality and quantity of the information derived from these measurements can be maximised.

The driving point residue (DPR) method was proposed by Kientzy and Richardson [2]. A driving point FRF is an FRF whose response and excitation degrees of freedom (dofs)

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are the same. It always has a resonance–antiresonance pattern [3], and the number of excited modes can be checked easily. Considering the residue and natural frequency together for each mode represents a great breakthrough in the problem of the sensor and actuator placement. This method is simple to use; however, it restricts the placement to the driving point, and this approach is not general enough when the number of sensors required is greater than the number of actuators or when the sensor and actuator are allowed to be non-collocated.

Maximum kinetic energy (MKE) was used by Chung and Moore [4] to place acceleration sensors according to the modal kinetic energy contribution of each dof. If the objective function is examined, it appears that is loses frequency information. The lack of temporal information (time or frequency) makes it unsuitable for comparison of multiple target modes and for other sensor types. When different locations are compared according to a certain mode (i.e. one target mode), the frequency component can be ignored. But when dofs which contain several modes are compared, the individual contributions from each mode at different frequencies need to be considered.

Niedbal and Klusowski [5] focused on force tuning for the phase resonant method but also discussed the placement of the exciters for the phase separation method. They defined the object function as the condition number, which should be as large as possible, of the target modal partition matrix and found the optimal value by an exhaustive search. The idea of maximising the condition number is equivalent to that of maximising the input force or energy. One problem with this method is that the number of actuators can only be reduced to the number of target modes. In most situations this restriction is too stringent to perform a modal survey.

Kammer [6] proposed a method called the effective independence method, to select the sensor locations. This method finds the optimal solution in an iterative manner instead of using an intractable exhaustive search algorithm. The original modal partition is very similar to that adopted by [5] for actuator selection. The basic idea is to reduce the analytic model to a best test analysis model (TAM), whose modal vectors are spatially independent, and can further be used to modfiy the original FEM by using test analysis correlation techniques. The number of sensors cannot be reduced to less than the number of the target modes, however, nor is any information provided about how to place the actuators. In fact, this method resembles a model order reduction method more than a sensor–actuator placement method.

Some papers have examined the sensor–actuator placement for control purposes; see, e.g. the survey in [7]. A degree of controllability definition was proposed by Viswanathan *et al*. [8] to select the number and locations of the control actuators. In order to control a structure's behaviour, one usually wishes to monitor or depress the response of the system, but for identification purposes, the response needs to be maximised to obtain a good SNR. Therefore, the suitability of controllability and observability measures for various purposes must be examined closely.

This paper proposes a method of placing sensors and actuators to obtain reliable measured data for estimating the modal parameters. The method is based on the proposed controllability and observability measures for second-order linear systems, which are suitable for identification purposes and have a physical interpretation on the FRF plots. The influence of different types of sensors is also accounted for in the proposed measures. When linked with an objective criterion, these measures provide us with a method for ranking the effectiveness of alternative actuator and sensor distributions and hence a rational basis for choosing actuator and sensor locations. A priori information from FEA (finite element analysis) is necessary for initialisation. Compared with other methods by simulation, the proposed method appears to be very promising.

2. THEORETICAL FORMULATION

2.1. CONTROLLABILITY AND OBSERVABILITY

The formal definition of controllability for linear time-invariant systems is given in state space [9]. Consider a first-order system defined by

$$\dot{x} = Ax + Bu$$
$$y = Cx \tag{1}$$

where x is an $2n \times 1$ state vector, u is an $m \times 1$ input vector, y is a $p \times 1$ output vector, A is a $2n \times 2n$ state matrix, B is a $2n \times m$ input coefficient matrix, and C is a $p \times 2n$ output coefficient matrix. The standard check for the controllability of a system is a rank test of a controllable matrix, defined by

$$W_c = [B \;\; AB \;\; A^2B \;\; \ldots \;\; A^{2n-1}B]. \tag{2}$$

If the rank of Wc is $2n$, i.e. full rank, the system is *state controllable*.

State observability is a dual concept to state controllability. A system is observable if information about each of the state variables can be determined by examining the output (response) of the system. The observability matrix is defined by

$$W_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{2n-1} \end{bmatrix}. \tag{3}$$

If the rank of W_o is $2n$, the system is *state observable*.

Although these rank-check conditions are conceptually simple, they may create computational difficulties for systems with many dofs. Therefore, other methods have been proposed to check these two system properties [10–12]. Note that there is no way to know how easily an actuator can control or excite each mode of the system even if the system is completely controllable. The similar statement concerning the response of the system is also true regarding observability.

2.2. THE MEASURES OF CONTROLLABILITY AND OBSERVABILITY

For identification purposes, the measures or degrees of controllability and observability defined here must meet the following criteria:

(1) the measure of controllability should be zero when the system is uncontrollable; the measure of observability should be zero when the system is unobservable;
(2) the measures must consider the dependence on total time T or frequency;
(3) the measure of controllability must be independent of the magnitude of input and the measure of observability must be independent of the gain of output;
(4) the joint modal controllability and observability can be used as an indication of the success of the identification process;
(5) ordinarily, the results of parameters are defined in second-order form. Thus these measures should be defined in terms of a second-order system rather than a first-order state space system.

Assume the second-order vector differential equations are of the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{B}\mathbf{f} \tag{4}$$

where $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{f} \in \mathbf{R}^m$, $\mathbf{y} \in \mathbf{R}^p$, $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{K}$ are $n \times n$ matrices $\mathbf{B}$, is an $n \times n$ matrix, and $\mathbf{C}$ and $\mathbf{T}$ are $p \times n$ matrices. Modal testing usually uses the same sensor type, so without loss of generality we can let $\mathbf{T}$ be a zero matrix. For modal parameter estimation, modal coordinates are naturally the best choice for measuring controllability and observability. This study is interested in how strongly the modes are excited defined in the modal coordinates, and in what information about the target modes contained in the system response. When the system equation is transformed into the modal coordinates, if the damping is proportional, the equation will be decoupled.

From equation (4), the FRF between the jth sensor and ith actuator can be represented as

$$h_{ji}(\omega) = \sum_{r=1}^{n} \frac{\mathbf{c}_j \boldsymbol{\varphi}_r \boldsymbol{\varphi}_r^T \mathbf{b}_i}{m_r \omega_{nr}^2 [(1 - \bar{\omega}_r^2) + j2\zeta_r \bar{\omega}_r]}$$

$$= \sum_{r=1}^{n} \frac{\boldsymbol{\varphi}_{jr} \boldsymbol{\varphi}_{ir}}{m_r \omega_{nr}^2 [(1 - \bar{\omega}_r^2) + j2\zeta_r \bar{\omega}_r]} \tag{5}$$

where

$$\bar{\omega}_r = \omega / \omega_{nr} \qquad r = 1, 2, \ldots, n \tag{6}$$

and ζ_r and ω_{nr} are the damping ratio and natural frequency of the rth mode, respectively. $\mathbf{b}_i$ is ith column vector of $\mathbf{B}$, $\mathbf{c}_j$ is jth row vector of $\mathbf{C}$, and φ_{jr} and φ_{ir} are the jth and ith elements of modal vector $\boldsymbol{\varphi}_r$ respectively, where modal vector $\boldsymbol{\varphi}_r$ is the rth column vector of modal matrix $\boldsymbol{\Phi}$.

The FRF matrix can be represented as

$$\mathbf{H}(\omega) = \sum_{r=1}^{n} \left(\frac{\boldsymbol{\varphi}_r \boldsymbol{\varphi}_r^T}{a_r (j\omega - \lambda_r)} + \frac{\boldsymbol{\varphi}_r^* \boldsymbol{\varphi}_r^{*T}}{a_r^* (j\omega - \lambda_r^*)} \right)$$

$$= \sum_{r=1}^{n} \left(\frac{\mathbf{A}_r}{j\omega - \lambda_r} + \frac{\mathbf{A}_r^*}{j\omega - \lambda_r^*} \right) \tag{7}$$

where

$$\lambda_r = -\zeta_r \omega_{nr} + j\omega_{nr} \sqrt{1 - \zeta_r^2} \tag{8}$$

$$a_r = j2m_r \omega_{nr} \sqrt{1 - \zeta_r^2} \tag{9}$$

$$a_r^* = -j2m_r \omega_{nr} \sqrt{1 - \zeta_r^2} \tag{10}$$

and $\mathbf{A}_r$ is the rth residue matrix, in which the jith element is $A_{jir} = \varphi_{jr} \varphi_{ir} / m_r \omega_{nr}$, $r = 1, 2, \ldots, n$. The residue is an invariant property and will not change under any coordinate transformation [13].

From the definitions of the controllability and observability matrices, it can be seen that the position selection matrices $\mathbf{B}$ and $\mathbf{C}$ are important in the transfer functions, or the FRFs. When the system is transformed to modal space and decoupled into many single dof sub-systems, each mode (sdof sub-system) behaves in its own way according to its parameters (natural frequency and damping ratio). The energy exerted by all actuators is divided into several small parts and then amplified (or attenuated) by matrix $\boldsymbol{\Phi}^T \mathbf{B}$ before being fed into several sdof sub-systems. Each sdof sub-system can be regarded as a band-pass filter. The outputs from all sdof sub-systems are amplified (or attenuated) by matrix $\mathbf{C\Phi}$ and then transferred to different locations for measurement. In this way, each actuator (or sensor) location will influence all sdof sub-systems with different weights.

Hamdan and Nayfeh [14] proposed measures of modal controllability and observability and discussed the influence of different input and output gain matrices on the state feedback control systems. The matrix of these measures is defined as $\cos \Theta$, and its rith element is

$$\cos \Theta_{ri} = \frac{|\boldsymbol{\varphi}_r^T \mathbf{b}_i|}{\|\boldsymbol{\varphi}_r\| \, \|\mathbf{b}_i\|}. \tag{11}$$

Similarly, the matrix of modal observability measures is denoted by $\cos \boldsymbol{\Psi}$, and its jrth entry is

$$\cos \Psi_{jr} = \frac{|\mathbf{c}_j \boldsymbol{\varphi}_r|}{\|\mathbf{c}_j\| \, \|\boldsymbol{\varphi}_r\|} \tag{12}$$

where $\|\mathbf{b}_i\|$, $\|\mathbf{c}_j\|$, and $\|\boldsymbol{\varphi}_r\|$ are the two-norms of $\mathbf{b}_i$, $\mathbf{c}_j$, and $\boldsymbol{\varphi}_r$, respectively.

If one focuses on the issue of actuator and sensor placement, all the actuator and sensor gains are equally weighted and have no effect on the determination of location. In this case, all elements of $\mathbf{c}_j$ will be 0 except for the jth element, which is 1. Similarly, all elements of $\mathbf{b}_i$ will be 0 except for the ith element, which is 1. In this way, $\mathbf{c}_j \boldsymbol{\varphi}_r$ and $\boldsymbol{\varphi}_r^T \mathbf{b}_i$ become φ_{jr} and φ_{ir}, where φ_{jr} and φ_{ir} are the jth and ith elements of modal vector $\boldsymbol{\varphi}_r$, respectively, and the norms of $\mathbf{c}_j$ and $\mathbf{b}_i$ are 1. $\cos \Theta_{ri}$ and $\cos \Psi_{jr}$ are the indices of the controllability and observability for the rth mode. Although these definitions have some drawbacks as discussed later, for comparison with the proposed method, they are adopted to choose the best locations of sensors and actuators by selecting the maximum values defined by equations (11) and (12). Henceforth, this method is refered to as the COMC method (controllability and observability measures for control purpose).

From equation (5) it is known that the FRF involves the modal controllability and observability definition given above, but the contribution of the joint measure (sometimes called the output controllability) is more important than those of the individual measures. Examining equations (11) and (12) and (5), one finds that the product of equations (11) and (12) as the joint measure of modal controllability and observability does not depict the input–output relationship (FRF) completely, since equation (5) shows that the FRF is the summation of many modal responses which are related to their modal frequencies. When identifying multiple target modes, all measures will be summed up for comparison to find the best sensor–actuator pair. Ignoring the frequency information makes these measures unsuitable for identification purposes (i.e. finding natural frequencies and damping ratios by curve fitting). Specifically, these measures lose frequency information, and the lack of frequency information makes them unsuitable for comparison of multiple target modes. Therefore, in modal testing the indices should be redefined to determine actuator and sensor locations.

2.3. PROPOSED MEASURES OF CONTROLLABILITY AND OBSERVABILITY AND JOINT MEASURE OF CONTROLLABILITY AND OBSERVABILITY

From equation (4), the relation between the rth modal displacement q_r and the force exerted by the ith actuator (f_i) can be derived as follows:

$$\frac{q_r}{f_i} = \frac{\boldsymbol{\varphi}_r^T \mathbf{b}_i}{m_r \omega_{nr}^2 [(1 - \bar{\omega}_r^2) + j2\zeta_r \bar{\omega}_r]}. \tag{13}$$

Similarly, the relation between the jth sensor output y_j and the rth modal displacement q_r can be derived as follows:

$$\frac{y_j}{q_r} = \mathbf{c}_j \boldsymbol{\varphi}_r \tag{14}$$

First, consider the case where the damping effect is neglected, which is reasonable since the damping is generally ignored in FEA in order to simplify the computations and there is no reliable damping information before testing. For such a case, if one regards the pre-amplifying effects before energy enters the sdof sub-systems as modal controllability and the post-amplifying effects before the energy is transferred to the sensors as modal observability, the modal controllability cnt_{ri}, observability obs_{jr}, and the joint measure of modal controllability and observability $obs_{jr}cnt_{ri}$ for the rth mode can be defined as follows:

$$cnt_{ri} = \left|\frac{\boldsymbol{\varphi}_r^T \mathbf{b}_i}{m_r \omega_{nr}^2}\right| = \left|\frac{\boldsymbol{\varphi}_r^T \mathbf{b}_i}{k_r}\right| \tag{15}$$

$$obs_{jr} = |\mathbf{c}_j \boldsymbol{\varphi}_r| \tag{16}$$

$$obs_{jr}cnt_{ri} = \left|\frac{\mathbf{c}_j \boldsymbol{\varphi}_r \boldsymbol{\varphi}_r^T \mathbf{b}_i}{m_r \omega_{nr}^2}\right| \tag{17}$$

or

$$obs_{jr}cnt_{ri} = \left|\frac{\mathbf{c}_j \boldsymbol{\varphi}_r \boldsymbol{\varphi}_r^T \mathbf{b}_i}{\omega_{nr}^2}\right| \quad \text{when} \quad m_r = 1 \tag{18}$$

In light of equation (13), equation (15) can be regarded as the gain between the actuator and modal displacement. This measure tells us how strongly the energy is transferred into some sdof sub-systems. By equation (14), equation (16) can be regarded as the gain between the sensor and modal displacement. This measure indicates to what extent energy is observed through some sdof sub-systems. The product of equations (15) and (16) yields equation (17) as the joint measure of modal controllability and observability, which directly relates to the FRFs in equation (5). It is obvious that cnt_{ri} and obs_{jr} will change when the modal vector $\boldsymbol{\varphi}_r$ is normalised by different methods [15]. In order to simplify the notation and calculation, the unity modal mass scaling method is used (i.e. $m_r = \boldsymbol{\varphi}_r^T \mathbf{M} \boldsymbol{\varphi}_r = 1$, $\boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi} = \mathbf{I}$, and $k_r = \omega_{nr}^2$), which is commonly adopted in commercial analytic programs. Then the modal controllability can be expressed more compactly as follows:

$$cnt_{ri} = \left|\frac{\boldsymbol{\varphi}_r^T \mathbf{b}_i}{k_r}\right| = \left|\frac{\boldsymbol{\varphi}_r^T \mathbf{b}_i}{\omega_{nr}^2}\right|. \tag{19}$$

However, the modal observability and the joint measure of modal controllability and observability do not change no matter which scaling method is adopted. This is a useful system property.

When information about damping is available, e.g. after one trial modal testing, the modal controllability is redefined as follows:

$$cnt_{ri} = \left|\frac{\boldsymbol{\varphi}_r^T \mathbf{b}_i}{2\zeta_r \omega_{nr}^2}\right|, \qquad m_r = 1, \quad r = 1, 2, \ldots, n \tag{20}$$

which from equation (13) can be regarded as the gain between the actuator and modal displacement at natural frequencies. This measure indicates how strongly the energy is transferred into some sdof sub-systems at natural frequencies. In order to obtain a good signal-to-noise ratio (SNR) in the frequency domain around several modal frequencies of interest, this measure should be made as large as possible.

If different types of sensors, such as velocity sensors, are used, the FRFs then take on a velocity/force form. The definition of modal observability for these velocity sensors should be

$$obs_{jr} = |\mathbf{c}_j \boldsymbol{\varphi}_r \omega_{nr}|, \qquad m_r = 1, \quad r = 1, 2, \ldots, n \tag{21}$$

and the joint measure of modal controllability and observability should be

$$obs_{jr}cnt_{ri} = \left|\frac{\mathbf{c}_j \boldsymbol{\varphi}_r \boldsymbol{\varphi}_r^T \mathbf{b}_i}{m_r \omega_{nr}}\right| = \left|\frac{\varphi_{jr}\varphi_{ir}}{m_r \omega_{nr}}\right| \tag{22}$$

or

$$obs_{jr}cnt_{ri} = \left|\frac{\mathbf{c}_j \boldsymbol{\varphi}_r \boldsymbol{\varphi}_r^T \mathbf{b}_i}{\omega_{nr}}\right| = \left|\frac{\varphi_{jr}\varphi_{ir}}{\omega_{nr}}\right| \qquad m_r = 1. \tag{23}$$

If accelerometers are used, the FRFs then take on a acceleration/force form. The definition of modal observability for these acceleration sensors should be

$$obs_{jr} = |\mathbf{c}_j \boldsymbol{\varphi}_r \omega_{nr}^2|, \qquad m_r = 1, \quad r = 1, 2, \ldots, n \tag{24}$$

and the joint measure of modal controllability and observability should be

$$obs_{jr}cnt_{ri} = \left|\frac{\mathbf{c}_j \boldsymbol{\varphi}_r \boldsymbol{\varphi}_r^T \mathbf{b}_i}{m_r}\right| = \left|\frac{\varphi_{jr}\varphi_{ir}}{m_r}\right| \tag{25}$$

or

$$obs_{jr}cnt_{ri} = |\mathbf{c}_j \boldsymbol{\varphi}_r \boldsymbol{\varphi}_r^T \mathbf{b}_i||\varphi_{jr}\varphi_{ir}| \qquad m_r = 1. \tag{26}$$

From the above definition a small value of joint measure for certain r indicates that the response of the corresponding mode r is too small to be identified accurately, and obviously the zero value of joint measure implies that the chosen sensor–actuator pair is either uncontrollable or unobservable for that particular mode. Recall that the above analysis is based on the assumption of proportional damping. Classifying structures into proportionally and non-proportionally damped systems is necessary when estimating the parameters of the system. In this research, FEA mode shapes information is used to place the sensors and actuators. Often FEA supplies real mode shapes only. For sensor–actuator placement, it makes little difference whether proportional or non-proportional damping is used. It can be assumed that proportional damping is used to simplify the calculation of the FRFs [16], especially when the system is lightly damped.

2.4. CRITERION FOR SENSOR AND ACTUATOR PLACEMENT

In order to employ the joint measures of modal controllability and observability defined above to choose sensor and actuator locations, one can define the cost Q as follows:

$$Q_{jir} \equiv obs_{jr}cnt_{ri} = \left|\frac{\mathbf{c}_j \boldsymbol{\varphi}_r \boldsymbol{\varphi}_r^T \mathbf{b}_i}{m_r \omega_{nr}^2}\right| = \left|\frac{\varphi_{jr}\varphi_{ir}}{m_r \omega_{nr}^2}\right| \quad \text{(COMI method)} \tag{27}$$

where the COMI method denotes the controllability and observability measures for identification purposes. Taking the damping ratios into consideration, from equations (16) and (20), Q_{jir} can be redefined as

$$Q_{jir} \equiv \left|\frac{\mathbf{c}_j \boldsymbol{\varphi}_r \boldsymbol{\varphi}_r^T \mathbf{b}_i}{m_r \zeta_r \omega_{nr}^2}\right| = \left|\frac{\varphi_{jr}\varphi_{ir}}{m_r \zeta_r \omega_{nr}^2}\right| \quad \text{(COMID method).} \tag{28}$$

Referring to Fig. 1 and the summation form of FRF $h_{ji}(\omega)$ defined in equation (5), Q_{jir} is the approximation of magnitude at the rth resonant frequency in FRF $h_{ji}(\omega)$. Each FRF contains n values of Q. If one wishes to estimate the rth mode with sufficient accuracy, one requires a FRF which contains a large Q value. Thus if this idea is extended to include several target modes, each FRF can have a simple index P_{ij} such as

$$P_{ij} = \min(Q_{ij1}, Q_{ij2}, \ldots, Q_{ijTM}) \tag{29}$$

where TM indicates the number of target modes. Let p and m represent the candidate number of sensors and actuators; then the number of P values is $p \times m$. Finding the maximum value of indices P obtains the best locations for sensor and actuator. For example, if the largest value of P is P_{23}, the best selection for sensors and actuators locations would be dof 2 and 3. If the structure obeys Maxwell–Betti's law, P_{32} will also be the largest value, i.e. exchanging the locations of sensors with actuators will not affect the P value. The object function matrix is expressed as follows:

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1p} \\ P_{21} & P_{22} & & \vdots \\ \vdots & & \ddots & \\ P_{l1} & \cdots & & P_{lp} \end{bmatrix}. \tag{30}$$

The best selection (a, b) is determined by

$$P_{ab} = \max\,(P_{11}, P_{12}, \ldots, P_{21}, P_{22}, \ldots, P_{lp}) \tag{31}$$

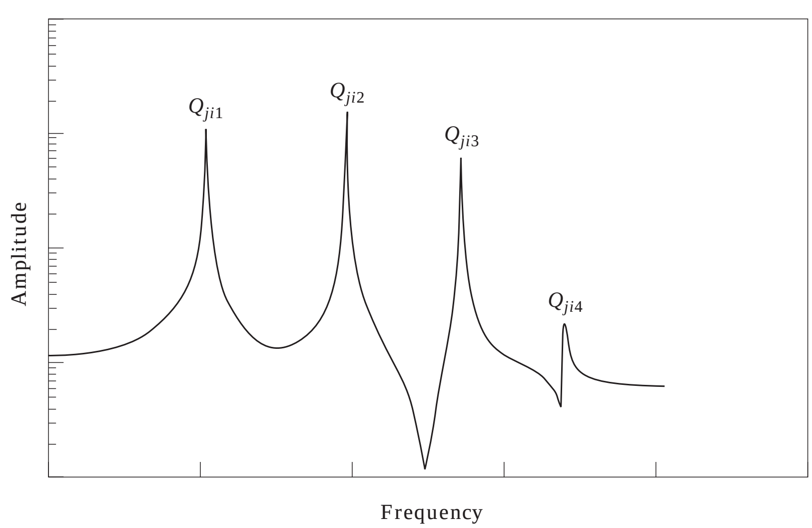

Figure 1. Example of Q_{jir}.

The definition in equation (29) ensures that all target modes will be excited to a certain level. But the level which is sufficient for successful identification depends on several factors, such as the SNR and the capability of the estimating program, and thus depends on the engineer's judgement.

Another simple index can be defined as the summation of all Q values, expressed as

$$P_{ij} = \text{sum}\,(Q_{ij1}, Q_{ij2}, \ldots, Q_{ijTM}) \quad (32)$$

which means that the total energy of the excited target modes will be maximised. The best selection of actuator and sensor locations is also determined by equation (31).

If one of the target modes is not excited, i.e. it is not output controllable, then it is not identifiable by FRF (j, i), and not all modes of interest can be identified at one time. One can then define the index in another way:

$$P_{ij} = \text{mean}(Q_{ij1}, Q_{ij2}, \ldots, Q_{ijTM})\,\min\,(Q_{ij1} Q_{ij2}, \ldots, Q_{ijTM}). \quad (33)$$

This index considers the total energy and minimum individual energy together. In response to various considerations, such as the capability of the estimation program, several different indices can be devised by changing the weights of these two values.

Finally, the procedures of selecting the best sensor/actuator locations can be summarised as follows:

(1) according to the accessibility constant, delete the impossible FRFs;
(2) according to target mode set, compute the Q value for each FRF;
(3) compute the **P** matrix. The component of P matirx can be defined as equations (29), (32) or (33);
(4) according to equation (31), find the maximum component of **P** matrix. The corresponding indexes a and b of the larger P_{ab} components are the sensor/actuator locations.

In this way, even for a large structure, the optimal solution can be found efficiently.

3. SIMULATION RESULTS

Figure 2 shows the simulated system, which has 7 dof, and the corresponding natural frequencies and damping ratios are listed in Table 1. The NSR (noise-to-signal ratio) used

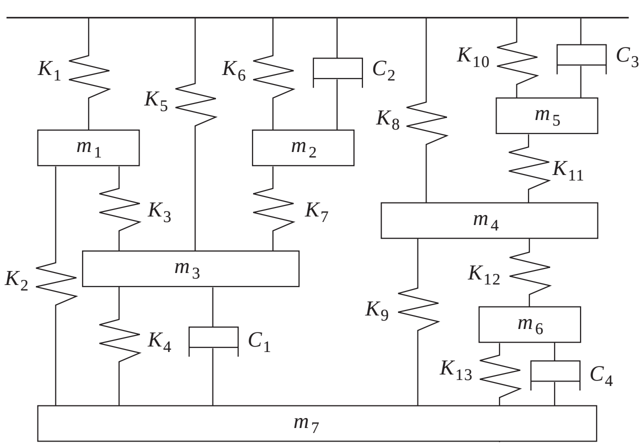

Figure 2. A lumped parameters test model. $M_{1\text{–}7} = 1$ kg; K_1, K_2, K_4, K_6, K_7, K_8, $K_{12} = 10\,000$ N/m; K_3, K_9, K_{10}, $K_{11} = 20\,000$ N/m; $K_5 = 5000$ N/m; $K_{13} = 30\,000$ N/m; $C_{1\text{–}4} = 20$ N/(m/s).

TABLE 1

Natural frequencies and damping ratios of a 7 dof lumped system

Mode	1	2	3	4	5	6	7
ω_n	12.5428	19.9756	26.1116	29.2773	40.276	43.0522	48.8616
ζ_n	0.0162	0.0739	0.0349	0.0433	0.0207	0.0283	0.0745

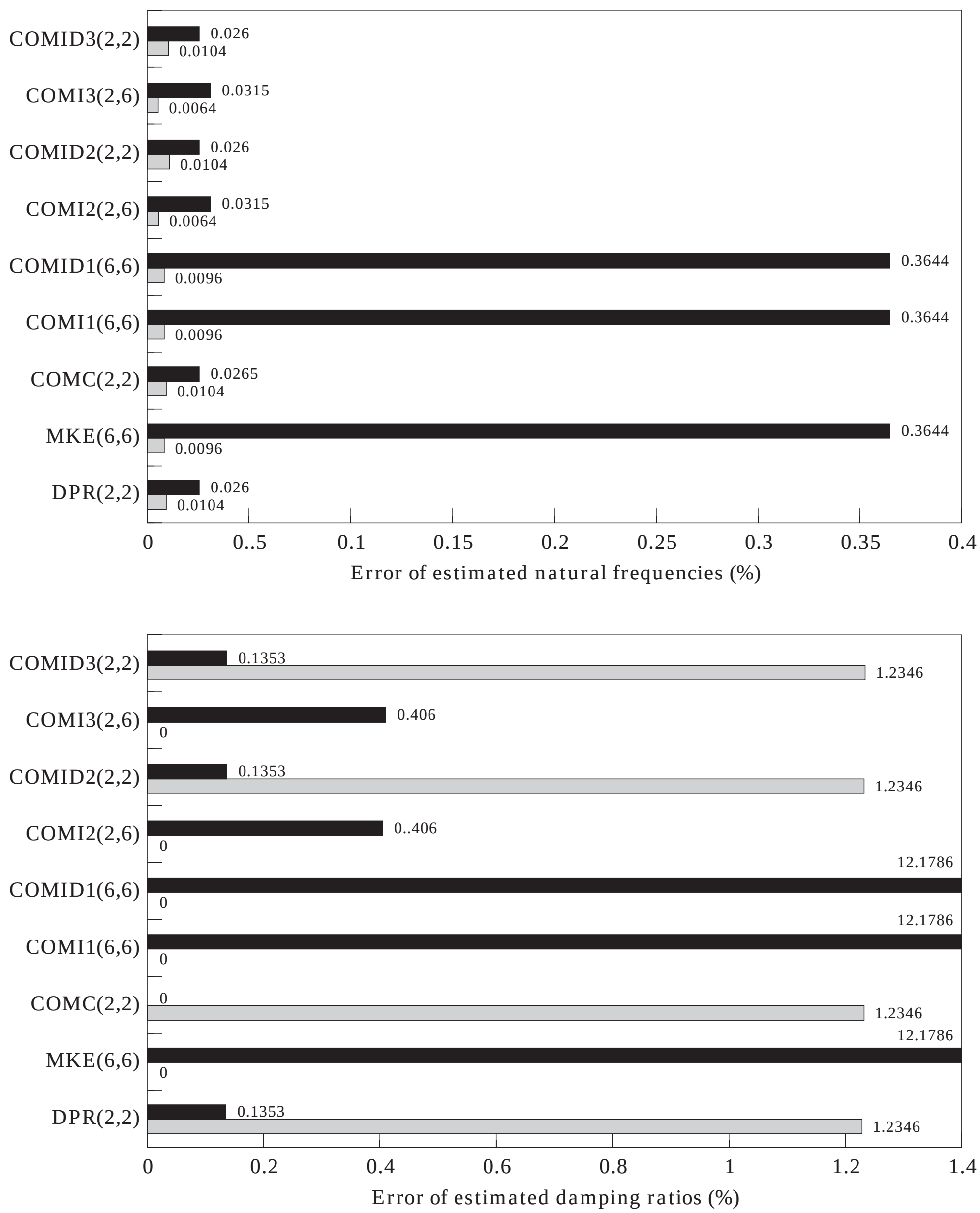

Figure 3. Comparison of different methods for target modes 1 (■), 2 (□).

in all figures and tables is defined as the ratio of the standard deviation of noise to that of the signal. The estimation is made by using an equation error method [17]. Programs were developed by using routines of the commercial program MATLAB, especially its signal processing toolbox [18]. In order to obtain a quantitative comparison, the estimated modal parameters from FRFs were used to examine the effectiveness of the sensor–actuator placement methods.

It can be found that if the target mode set contains only a single mode, the FRFs selected by all methods are the same and give the good estimates (not shown). Figures 3–5 show the estimated errors for identifying two target modes (1,2), (3,6), and (5,6), respectively. In these figures, each method and its selected sensor–actuator pair are denoted together for convenience of representation. For example, COMI3(2,6) represents that the COMI3 method was used to select sensor and actuator for the given target set, and the selected positions of sensor and actuator are at dof 2 and dof 6, respectively. The DPR and MKE shown here denote the methods proposed by Kientzy [2] and Chung [4]. The COMI1 to COMI3 methods denote that different indices, $P_{ij} = \text{sum}\,(Q_{ij1}, Q_{ij2}, \ldots, Q_{ijTM})$, $P_{ij} = \min\,(Q_{ij1}, Q_{ij2}, \ldots, Q_{ijTM})$, or $P_{ij} = \text{mean}\,(Q_{ij1}, Q_{ij2}, \ldots, Q_{ijTM})\min\,(Q_{ij1}, Q_{ij2}, \ldots, Q_{ijTM})$ are used, respectively. The same denotations are used for COMID1 to COMID3 methods. From these figures, it can be seen that if target modes are excited easily, e.g. modes (1,2), then all methods work well. For target modes (3,6), the COMI2, COMI3, COMID2 and COMID3 methods work better than the others. The COMI1 and COMID1 methods are not suitable for one-step identification; however, one mode (the third) can be estimated accurately. Some combinations of two target modes do not have sufficiently large responses simultaneously at any FRFs, e.g. modes (5,6), and it is difficult to find a method that can identify any two target modes at one time. In this situation, the number of target modes must be reduced and each mode identified individually.

When more than two modes are to be identified simultaneously, the results from different methods become more complicated. The case shown in Figs 6 and 7 illustrates different FRFs selected by different methods. No method obtained accurate modal parameters for all target modes. If some of the target modes are estimated accurately, it is regarded as successful sensor–actuator placement. COMI and COMID methods are found to work better than the other methods, and the COMI1 and COMID1 methods, provide better results than do the COMI2 and COMID2 methods. The results of the COMI3 and COMID3 methods fall between the above two indices.

4. A SCHEME FOR MODAL SURVEY

When performing a modal survey, one wants to identify all possible modes shown in FEA. Since the index P, defined as the summation of Q values (COMI1 or COMID1 methods), ensures that accurate parameters are obtained at least for one target mode, it is selected for the modal survey. To illustrate the proposed scheme for modal survey, consider Fig. 8. The procedure for identifying a simulated system is as follows.

(1) First choose the target modes 1, 2, 3, and 4, which are about half of the total number of modes. By using the COMI1 method (because damping ratio information is not available), the best selection is found to be (2,2). Since the mode 4 in FRF(2,2) in Fig. 9 is missing, so it must be included in the next target set (since a priori information, i.e. information on mode shapes and natural frequencies, is available from FEA, one can judge which mode is missing).

(2) Select modes 4, 5, 6 and 7 as the target set. If one uses the COMI1 method again, the best pair is (5,5). Modes 5 and 7 in FRF(5,5) are not clearly observed.

(3) Modes 5 and 7 now become the target modes for this estimation step. The best result is found to be FRF(3,3) and mode 7 is still not obvious.
(4) The best FRF for estimating mode 7 is FRF(7,7). All modes have been estimated at this point. The results are shown in Table 2.
(5) Now switch the sensor-actuator placement method to COMID1 by using the damping values just identified and repeat the estimation process from step 1. The first selection is FRF(6,6). Although the target modes are 1, 2, 3, and 4, the modes identified are 1, 2, and 4.

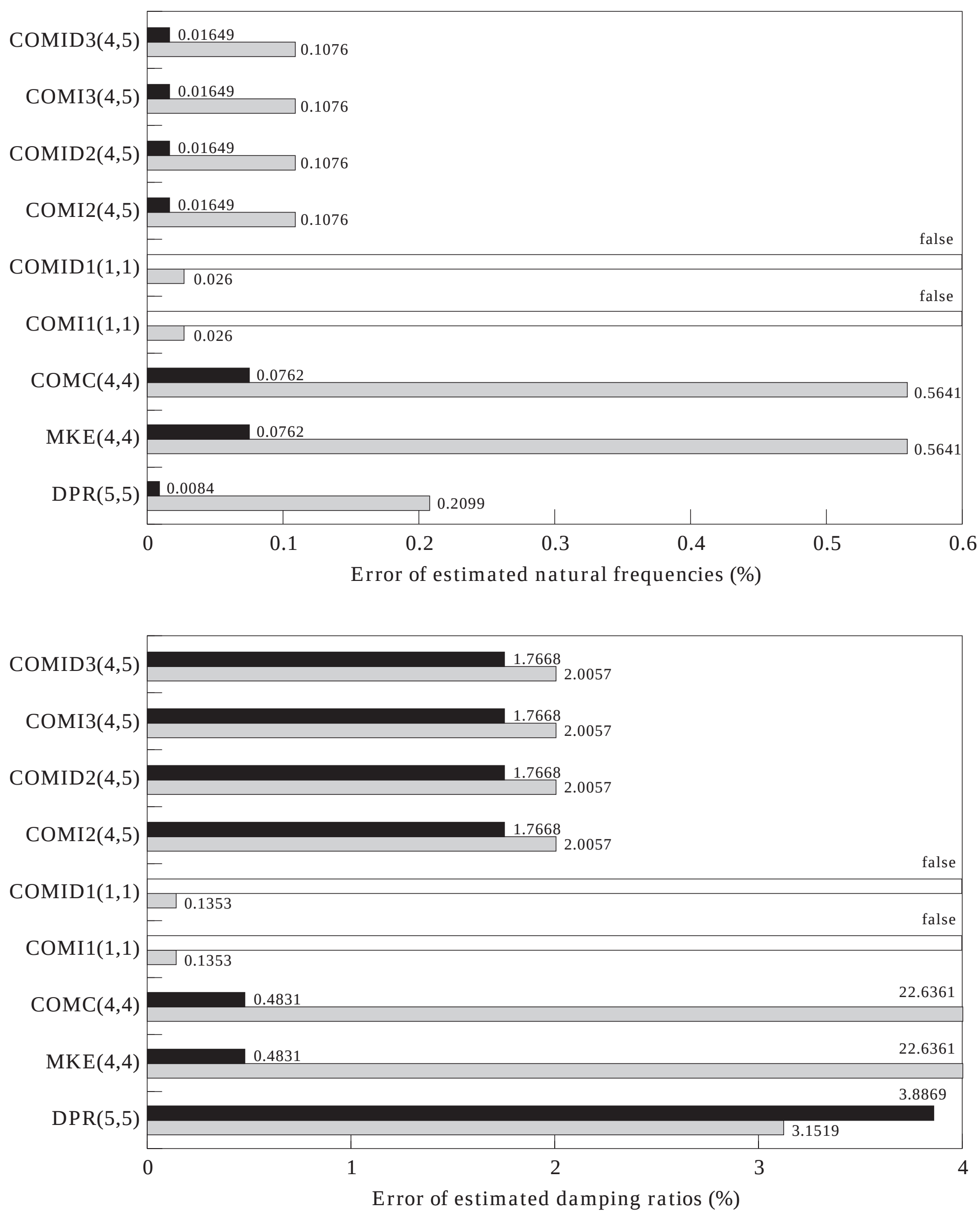

Figure 4. Comparison of different methods for target modes 3 (□), 6 (■).

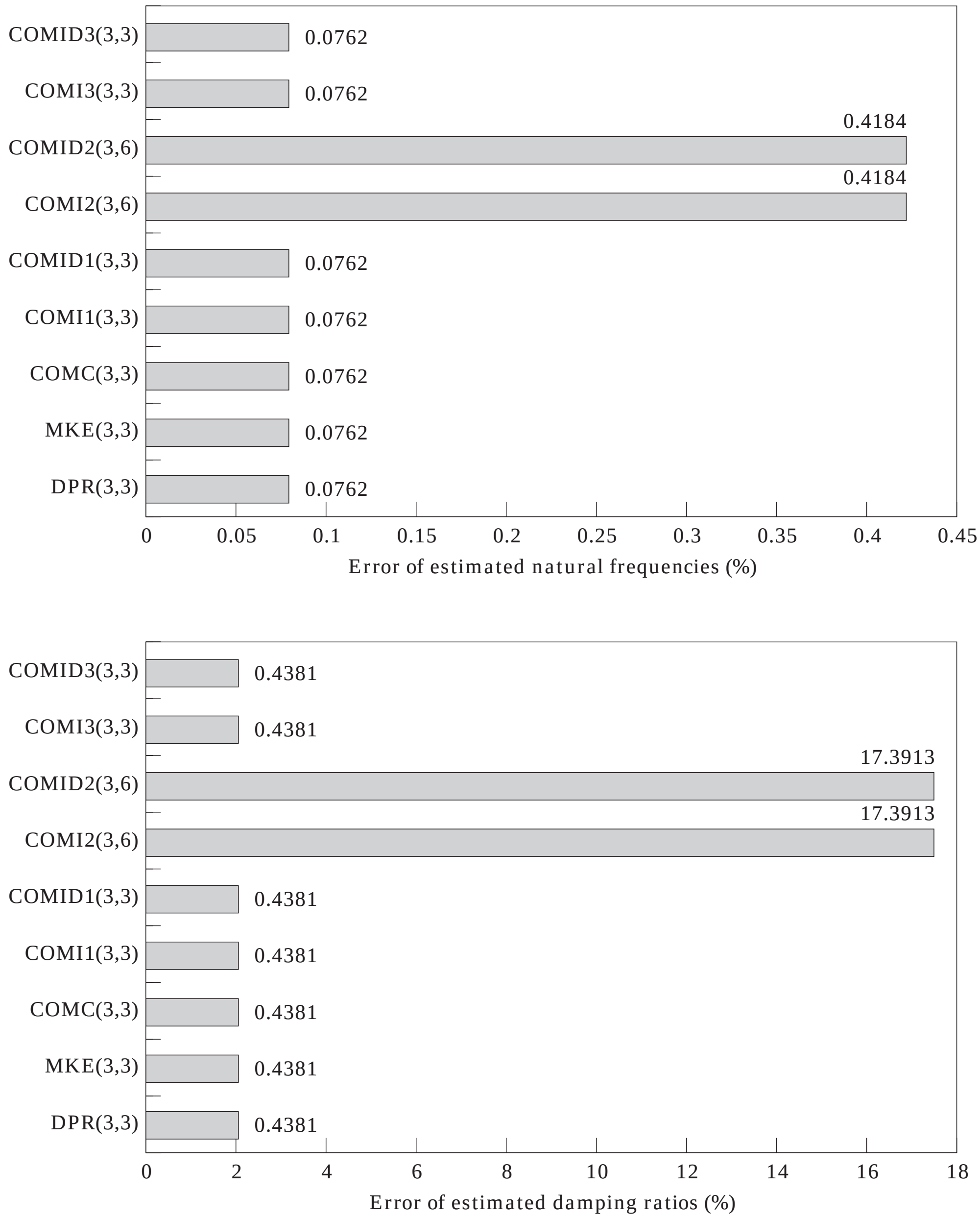

Figure 5. Comparison of different methods for target mode 5 (■).

(6) Now the target modes change to 3, 5, 6, and 7. The best selection is FRF(1,1); but only modes 3 and 5 are identified.
(7) The target modes now are 6 and 7. FRF(4,4) is chosen and mode 6 is estimated, but mode 7 is missing. Thus this estimation is similar to an estimation of a single target mode.
(8) Treating the single mode 7 as the target mode, one finds the best selection is FRF(7,7), which is the same as before. Thus there is no need to estimate this mode again.

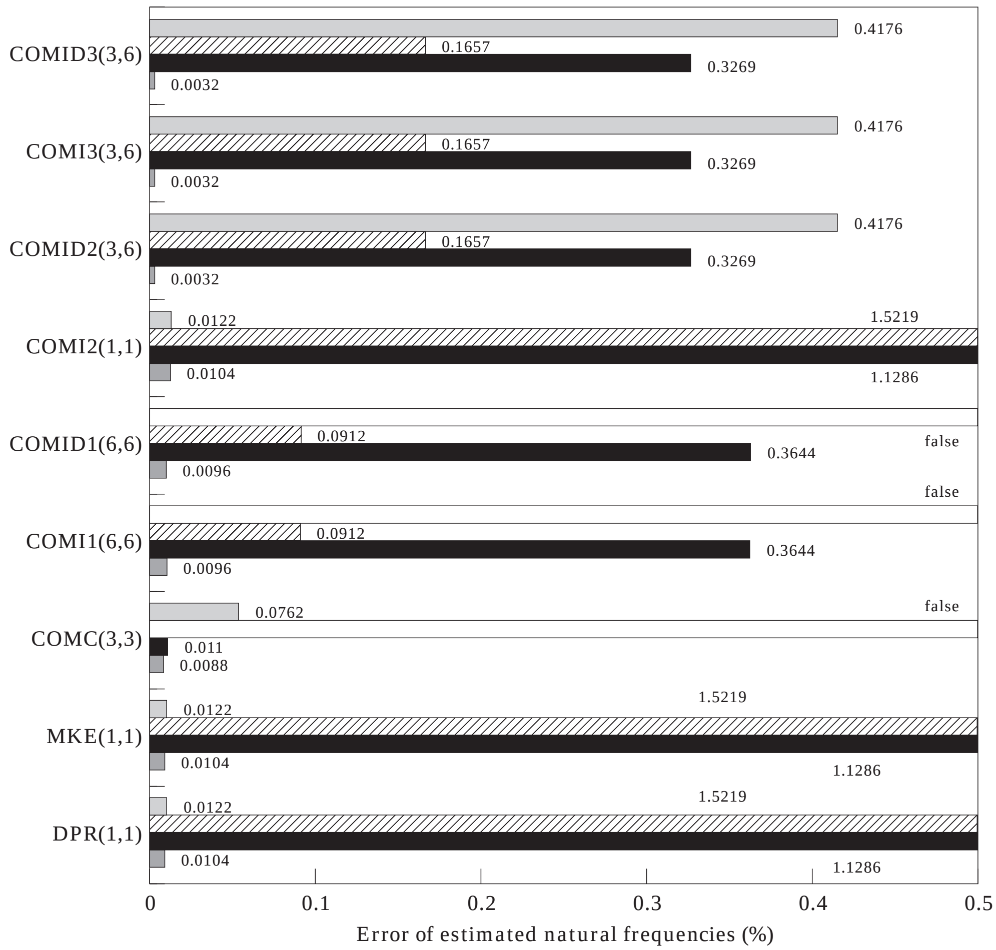

Figure 6. Comparison of different methods for target modes 1 (■), 2 (■), 4 (▨), 5 (■) (natural frequencies).

(9) All estimated parameters obtained by the COMI1 and COMID1 methods are listed in Table 2. The differences between them are small except for mode 2. Therefore, the mode 2 is estimated as a single target mode and find that the best selection is FRF(2,2).

(10) If the difference between the parameters estimated in two successive estimations is small, one can take the mean values of the two successive estimated parameters as the final values. When parameters are identified using the single target mode, they should be taken directly as the final values. In this way, a better and meaningful parameter set can be obtained. The final estimates are compared with the true values in Table 3.

5. DISCUSSION AND CONCLUSIONS

Generally, the COMI and COMID methods yield better results than the others. Different target mode sets and different definitions of index P produce different results. When the number of target modes is small, the index P defined as the minimum of Q values is recommended. On the other hand, if the number of target modes is large, taking the

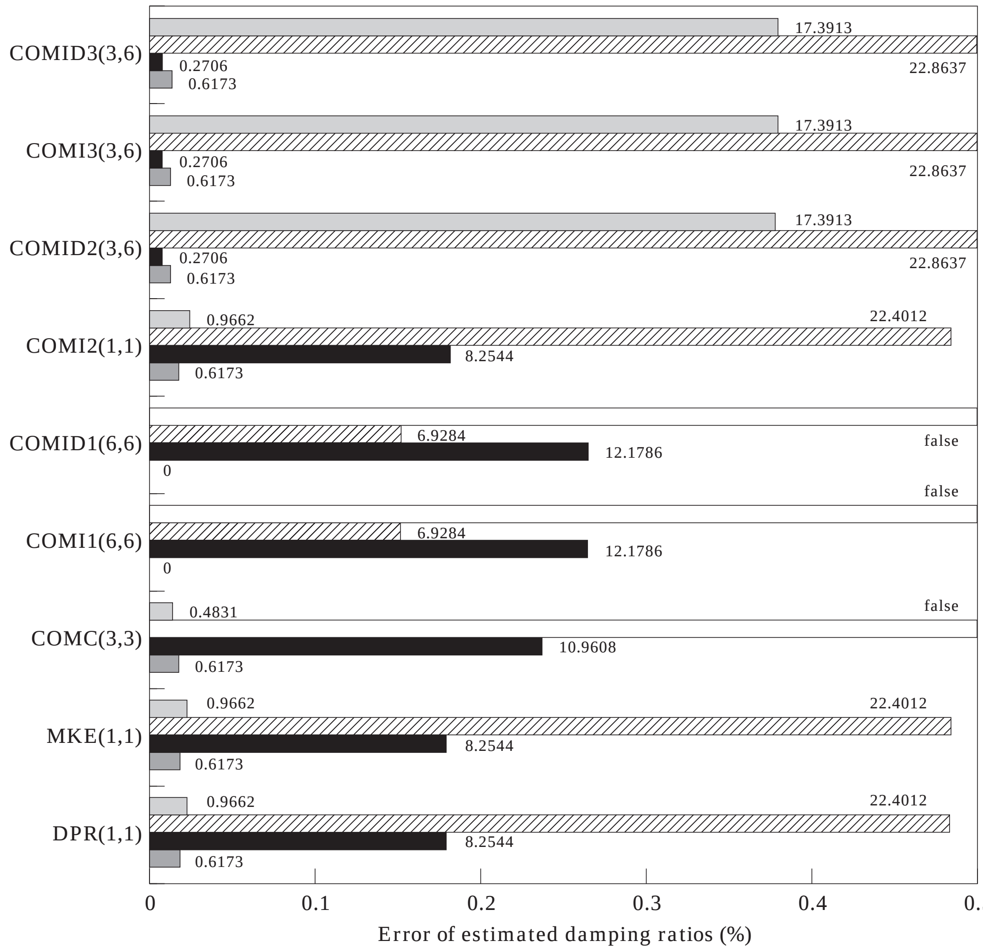

Figure 7. Comparison of different methods for target modes 1, 2, 4, 5. (damping ratios). Legend as for Fig. 6.

summation of Q values as an index is more appropriate. A compromise between the two indices is to define (mean Q^* min Q) as an index.

When the index P is defined as the summation of the Q values, if the Q value of a certain mode in the target set is extremely large compared with the others, then the COMI and COMID methods will degenerate into identifying that mode only. In contrast, when the index P is defined as the minimum of the Q values, then these two placement methods will ignore the large Q value and identify the smallest one. Thus, the former index ensures that the parameters of at least one target mode will be obtained, and the latter index works well if the smallest target mode Q value is not too small. When the index P is defined as (mean Q^* min Q), the COMI and COMID methods sometimes behave as when P is defined as the summation of the Q values, and sometimes behave as when P is defined as the minimum of Q values. It is difficult to decide which definition is better. One can select an appropriate index P by considering the test time, instruments, and accuracy required.

Performing a modal survey in an iterative way ensures quality of the estimated parameters. Simulation results indicate that the COMI and COMID methods work very well. All of the methods considered here require a priori information (mode shapes and natural frequencies, and sometimes the mass matrix), which can be obtained from FEA. These a priori data are used as initial values for the iterative identification methods.

Some modes cannot be clearly observed simultaneously because they are closely coupled or because of other structural characteristics, and thus these modes are hard to identify

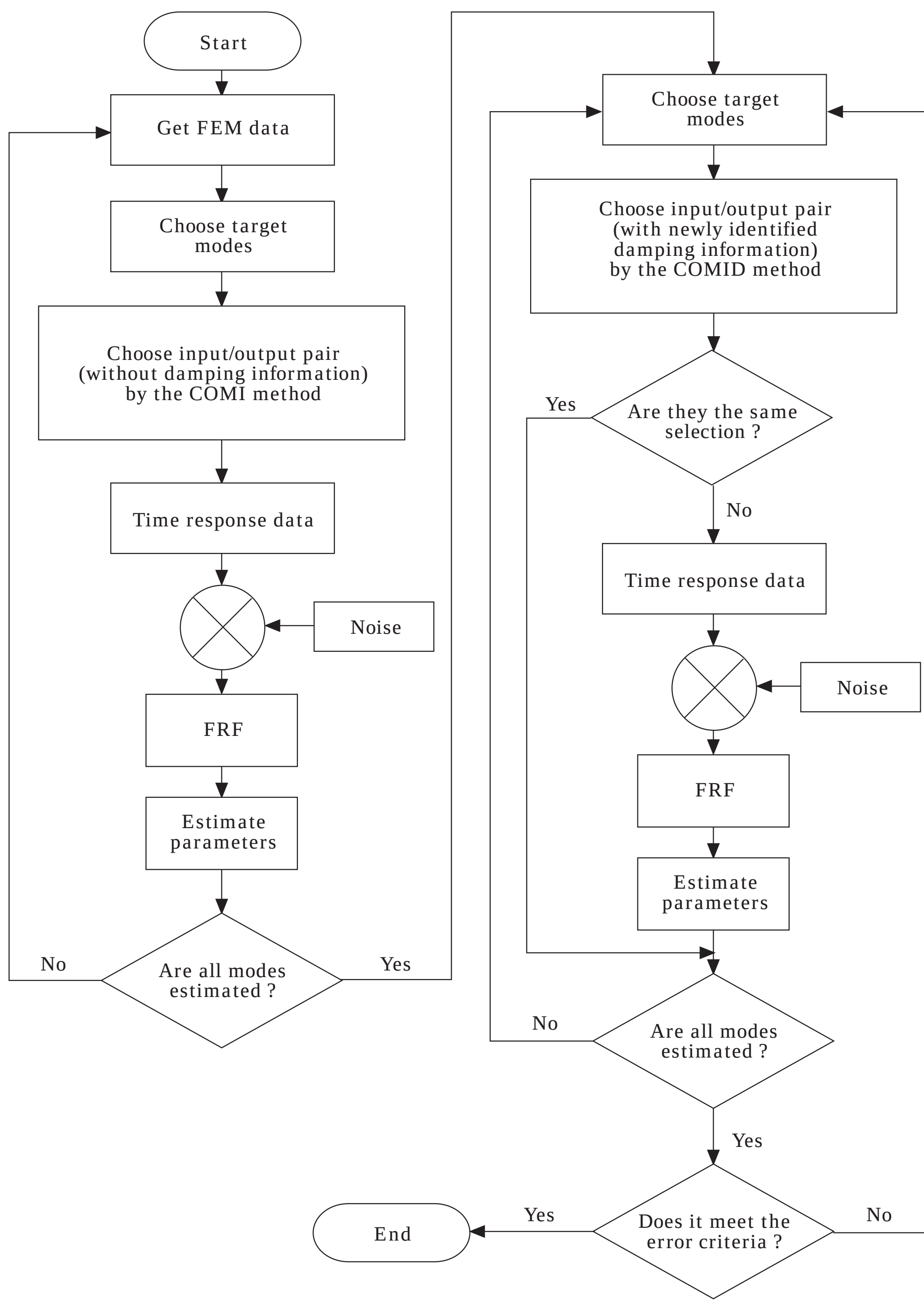

Figure 8. Flow chart for modal survey.

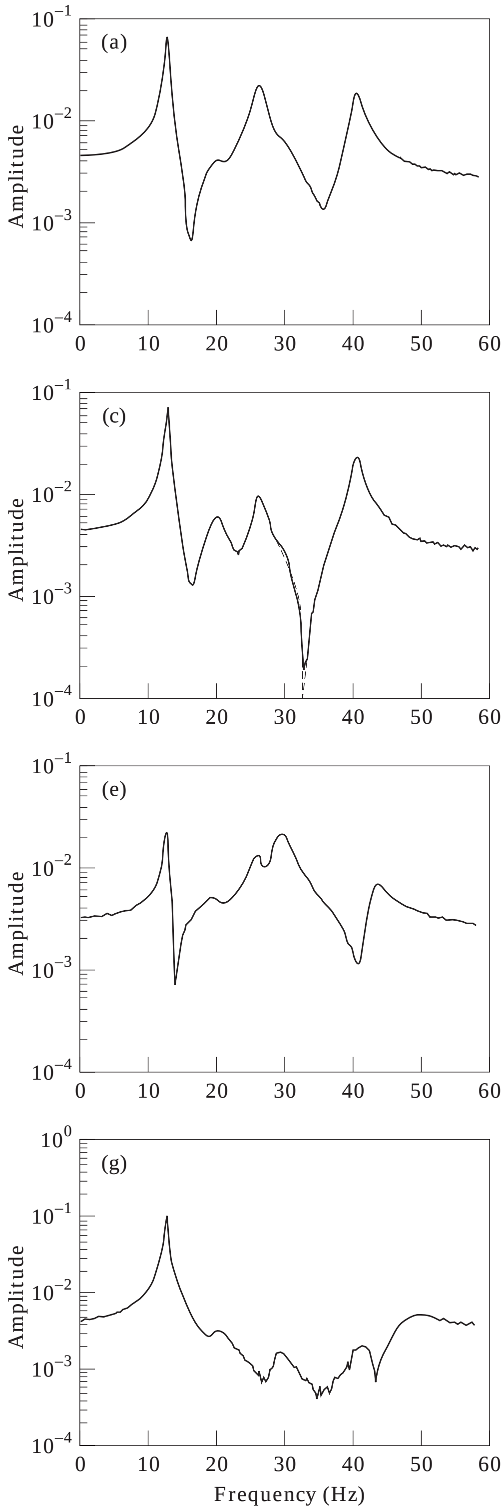

Figure 9. Amplitude *vs* frequency plot of FRFs: (a) (1,1), (b) (2,2), (c) (3,3), (d) (4,4), (e) (5,5), (f) (6,6), (g) (7,7). NSR = 1%.

TABLE 2

Estimates of the first and second iterations

No.	Natural frequency (first estimate) (Hz)	Natural frequency (second estimate) (Hz)	Difference (%)	Damping ratio (first estimate)	Damping ratio (second estimate)	Difference (%)
1	12.5418	12.5403	0.012	0.0159	0.0163	2.5157
2	19.9757	19.861	0.0574	0.0731	0.0645	11.7647
3	26.1262	26.1135	0.0486	0.0354	0.0344	2.8249
4	29.301	29.3164	0.0526	0.0427	0.0455	6.5574
5	40.2453	40.2802	0.0867	0.0206	0.0205	0.4854
6	43.0104	43.0255	0.0351	0.0295	0.0286	3.0508
7	48.579	48.579	0	0.0626	0.0626	0

NSR = 1%

simultaneously. For example, it is difficult to estimate modes 5 and 6 together, and there are some modes which cannot be sufficiently excited, such as mode 7 in this simulation. The error of the best damping ratio estimated from FRF(7,7) can be as large as 16%.

Although our simulations did not show this, it is necessary to modify the object function for different types of sensors. Furthermore, it is possible to extend this study to determine the positions of multiple sensors and actuators. Usually, fewer actuators than sensors are needed. One way to solve the placement problem is to choose actuator and sensor locations according to their total degrees of controllability and observability. Another way is similar to the COMI method: to determine the best dofs for actuators and sensors, one first finds the Q and P values and then sorts each column of the $\mathbf{P}$ matrix, summing the values according to the number of sensors required.

Finally, it should be noted that this paper is not trying to perform a modal reduction, so there is no need to perform any rank test. Usually, if a modal reduction is required, the rank tests of the controllability and observability matrices are necessary. The larger condition number of the reduced system indicates that more information is preserved. But here, the aim is to get the precise structural parameters of one or a few specific modes through limited number of sensors and actuators. The proposed measure provides us a convenient tool in the advance determination of sensor–actuator location for modal identification.

TABLE 3

Final estimates compared with the true values

No.	Natural frequency (theoretical) (Hz)	Natural frequency (estimated) (Hz)	Error (%)	Damping ratio (theoretical)	Damping ratio (estimated)	Error (%)
1	12.5428	12.5411	0.0136	0.0162	0.0162	0
2	19.9756	19.9757	0.0005	0.0739	0.0731	1.0825
3	26.1116	26.1199	0.0318	0.0349	0.0349	0
4	29.2773	29.3087	0.1073	0.0433	0.0441	1.8476
5	40.276	40.2628	0.0328	0.0207	0.0206	0.4831
6	43.0522	43.0255	0.062	0.0283	0.0286	1.0601
7	48.8616	48.579	0.5784	0.0745	0.0626	15.8732

NSR = 1%

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