

DILATION TO THE UNILATERAL SHIFTS

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The classical result of Foias says that an operator power dilates to a unilateral shift if and only if it is a C_0 contraction. In this paper, we consider the corresponding question of dilating to a unilateral shift. We show that for contractions with at least one defect index finite, dilation and power dilation to some unilateral shift amount to the same thing. The only difference is on the minimum multiplicity of the unilateral shift to which the contraction can be (power) dilated. We also obtain a characterization of contractions which are finite-rank perturbations of a unilateral shift, generalizing the rank-one perturbation result of Nakamura.

1. INTRODUCTION

The purpose of this paper is to address the problem, which bounded linear operator on a complex separable Hilbert space can be dilated to a unilateral shift. Recall that an operator A on space H is said to *dilate* (resp. *power dilate*) to operator B on K if there is an isometry V from H to K such that $A = V^*BV$ (resp. $A^n = V^*B^nV$ for all $n = 1, 2, \dots$) or, equivalently, if B is unitarily equivalent to a 2×2 operator matrix

$$\begin{bmatrix} A & * \\ * & * \end{bmatrix}$$

with A in its upper left corner (resp. B^n is unitarily equivalent to

$$\begin{bmatrix} A^n & * \\ * & * \end{bmatrix},$$

under the same unitary operator for all $n = 1, 2, \dots$). The *unilateral shift* S_k of multiplicity k ($1 \leq k \leq \infty$) is the operator $S_k(x_0, x_1, x_2, \dots) = (0, x_0, x_1, \dots)$ on $\sum_{n=0}^{\infty} \oplus H$ with $\dim H = k$.

The classical result of Foias settles the corresponding power dilation problem completely: an operator T power dilates to some unilateral shift S_k if and only if T is a contraction ($\|T\| \leq 1$) of class C_0 , that is, T satisfies $T^{**n} \rightarrow 0$ in the strong operator topology, and, moreover, in this case the minimum value of k is $\dim \text{ran} (1 - TT^*)^{\frac{1}{2}}$ (cf. [4, Problem 152]). In this paper, we consider the dilation problem for various classes of operators. In

Section 2 below, we will show that in all the cases we investigated (contractions with at least one defect index finite, C_0 contractions, strict contractions, normal contractions and compact contractions) the two classes, one consisting of those which dilate to a unilateral shift and the other those which power dilate, coincide. The only difference is on the minimum multiplicity of the unilateral shift which can be (power) dilated to. In the dilation case, this multiplicity can be as small as 1 if the two defect indices $d_T = \dim \text{ran}(1 - T^*T)^{\frac{1}{2}}$ and $d_{T^*} = \dim \text{ran}(1 - TT^*)^{\frac{1}{2}}$ of the contraction T under consideration are equal, and $d_{T^*} - d_T$ if otherwise. This is in contrast to the multiplicity d_{T^*} of the unilateral shift to which T can be power dilated. Our proof depends on the result of Nakamura [5, Corollary 3] on the rank-one perturbation of unilateral shifts and that of Carey [1, Proposition] on the finite-rank perturbation of isometries.

In Section 3, we take up the problem of characterizing contractions which are finite-rank perturbations of unilateral shifts. We prove in Theorem 3.1 that the completely nonunitary ones among such perturbations are exactly those C_0 contractions T with $d_T < d_{T^*}$. This generalizes the rank-one perturbation result of Nakamura [5] although our proof is built upon his.

The monograph [7] by Sz.-Nagy and Foias is our standard reference for the terminology and results of their contraction theory. We will also refer to some basic Fredholm theory from time to time. For this, the reader can consult [2, Chapter XI].

2. UNILATERAL SHIFT DILATION

We say that operator A on H dilates to operator B on K by n -dimension ($0 \leq n \leq \infty$) if $A = V^*BV$ and $\dim(K \ominus VH) = n$ for some isometry $V : H \rightarrow K$. We start this section with a characterization of operators which dilate to a unilateral shift by finite dimension.

THEOREM 2.1. *An operator T dilates to S_k by n -dimension ($1 \leq k \leq \infty, 0 \leq n < \infty$) if and only if T is a C_0 contraction with $d_T < \infty$ and $d_T \neq d_{T^*}$. In this case, $k = d_{T^*} - d_T$ and the minimum value for n is d_T .*

Note that if T is a C_0 contraction, then $d_T \leq d_{T^*}$ always holds (cf. [7, Proposition VI.3.5]).

To prove the necessity part of this theorem, we need the following proposition. Its proof we omit since it is analogous to that of [10, Proposition 3.5]. Recall that a contraction T is of class C_{11} if $T^n x \not\rightarrow 0$ and $T^{*n}x \not\rightarrow 0$ in norm for any nonzero vector x . A C_{11} contraction has equal defect indices (cf. [7, Proposition VI.3.5]).

PROPOSITION 2.2. *If T is a C_{11} contraction with finite defect indices or a contraction with $d_{T^*} < d_T$, then it has no unilateral shift dilation.*

Now we can prove the

NECESSITY OF THEOREM 2.1. Assume that

$$S_k = \begin{bmatrix} T & A \\ B & C \end{bmatrix} \quad \text{on } K = H \oplus C^n.$$

A simple computation yields $T^*T + B^*B = 1$. Hence $d_T = \text{rank}(1 - T^*T) = \text{rank } B^*B \leq n < \infty$. If T is not of class C_0 , then it has the canonical triangulation

$$T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix} \quad \text{of type } \begin{bmatrix} C_{.1} & * \\ 0 & C_{.0} \end{bmatrix}$$

with T_1 present (cf. [7, Theorem II.4.1]). Since $d_{T^*} \leq d_{T_1}$ (for T_1 is of class $C_{.1}$) and $d_{T_1} \leq d_T < \infty$ (cf. [7, Proposition VII.3.6]), we infer from Proposition 2.2 that T_1 has no unilateral shift dilation. Thus the same is true for T , which contradicts our assumption. Hence T must be of class C_0 . To prove that $d_T \neq d_{T^*}$, we appeal to the Fredholm theory. Since S_k and $T \oplus 0$ differ by a finite-rank operator, we have that T is left Fredholm and $\text{ind } T = \text{ind } S_k = -k$. The identity $d_T + \dim \ker T^* = d_{T^*} + \dim \ker T$ (cf. [3, Lemma 4]) then implies that $d_{T^*} - d_T = k$ and hence $d_T < d_{T^*}$. ■

The sufficiency part of Theorem 2.1 will be proved through a series of lemmas. The first one appeared in [5, Corollary 3]; the assertion on $\text{ran } F$ follows from the proof in there.

LEMMA 2.3. *If T is a C_0 contraction with $d_T = 1$ and $d_{T^*} \neq 1$, then there is a rank-one operator F with $\text{ran } F \subseteq \ker T^*$ such that $T + F$ is unitarily equivalent to $S_k, k = d_{T^*} - 1$.*

We next transfer this result into one concerning dilations

LEMMA 2.4. *If T is a C_0 contraction on H with $d_T = 1$ and $d_{T^*} \neq 1$, then T dilates to the unilateral shift $S_k, k = d_{T^*} - 1$, by one dimension.*

PROOF. Note that T is a left Fredholm operator with $\text{ind } T = 1 - d_{T^*}$ (cf. [10, Lemma 3.3]). Let $K = \ker T^*, J$ be the inclusion map from K to H , and

$$T' = \begin{bmatrix} T & J \\ 0 & 0 \end{bmatrix} \quad \text{on } H \oplus K.$$

Since

$$1 - T'^*T' = 1 - \begin{bmatrix} T^*T & T^*J \\ J^*T & J^*J \end{bmatrix} = \begin{bmatrix} 1 - T^*T & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$T^{j^{*n}} = \begin{bmatrix} T^{*n} & 0 \\ J^{*}T^{*(n-1)} & 0 \end{bmatrix},$$

it is easily seen that T' is also a C_0 contraction with $d_{T'} = 1$. We next show that $d_{T'^*} = d_{T^*}$. This is done by considering the following two separate cases:

(1) $d_{T^*} = \infty$. In this case, since $d_{T^*} \leq d_{T'} + d_{T'^*} = 1 + d_{T'^*}$ (cf. [7, Proposition VII.3.6]), we have $d_{T'^*} = \infty = d_{T^*}$.

(2) $1 < d_{T^*} < \infty$. Since $K \subseteq \text{ran}(1 - T'^*T^*)$, the assertion $d_{T^*} < \infty$ implies that K is finite-dimensional. Then T' and $T \oplus 0$ differ by a finite-rank operator. Since T is a Fredholm operator with $\text{ind } T = 1 - d_{T^*}$, we conclude that T' is also Fredholm and $\text{ind } T' = \text{ind } T = 1 - d_{T^*} = d_{T'} - d_{T^*}$. The identity $d_{T'} + \dim \ker T'^* = d_{T'^*} + \dim \ker T'$ then implies that $d_{T'^*} = d_{T^*}$.

By Lemma 2.3, there is a rank-one operator F' with $\text{ran } F' \subseteq \ker T'^* = \{0\} \oplus K$ such that $T' + F'$ is unitarily equivalent to S_k . If

$$F' = \begin{bmatrix} 0 & 0 & 0 \\ F_1 & F_2 & F_3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad T' = \begin{bmatrix} T & J_1 & J_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

on $H \oplus \text{ran } F' \oplus (K \ominus \text{ran } F')$, then

$$T'' \equiv \begin{bmatrix} T & J_1 \\ F_1 & F_2 \end{bmatrix}$$

is unitarily equivalent to some S_j ($1 \leq j \leq k$). But T'' and T differ by a finite-rank operator. Fredholm theory implies that $\text{ind } T'' = \text{ind } T$ or $j = k$. Hence T'' is unitarily equivalent to S_k . Since $\text{rank } F' = 1$, this shows that T dilates to S_k by one dimension, completing the proof. ■

The next lemma generalizes Lemma 2.4 to C_0 contractions T with $d_T < \infty$.

LEMMA 2.5. *If T is a C_0 contraction on H with $1 \leq d_T < \infty$ and $d_T \neq d_{T^*}$, then T dilates by one dimension to another C_0 contraction \tilde{T} with $d_{\tilde{T}} = d_T - 1$ and $d_{\tilde{T}^*} = d_{T^*} - 1$.*

PROOF. The case $d_T = 1$ is by Lemma 2.4. For the remaining part of the proof, we assume that $d_T > 1$. Since T is of class C_0 , its minimal isometric power dilation V is a unilateral shift with multiplicity d_{T^*} (cf. [7, Theorem VI.3.1]). We may assume that V has the form

$$V = \begin{bmatrix} S_n & * \\ 0 & T \end{bmatrix} = \begin{bmatrix} S_1 & 0 & D_1 \\ 0 & S_{n-1} & D_2 \\ 0 & 0 & T \end{bmatrix}$$

on $H_1 \oplus H_2 \oplus H$, where $n = d_T$. We claim that $\text{ran } D_1 = \ker S_1^*$ and $\text{ran } D_2 = \ker S_{n-1}^*$. Indeed, from $V^*V = 1$ we can compute that $S_1^*D_1 = 0$ and $S_{n-1}^*D_2 = 0$. Hence $\text{ran } D_1 \subseteq \ker S_1^*$ and $\text{ran } D_2 \subseteq \ker S_{n-1}^*$ hold, and, in particular, $\text{rank } D_1 \leq 1$. If $\text{rank } D_1 = 0$, that is, if $D_1 = 0$, then

$$\begin{bmatrix} S_{n-1} & D_2 \\ 0 & T \end{bmatrix}$$

is an isometric power dilation of T smaller than V , which contradicts our assumption on the minimality of V . Hence we must have $\text{rank } D_1 = 1$ or $\text{ran } D_1 = \ker S_1^*$. On the other hand, if $\text{ran } D_2 \neq \ker S_{n-1}^*$, then, expressing V as

$$V = \begin{bmatrix} S_1 & 0 & 0 & 0 & D_1 \\ 0 & A & B & C & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_3 \\ 0 & 0 & 0 & 0 & T \end{bmatrix}$$

on $H_1 \oplus (H_2 \ominus \ker S_{n-1}^*) \oplus (\ker S_{n-1}^* \ominus \text{ran } D_2) \oplus \text{ran } D_2 \oplus H$, the operator

$$\begin{bmatrix} S_1 & 0 & 0 & D_1 \\ 0 & A & C & 0 \\ 0 & 0 & 0 & D_3 \\ 0 & 0 & 0 & T \end{bmatrix}$$

is an isometric power dilation of T smaller than V , again contradicting the minimality of V . Therefore, $\text{ran } D_2 = \ker S_{n-1}^*$ as asserted. If we let

$$T' = \begin{bmatrix} S_{n-1} & D_2 \\ 0 & T \end{bmatrix},$$

then T' is a C_0 contraction. From $V^*V = 1$, we have

$$\begin{bmatrix} 0 & 0 \\ 0 & D_1^*D_1 \end{bmatrix} + T'^*T' = 1,$$

and hence $d_{T'} = \text{rank } (1 - T'^*T') = \text{rank } D_1^*D_1 = \text{rank } D_1 = 1$. On the other hand, we also have $d_{T'^*} > 1$ since if $d_{T'^*} = 1$ then T' is a $C_0(N)$ contraction which would imply that S_{n-1} is also of class $C_0(N)$, certainly a contradiction. Hence Lemma 2.4 is applicable, which yields a unilateral shift dilation V' of T' by one dimension. Assume that

$$V' = \begin{bmatrix} S_{n-1} & D_2 & T_{13} \\ 0 & T & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \quad \text{on } H_2 \oplus H \oplus C.$$

Since V' and S_{n-1} are both unilateral shifts, we have $T_{31} = 0$. Hence the contraction

$$\tilde{T} = \begin{bmatrix} T & T_{23} \\ T_{32} & T_{33} \end{bmatrix}$$

is of class C_0 . Letting $D = [D_2 \ T_{13}]$, we next check that $\text{ran } D = \ker S_{n-1}^*$. Since $V'^*V' = 1$, we have $S_{n-1}^*D = 0$ and hence $\text{ran } D \subseteq \ker S_{n-1}^*$. For the converse inclusion, let $y \in \ker S_{n-1}^* = \text{ran } D_2$. Then $y = D_2x$ for some x in H and hence $y = D \begin{bmatrix} x \\ 0 \end{bmatrix}$ is in $\text{ran } D$. Now we show that $d_{\tilde{T}} = d_T - 1$. Since

$$V' = \begin{bmatrix} S_{n-1} & 0 \\ 0 & \tilde{T} \end{bmatrix}$$

is an isometry, a simple computation yields $1 - \tilde{T}^*\tilde{T} = D^*D$. Thus $d_{\tilde{T}} = \text{rank } D^*D = \text{rank } D = n - 1 = d_T - 1$. Finally, $d_{\tilde{T}^*} = d_{T^*} - 1$ follows from the Fredholm theory and the identity $d_{\tilde{T}} + \dim \ker \tilde{T}^* = d_{\tilde{T}^*} + \dim \ker \tilde{T}$ as before. This shows that \tilde{T} is the desired dilation of T . ■

We are now ready for the proof of the

SUFFICIENCY OF THEOREM 2.1. The assertion is trivial if $d_T = 0$ and is a consequence of Lemma 2.4 if $d_T = 1$. In general, if $d_T > 1$, then apply Lemma 2.5 repeatedly to obtain a dilation \tilde{T} of T by d_T -dimension which is unitarily equivalent to S_k , $k = d_{T^*} - d_T$. ■

The next theorem, the main result of this section, shows that among contractions with at least one defect index finite, dilation to a unilateral shift and power dilation to one are about the same. The only difference is on the multiplicity of the unilateral shift to which the contraction is (power) dilated.

THEOREM 2.6. *Let T be a contraction with at least one defect index finite. Then the following conditions are equivalent:*

- (a) T dilates to some S_k , $1 \leq k \leq \infty$;
- (b) T power dilates to some S_ℓ , $1 \leq \ell \leq \infty$;
- (c) T is of class C_0 .

Moreover, in this case, the minimum value of k in (a) is 1 if $d_T = d_{T^*}$ and $d_{T^*} - d_T$ if otherwise, and the minimum value of ℓ in (b) is d_{T^*} .

Since a C_0 contraction with equal and finite defect indices must be of class $C_0(N)$, we now recall the relevant definitions of such operators. Let T be a *completely nonunitary* (c.n.u.) contraction, that is, T has no nontrivial reducing subspace on which T is unitary. T is of class C_0 if there is a nonzero function f in H^∞ such that $f(T) = 0$, where $f(T)$ is formed according to the Sz.-Nagy-Foias functional calculus. In this case, there is a (nonconstant) inner function ϕ in H^∞ such that $\phi(T) = 0$ and it divides any function f in H^∞ with

$f(T) = 0$. Such a ϕ is unique up to a scalar multiple with modulus one and is called the *minimal function* of T . Note that the defect indices of any C_0 contraction are equal. A C_0 contraction is of class $C_0(N)$ (N a positive integer) if its defect indices are at most N . An example of $C_0(N)$ contractions is the operator $S(\phi)$, where ϕ is any (nonconstant) inner function, on $H(\phi) = H^2 \ominus \phi H^2$ defined by $S(\phi)f = P(zf(z))$, where P is the (orthogonal) projection onto $H(\phi)$. The defect indices of $S(\phi)$ are equal to 1 and the minimal function is ϕ .

For convenience, we denote the unilateral shift of multiplicity 1 by S . The next lemma facilitates the dilation to S . It slightly generalizes [9, Theorem 2.7].

LEMMA 2.7. *For (nonconstant) inner functions $\phi_n, n = 1, 2, \dots$, the direct sum $\sum_n \oplus S(\phi_n)$ dilates to S .*

PROOF. As proved in [9, Theorem 2.7], for an inner function ϕ_1 there is $a_1, 0 < a_1 \leq 1$, such that the unilateral shift S on ℓ^2 is unitarily equivalent to

$$\left[\begin{array}{cccc|ccc} & & & & 0 & & \\ & & & & \vdots & & \\ & & & & \vdots & & \\ S(\phi_1) & & & & \vdots & & \\ & & & & \vdots & & \\ 0 & 0 & \cdots & a_1 & 0 & & \\ \hline & & & & 1 & 0 & \\ & & & & & 1 & 0 \\ & & & & & & 1 & \ddots \\ & & & & & & & \ddots \end{array} \right],$$

where the unspecified entries are all zero. Hence there are orthonormal vectors $\{e_{11}, e_{12}, \dots\}$ in ℓ^2 such that the compression $P_1 S|_{K_1}$, where $K_1 = \vee\{e_{11}, e_{12}, \dots\}$ and P_1 is the (orthogonal) projection onto K_1 , is unitarily equivalent to $S(\phi_1)$. Applying this to the operator in the lower right corner of the above matrix, we obtain that S is unitarily equivalent to

$$\left[\begin{array}{cccc|ccc|ccc} & & & & 0 & & & & & \\ & & & & \vdots & & & & & \\ & & & & \vdots & & & & & \\ S(\phi_1) & & & & \vdots & & & & & \\ & & & & \vdots & & & & & \\ 0 & 0 & \cdots & a_1 & 0 & & & & & \\ \hline & & & & * & & & & 0 & \\ & & & & \vdots & & & & \vdots & \\ & & & & \vdots & & & & S(\phi_2) & \\ & & & & \vdots & & & & \vdots & \\ & & & & * & 0 & 0 & \cdots & a_2 & 0 \\ \hline & & & & * & & & & 1 & 0 \\ & & & & \vdots & & & & & 1 & 0 \\ & & & & \vdots & & & & & & 1 & \ddots \\ & & & & \vdots & & & & & & & \ddots \end{array} \right]$$

Hence there exist orthonormal vectors $\{e_{21}, e_{22}, \dots\}$ in ℓ^2 all orthogonal to $\{e_{11}, e_{12}, \dots\}$ such that $P_2 S|_{K_2}$, where $K_2 = \vee\{e_{21}, e_{22}, \dots\}$ and P_2 is the (orthogonal) projection onto K_2 , is unitarily equivalent to $S(\phi_2)$. Repeating this argument indefinitely results in the orthonormal vectors $\{e_{ij}, i, j \geq 1\}$ with the property that on $\ell^2 = K \oplus K^\perp, K = \vee\{e_{ij}, i, j \geq 1\}$, S has the representation

$$\begin{bmatrix} \sum_n \oplus S(\phi_n) & * \\ * & * \end{bmatrix}$$

This completes the proof. ■

LEMMA 2.8. *Every C_0 contraction dilates to the unilateral shift S . Moreover, any dilation of a C_0 contraction to some $S_k, 1 \leq k \leq \infty$, must be by infinite dimension.*

PROOF. It was shown in [6, Lemma 4] that every C_0 contraction T with minimal function ϕ can be extended to the operator $\underbrace{S(\phi) \oplus \dots \oplus S(\phi)}_{d_T}$. Hence our first assertion follows from Lemma 2.7.

If the C_0 contraction T dilates to S_k by finite dimension, then from the fact that T is Fredholm with index 0 and that S_k and $T \oplus 0$ differ by a finite-rank operator we arrive at $-k = \text{ind } S_k = \text{ind } T = 0$, a contradiction. This proves the second assertion. ■

PROOF OF THEOREM 2.6. The equivalence of (b) and (c), together with the assertion on the minimum value of ℓ , is the classical result of Foias (cf. [4, Problem 152]). The implication (c) \Rightarrow (a) and the assertion on the minimum value of k , other than the fact that if the C_0 contraction T with $d_T < d_{T^*}$ dilates to S_k by infinite dimension, then $d_{T^*} - d_T \leq k$, are consequences of Theorem 2.1 and Lemma 2.8. We now proceed to prove this claim. Indeed, under the assumption $d_T < d_{T^*}$, Theorem 2.1 implies that T also dilates to S_d by n -dimension, where $d = d_{T^*} - d_T$ and $n = d_T$. Hence we may assume that

$$(*) \quad S_k = \begin{bmatrix} S_d + F & A \\ B & C \end{bmatrix}$$

for some finite-rank operator F . Since $S_k^* S_k = 1$, a simple computation yields (1) $S_d^* S_d + S_d^* F + F^* S_d + F^* F + B^* B = 1$, (2) $S_d^* A + F^* A + B^* C = 0$ and (3) $A^* A + C^* C = 1$. From (1), we have $B^* B = -S_d^* F - F^* S_d - F^* F$ and hence $B^* B$, together with B , is of finite rank. Assume first that $d < \infty$. Since $A = S_d S_d^* A + (1 - S_d S_d^*) A = -S_d (F^* A + B^* C) + (1 - S_d S_d^*) A$ by (2), A is also of finite rank. Hence S_k and $S_d \oplus C$ differ by a finite-rank operator. Therefore, $-k = \text{ind } S_k = \text{ind } S_d + \text{ind } C = -d + \text{ind } C$ or $\text{ind } C = d - k$. Hence if $d > k$, then C has the polar decomposition $C = V(C^* C)^{\frac{1}{2}}$, where V is a coisometry with $\dim \ker V = d - k$. From (3), we obtain $C = V(1 - A^* A)^{\frac{1}{2}}$. It is then easy to infer that C and V differ by a finite-rank operator, and thus the same is true for S_k and $S_d \oplus V$. Since V is the direct sum of a unitary

operator U and the adjoint of some S_m ($m = d - k$) and $S \oplus S^*$ is a rank-one perturbation of the simple bilateral shift W , we infer that S_k and $S_k \oplus U \oplus \underbrace{W \oplus \cdots \oplus W}_m$ differ by a finite-rank operator. Carey's result [1, Proposition] implies that these two isometries have unitarily equivalent absolutely continuous unitary parts, which is a contradiction. Thus in the case $d < \infty$, we have $d \leq k$. On the other hand, if $d = \infty$, then we infer from (*) that for any positive integer ℓ there is a finite-rank F_ℓ such that $S_\ell + F_\ell$ dilates to S_k . Hence from what we've proved above, $\ell \leq k$ holds. This implies that k is also infinity. Thus $d = d_{T^*} - d_T \leq k$ is always true.

To complete the proof, we need show that (a) \Rightarrow (c). The argument is analogous to the one in proving the necessity part of Theorem 2.1. If T is not of class C_0 , then it has the triangulation

$$T = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix} \quad \text{of type} \quad \begin{bmatrix} C_{.1} & * \\ 0 & C_0 \end{bmatrix}$$

with T_1 present. Since $d_{T_2^*} \leq d_{T_1} \leq d_T$ (because T_1 is of class $C_{.1}$) and $d_{T_1^*} \leq d_{T^*}$ (because T_2 is of class C_0), we have $d_{T_1^*} < \infty$. Proposition 2.2 implies that T_1 has no unilateral shift dilation and thus the same is true for T . This proves (a) \Rightarrow (c). ■

The following is an easy consequence of Theorem 2.6.

COROLLARY 2.9. *Let T be a contraction with at least one defect index finite. Then T dilates to the unilateral shift S if and only if T is of class C_0 and $d_{T^*} - d_T = 0$ or 1.*

An operator T is algebraic if $p(T) = 0$ for some polynomial p . Since every algebraic contraction can be written as the direct sum of a unitary operator with finitely many points in its spectrum and a C_0 contraction with minimal function a finite Blaschke product, the next corollary follows easily from Theorem 2.6. It generalizes [9, Theorem 2.7] for finite-dimensional operators.

COROLLARY 2.10. *The following conditions are equivalent for an algebraic operator T :*

- (a) T dilates to S ;
- (b) T dilates to some S_k , $1 \leq k \leq \infty$;
- (c) T power dilates to some S_ℓ , $1 \leq \ell \leq \infty$;
- (d) T is a C_0 contraction;
- (e) T is a contraction with spectrum contained in $\mathbf{D}(= \{z \in \mathbf{C}: |z| < 1\})$.

COROLLARY 2.11. *If T is a strict contraction ($\|T\| < 1$) or has numerical radius at most $1/2$, then T dilates to the unilateral shift S .*

Recall that the numerical radius of an operator T on H is the quantity sup

$\{|\langle Tx, x \rangle| : x \in H, \|x\|=1\}$.

PROOF. It was proved in [10, Proposition 2.7] that for every strict contraction T there is an inner function ϕ , $\phi(z) = z^n$, such that T dilates to $S(\phi) \oplus S(\phi) \oplus \cdots$. Hence T dilates to S by Lemma 2.7. On the other hand, it was noted in [10] that if T has numerical radius at most $1/2$, then it dilates to $S(\varphi) \oplus S(\varphi) \oplus \cdots$, where $\varphi(z) = z^2$. Hence T dilates to S as above. ■

COROLLARY 2.12. *Every operator dilates to a scalar multiple of the unilateral shift S .*

PROOF. Since for any operator some scalar multiple of it is a strict contraction, the assertion follows from Corollary 2.11. ■

To conclude this section, we consider dilating normal operators and compact operators to a unilateral shift.

COROLLARY 2.13. *The following conditions are equivalent for a normal operator T :*

- (a) T dilates to S ;
- (b) T dilates to some $S_k, 1 \leq k \leq \infty$;
- (c) T power dilates to some $S_\ell, 1 \leq \ell \leq \infty$;
- (d) T is a C_{00} contraction, that is, $T^n \rightarrow 0$ and $T^{*n} \rightarrow 0$ in the strong operator topology;
- (e) T is a c.n.u. contraction.

PROOF. The implications (a) \Rightarrow (b) and (b) \Rightarrow (e) are trivial. The equivalences (c) \Leftrightarrow (d) and (d) \Leftrightarrow (e) are proved in [4, Problem 152] and [8, Lemma], respectively. To complete the proof, we need only check (e) \Rightarrow (a). Indeed, if T is a c.n.u. contraction, then by the spectral theorem $T = \sum_n \oplus T_n$, where the T_n 's are all strict contractions. A similar argument as in the proof of Corollary 2.11 shows that T dilates to an operator of the form $\sum_j \oplus S(\phi_j)$, where each ϕ_j is of the form $\phi_j(z) = z^{n_j}, n_j \geq 1$. (a) then follows from Lemma 2.7. ■

COROLLARY 2.14. *The following conditions are equivalent for a compact operator T :*

- (a) T dilates to some $S_k, 1 \leq k \leq \infty$;
- (b) T power dilates to some $S_\ell, 1 \leq \ell \leq \infty$;
- (c) T is a C_{00} contraction;
- (d) T is a contraction with spectrum contained in \mathbf{D} .

PROOF. If (a) holds, then obviously T is a c.n.u. contraction. Since T is compact, the intersection $\sigma(T) \cap \sigma\mathbf{D}$ consists of eigenvalues of T . Hence T is c.n.u. implies that $\sigma(T) \cap \sigma\mathbf{D} = \phi$, that is, $\sigma(T)$ is contained in \mathbf{D} so (a) \Rightarrow (d). Since the implications (d)

\Rightarrow (c) and (c) \Rightarrow (b) are proved in [4, Problem 153] and [4, Problem 152], respectively, and (b) \Rightarrow (a) is trivial, the proof is complete. ■

We conclude this section by asking whether contractions which dilate to a unilateral shift can always power dilate to one. In the cases we considered here, this is indeed true. The general case remains open. Another question worth investigating is whether a contraction with equal defect indices always dilates to the unilateral shift S . In particular, this is unknown for compact contractions. (Note that every compact contraction has both defect indices equal to infinity.) Since every finite-rank operator is algebraic, Corollary 2.10 affirms this for finite-rank contractions.

3. FINITE-RANK PERTURBATION

As we have seen in Section 2, the result of Nakamura, Lemma 2.3, on the rank-one perturbation of unilateral shifts plays a prominent role in our derivation of the dilation theorems. In this section, we generalize his result to a characterization of contractions which are finite-rank perturbations of unilateral shifts. Our main result is the following.

THEOREM 3.1 *Let T be a contraction and $1 \leq k \leq \infty$. Then $T = S_k + F$ for some finite-rank operator F if and only if T is the direct sum of a multicyclic singular unitary operator U and a C_0 contraction T_1 with $d_{T_1} < \infty$ and $d_{T_1^*} - d_{T_1} = k$. In this case, both $\mu(U)$, the multiplicity of U , and d_{T_1} are no bigger than $\text{rank } F$, and furthermore the finite-rank operator F can be chosen so that $\text{rank } F \leq \mu(U) + d_{T_1}$.*

Recall that the multiplicity $\mu(T)$ of an operator T on H is the minimum cardinality of a subset X of H for which $H = \vee\{T^n x : n \geq 0, x \in X\}$. T is *multicyclic* if $\mu(T) < \infty$ and *cyclic* if $\mu(T) = 1$.

The necessity part of Theorem 3.1, together with the assertion $\mu(U), d_{T_1} \leq \text{rank } F$, can be proved analogously as in [10, Theorem 4.1]; we omit the proof. For the sufficiency, we need the following.

LEMMA 3.2. *If T is a C_0 contraction with $d_T < \infty$ and $d_T \neq d_{T^*}$, then there is an operator F with $\text{rank } F \leq d_T$ such that $T + F$ is unitarily equivalent to $S_k, k = d_{T^*} - d_T$.*

PROOF. We prove this by induction on $n = d_T$. The case $n = 0$ is trivial while $n = 1$ is the result of Nakamura (cf. Lemma 2.3). Now we assume that the assertion is true for $n - 1$ and let T be a C_0 contraction on H with $d_T = n \geq 1$ and $d_{T^*} \neq n$. By Lemma 2.5, T dilates to a C_0 contraction \tilde{T} on $H \oplus \mathbb{C}$ with $d_{\tilde{T}} = d_T - 1$ and $d_{\tilde{T}^*} = d_{T^*} - 1$. Since $d_{\tilde{T}} \neq d_{\tilde{T}^*}$, the induction hypothesis can be applied to yield an operator G with rank

$G \leq n - 1$ such that $\tilde{T} + G$ is unitarily equivalent to $S_k, k = d_{\tilde{T}^*} - d_{\tilde{T}} = d_{T^*} - d_T$. Let

$$G = \begin{bmatrix} G_1 & * \\ * & * \end{bmatrix} \quad \text{on} \quad H \oplus \mathbb{C}.$$

Since $T + G_1$ dilates to $\tilde{T} + G$ by one dimension, Theorem 2.1 implies that $T_1 \equiv T + G_1$ is of class C_0 with $d_{T_1} \leq 1$ and $d_{T_1^*} - d_{T_1} = k = d_{T^*} - d_T$. Now apply Lemma 2.3 to T_1 to yield an operator F_1 with $\text{rank } F_1 \leq 1$ such that $T_1 + F_1 = T + G_1 + F_1$ is unitarily equivalent to S_k . Since $\text{rank } (G_1 + F_1) \leq \text{rank } G_1 + \text{rank } F_1 \leq n$, this completes the proof of the induction. ■

SUFFICIENCY OF THEOREM 3.1 The sufficiency and the choice of F satisfying $\text{rank } F \leq \mu(U) + d_{T_1}$ follows from Lemma 3.2 and Nakamura's result [5, Proposition 2] that for any cyclic singular unitary operator U and $1 \leq k \leq \infty$, there is a rank-one operator F such that $S_k + F$ is unitarily equivalent to $U \oplus S_k$. ■

We remark that in [5] Nakamura actually proved the following more precise rank-one perturbation result:

Let T be a contraction and $1 \leq k \leq \infty$. Then $T = S_k + F$ for some rank-one operator F if and only if either (1) T is the direct sum of a cyclic singular unitary operator and S_k , or (2) T is of class C_0 with $d_T = 1$ and $d_{T^} = k + 1$.*

(The necessity is by [5, Theorems 2 and 3] and the sufficiency by [5, Proposition 2 and Corollary 3].) Our Theorem 3.1 is not strong enough to cover this case. It suggests that under the conditions of Theorem 3.1, the inequality $\mu(U) + d_{T_1} \leq \text{rank } F$ is probably true. This we are unable to prove at the present time.

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