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Representations and characterizations of vertices of bounded-shape partition polytopes

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Abstract

Consider a finite set whose elements are associated with vectors of common dimension. A partition of such a set is associated with a matrix whose columns are the sums of the vectors corresponding to each part. The *partition polytope* associated with a class of partitions (that share the number of parts) is then the convex hull of the corresponding matrices. We derive representations and characterizations of these polytopes and their vertices.

Keywords: Partitions; Polytopes; Vertices

1. Introduction

Following Barnes et al. (1992), hereafter referred to as BHR, we study partitions where each element of the partitioned set is associated with a (fixed) number of numerical attributes. So, vectors $A^1, \ldots, A^n$ are given, say of

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dimension k, and we consider *(ordered) partitions* $\pi = (\pi_1, \ldots, \pi_p)$ of $N \equiv \{1, \ldots, n\}$. Given a partition $\pi = (\pi_1, \ldots, \pi_p)$, we refer to the integer p as the *size* of π and to the integer vector $(|\pi_1|, \ldots, |\pi_p|)$ as the *shape* of π. Also, π is associated with the $k \times p$ matrix $A^\pi \equiv (\sum_{t\in\pi_1} A^t, \ldots, \sum_{t\in\pi_p} A^t)$. The *partition polytope* associated with a set Π of partitions sharing a common size p is defined as the convex hull of all $k \times p$ matrices A^π corresponding to partitions π in Π. Of particular interest are *constrained-shape partition polytopes* where Π is determined by constraints over shapes; if such constraints are in terms of lower and upper bounds, we refer to *bounded-shape partition polytopes*.

In this paper we explore vertices of partition polytopes. A motivation for one's interest in the vertices of convex hulls of finite sets is the following standard result.

Proposition 1.1. *Let Ψ be a finite set of vectors of common size and $P \equiv \operatorname{conv} \Psi$. Then Ψ contains all vertices of P. Further, if $h(\cdot)$ is a convex function on P, a maximum of $h(\cdot)$ over P is attained at a vertex of P and such a vertex maximizes $h(\cdot)$ over Ψ.* [3]

Proposition 1.1 is relevant to the study of maximization problems over sets of partitions where the objective $F(\pi)$ associated with a partition π has the representation $F(\pi) = h(A^\pi)$ with $h(\cdot)$ as a real-valued convex function on the convex hulls of the A^π's; see Hwang and Rothblum (in progress), Gao et al. (1998) and references therein for specific applications of such partitioning problems in diverse fields that include clustering, statistics, scheduling, reliability, inventory and system assembly. Specifically, Proposition 1.1 suggests that these partitioning problems be embedded in the problem of maximizing $h(\cdot)$ over the corresponding partition polytope, or restricted to optimization over partitions corresponding to vertices of that polytope. Our study of the vertices of partition polytopes is motivated by the second approach.

We mention that the above embedding and restriction of partitioning problems correspond to the two fundamental approaches used in the study of polytopes and optimization problems thereupon – one focusing on facets, that is, on defining systems for linear inequalities, while the other focusing on vertices. Neither approach necessarily dominates the other, and moving from one to the other is generally nontrivial computationally. The solution of optimization problems over partitions is explicitly addressed in Hwang et al. (unpublished manuscript).

[3] It is well-known that the conclusions of Proposition 1.1 extend to functions which are *quasi-convex*, that is, functions $h(.)$ satisfying $h[\alpha a + (1-\alpha)b] \leqslant \max\{h(a), h(b)\}$ for vectors a and b in their domain and $0 \leqslant \alpha \leqslant 1$. Further, these results were recently extended to an even larger class of functions called *edge-quasi-convex*; see Hwang and Rothblum (1996).

BHR explored partitioning problems and partitioning polytopes under a nondegeneracy assumption asserting that the columns of the underlying matrix A are nonzero and distinct. Some of their results were extended in Hwang et al. (submitted) our goal herein is to extend the results of BHR with full generalization to degenerate cases.

One issue of interest concerns uniqueness of the representation of vertices. For non-degenerate bounded-shape partition polytopes the issue was settled in Theorem 5 of BHR as follows:

Proposition 1.2. *Let P be a bounded-shape partition polytope where the columns of the underlying matrix A are nonzero and distinct and let V be a vertex of P. Then V has a unique representation as A^π with π a partition in the underlying set of partitions.*

Obviously, the unique representation of vertices does not hold in generate cases where A's columns include repeated vectors and/or zero vectors. For example, if all the columns of A coincide and are all nonzero, there is a one-to-one correspondence between the potential shapes of partitions and the associated vectors, but, with $p \geqslant 2$ and $n > p$ there will be multiple partitions with any given shape. More generally, we have that switches of indices with identical corresponding vectors between the parts as well as shifts of zero vectors between the parts will not change the associated vector. In Section 3 (Theorem 3.5) we demonstrate that the above are the only degrees of freedom in multiple representations of vertices of bounded-shape partition polytopes. We also demonstrate that Proposition 1.2 cannot be extended to constrained-shape partition polytopes.

A (geometric) necessary condition and an (algebraic) necessary and sufficient condition for vertices of bounded-shape partition polytope were obtained in BHR. The necessary condition was generalized to degenerate cases and to constrained-shape polytopes in Hwang et al. (submitted). The extended result is given in Proposition 1.3.

Proposition 1.3. *Let P be a constrained-shape partition polytope with vertex A^π where $\pi = (\pi_1, \ldots, \pi_p)$ is a corresponding partition. For $t = 1, \ldots, p$, let $\sigma_t \equiv \operatorname{conv}\{A^j\colon j \in \pi_t\}$. Then for each pair of distinct indices $r, s \in \{1, \ldots, p\}$, $\sigma_r \cap \sigma_s$ is either empty, or contains a single point which is a common vertex of σ_r and σ_s; such a common vertex is necessarily one of A's columns.*

The sufficient condition for vertices given in Proposition 1.3 is appealing because of its geometric expression. Further, the condition is used in Hwang et al. (submitted) to enumerate, in polynomial time, the vertices of constrained-shape polytopes. Still, an example in BHR demonstrates that, even in nondegenerate cases, the condition is not necessary. A condition which is both necessary and sufficient for vertices of nondegenerate bounded-shape partition polytopes in terms of solvability of linear systems was developed in BHR (it is included in Theorem 3.1). In Section 4 (Theorem 4.2) we extend the characterization to degenerate cases and demonstrate that its verification can be executed in effort which is polynomial in the parameters of the problem; the result facilitates a polynomial test for vertices.

Partition polytopes and preliminaries are formally introduced in Section 2. Representations of partition polytopes and their vertices are given in Section 3, with particular emphasis on the degrees of freedom in such representations. Finally, an algebraic characterization of the vertices is provided in Section 4.

2. Preliminaries: Partition polytopes

Throughout, we let k and n be positive integers. These parameters will be fixed throughout this section.

Superscripts are used to denote columns of matrices, subscripts for rows and double indices for elements, e.g., U^t, U_i and U_i^t. The vector of 1's of appropriate dimension is denoted e. For matrices U and V of common dimension, say $m \times p$, the *inner product* of U and V is defined by $\langle U, V\rangle \equiv \sum_{i=1}^{m} \sum_{j=1}^{p} U_i^j V_i^j$. We recall that for matrices U, V and W of dimension $m \times p, m \times q$ and $q \times p$, respectively, we have that $\langle U, VW\rangle = \langle V^{\mathrm{T}} U, W\rangle = \langle UW^{\mathrm{T}}, V\rangle$.

A *partition* is an ordered collection of sets $\pi = (\pi_1, \ldots, \pi_p)$, where $\pi_1, \ldots, \pi_p$ are disjoint, nonempty subsets of N whose union is N. Given such a partition π, we refer to p as its *size* and to the sets $\pi_1, \ldots, \pi_p$ as its *parts*. Also, if the number of elements in the parts of a partition $\pi = (\pi_1, \ldots, \pi_p)$ are $n_1, \ldots, n_p$, respectively, we refer to $(n_1, \ldots, n_p)$ as the *shape* of π; of course, in this case $\sum_{j=1}^{p} n_j = |N| = n$. Partitions of size p are called *p-partitions* and partitions of shape $(n_1, \ldots, n_p)$ are called $(n_1, \ldots, n_p)$*-partitions*.

Sets of partitions of particular interest are those whose shape is constrained to be in a prescribed set. Specifically. if Γ is a set of positive integer p-vectors with coordinate-sum n (that is, Γ is a set of potential shapes of p-partitions) we refer to the set of all p-partitions whose shape is in Γ as *the set of Γ-shape partitions*; at convenience, we suppress the explicit dependence on Γ and refer generically to *constrained-shape partition-sets*. If L and U are positive integer p-vectors satisfying $L \leqslant U$ and $\sum_{j=1}^{p} L_j \leqslant n \leqslant \sum_{j=1}^{p} U_j$, the (nonempty) set of positive integer p-vectors $(n_1, \ldots, n_p)$ with coordinate-sum n that satisfy $L_j \leqslant n_j \leqslant U_j$ for each $j = 1, \ldots, p$ is denoted $\Gamma^{(L,U)}$; the corresponding set of partitions is denoted $\Pi^{(L,U)}$ and, with the dependence of L and U suppressed, referred to as a *bounded-shape partition-set*.

Let A be a $k \times n$ real matrix. For a p-partition $\pi = (\pi_1, \ldots, \pi_p)$ we define the *π-summation-matrix of A*, denoted A^π, by

$$A^{\pi} \equiv \left[\sum_{t\in\pi_1} A^t, \ldots, \sum_{t\in\pi_p} A^t\right] \in \mathbb{R}^{k\times p}. \tag{2.1}$$

With $e^1, \ldots, e^k$ as the unit vectors in $\mathbb{R}^p$, we note that

$$\begin{aligned} A^{\pi} &= \left[\sum_{t\in\pi_1} A^t, \ldots, \sum_{t\in\pi_p} A^t\right] = \sum_{j=1}^{p}\left(\sum_{t\in\pi_j} A^t\right)(e^j)^{\mathrm{T}} \\ &= \sum_{j=1}^{p}\sum_{t\in\pi_j} A^t (e^j)^{\mathrm{T}}. \end{aligned} \tag{2.2}$$

We recall that a *polytope* is the convex hull of a finite set. For a matrix $A \in \mathbb{R}^{k\times n}$ and a set of p-partitions Π, the *partition polytope with data-matrix A corresponding to Π*, denoted P_A^{Π}, is the convex hull of $\{A^{\pi}\colon \pi \in \Pi\} \subseteq \mathbb{R}^{k\times n}$. While the notational dependence of P_A^{Π} on A and Π is always preserved, we sometimes refer to *partition polytopes* or to *the partition polytopes corresponding to Π*. If $\Pi = \Pi^{(L,U)}$ for corresponding positive integer p-vectors L and U we use the notation $P_A^{(L,U)}$ for P_A^{Π} and refer to this polytope as a *bounded-shape partition polytope*.

We recall that a *vertex* of a polytope P is a point V in P having the property that the only representation of v as $\frac{1}{2}(a+b)$ with $a, b \in P$ has $a = b$. It is well known that $V \in P$ is a vertex of P if and only if there is a linear function that attains a unique maximum over P at V, and we use this property interchangeably with the above definition. The important role vertices play in convex maximization problems is discussed in the introduction.

We next consider the case where the data-matrix is the identity $I \in \mathbb{R}^{n\times n}$. In this case, for each p-partition $\pi, I^{\pi} \in \mathbb{R}^{n\times p}$ is given by

$$\begin{aligned} (I^{\pi})^j_t &\equiv \begin{cases} 1 & \text{if } t \in \pi_j, \\ 0 & \text{otherwise}, \end{cases} \\ &= \sum_{j=1}^{p}\sum_{t\in\pi_j} e^t (e^j)^{\mathrm{T}}, \end{aligned} \tag{2.3}$$

where e^t for $t = 1, \ldots, n$ and e^j for $j = 1, \ldots, p$ denote the unit vectors in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively; consequently (using (Eq. (2.2)) with I as the underlying matrix)

$$\begin{aligned} AI^{\pi} &= A\left[\sum_{j=1}^{p}\sum_{t\in\pi_j} e^t (e^j)^{\mathrm{T}}\right] = \sum_{j=1}^{p}\sum_{t\in\pi_j} Ae^t (e^j)^{\mathrm{T}} \\ &= \sum_{j=1}^{p}\sum_{t\in\pi_j} A^t (e^j)^{\mathrm{T}} = A^{\pi}. \end{aligned} \tag{2.4}$$

An explicit representing systems of linear inequalities is next derived for bounded-shape partition polytopes with the identity as the data-matrix.

Lemma 2.1. *Let L and U be positive integer p-vectors satisfying* $L \leqslant U$ *and* $\sum_{j=1}^{p} L_j \leqslant n \leqslant \sum_{j=1}^{p} U_j$, *and let* Π *be the set of* $\Gamma^{(L,U)}$*-partitions. Then* P_I^{Π} *is the solution set of the linear system:*

$$X_t^j \geqslant 0 \quad \text{for } t = 1, \ldots, n \text{ and } j = 1, \ldots, p, \tag{2.5a}$$

$$\sum_{j=1}^{p} X_t^j = 1 \quad \text{for } t = 1, \ldots, n, \tag{2.5b}$$

$$L_j \leqslant \sum_{t=1}^{n} X_t^j \leqslant U_j \quad \text{for } j = 1, \ldots, p. \tag{2.5c}$$

Proof. Let K be the solution set of (2.5). Trivially, $I^{\pi} \in K$ for each $\pi \in \Pi$, implying that the convex hull of these matrices, namely P_I^{Π}, is contained in K. Next, standard results (that rely on the fact that the constraint matrix of the inequality system (2.5) is totally unimodular) assure that the vertices of K are integer solutions of (2.5) (cf., Schrijver, 1986); as integer solutions of (2.5) correspond to p-partitions in Π, that is, have representation as I^{π} for some $\pi \in \Pi$, each vertex of K is in P_I^{Π}. By another standard result, K is the convex hull of its vertices, and consequently K is contained in P_I^{Π}. □

3. Vertex representation

In the current section we derive representations of partition polytopes and their vertices. We recall Proposition 1.2 which asserts unique representations as A^{π} of the vertices of bounded-shape partition polytopes when the vectors $A^1, \ldots, A^n$ are nonzero and distinct. The next example demonstrates that with the columns of A nonzero and distinct, multiple representations of interior vectors of bounded-shape partition polytopes and of vertices of constrained-shape partition polytopes (which are not bounded-shape) are possible.

Example. Let $k = 1, n = 4, A = (-2, -1, 1, 2)$ and $p = 2$. For positive indices i and j with $i + j = 4$, we let $\Pi^{(i,j)}$ be the partitions with shape (i, j) and we let $P^{(i,j)}$ be the corresponding partition polytopes. Now, $\pi^1 = (\{1, 4\}, \{2, 3\})$ and $\pi^2 = (\{2, 3\}, \{1, 4\})$ are two distinct partitions in $\Pi^{(2,2)}$ that satisfy $A^{\pi^1} = A^{\pi^2} = (0, 0)$. Of course, (0,0) is not a vertex of $P^{(2,2)} = \{(\alpha, -\alpha)\colon -3 \leqslant \alpha \leqslant 3\}$ (in fact, in view of Proposition 1.2, (0,0) is not a vertex of any bounded-shape partition polytope). Also, $\{A^{\pi}\colon \pi \in \Pi^{(1,3)}\} = \{A^{\pi}\colon \pi \in \Pi^{(3,1)}\} =$

$\{(-2,2),(-1,1),(1,-1),(2-2)\} = \{A^\pi\colon \pi \in \Pi^{(1,3)\cup\Pi^{(3,1)}}\}$. So, the vertices of the partition polytope corresponding to $\Pi^{(1,3)} \cup \Pi^{(3,1)}$ are $(-2,2)$ and $(2,-2)$ and each is realizable by two partitions. Of course, $\Pi^{(1,3)} \cup \Pi^{(3,1)}$ is a set of partitions which is constrained-shaped, but not bounded-shape.

The next result provides three necessary and sufficient conditions for vectors corresponding to given partitions to be vertices of bounded-shape partition polytopes. One of these (condition (d) in Theorem 3.1) was introduced in BHR; another (condition (b)) tightens the necessary condition for being a vertex stated in Proposition 1.2.

Theorem 3.1. *Let $A \in \mathbb{R}^{k\times n}$ have nonzero and distinct columns, let $L_1,\ldots,L_p, U_1,\ldots,U_p$ be positive integers satisfying $L_j \leqslant U_j$ for $j = 1,\ldots,p$ and $\sum_{j=1}^p L_j \leqslant n \leqslant \sum_{j=1}^p U_j$, and let $\pi \in \Pi^{(L,U)}$. Then the following are equivalent:*

(a) *A^π is a vertex of $P_A^{(L,U)}$,*

(b) *$A^\pi = AI^\pi$ is a unique representation of A^π as $A^\pi = AX$ with $X \in P_I^{(L,U)}$,*

(c) *$\{Y \in P_I^{(L,U)}\colon AY = A^\pi$ and $\langle I^\pi, Y\rangle \leqslant n-1\} = \emptyset$, and*

(d) *there exists a matrix $C \in \mathbb{R}^{k\times p}$ and vector $\alpha \in \mathbb{R}^p$ such that:*

(1) *$(C^r - C^s)^{\mathrm{T}} A^t > \alpha_s - \alpha_r$ for $r,s \in \{1,\ldots,p\}$ with $r \neq s$ and $t \in \pi_r$,*

(2) *$\alpha_r \leqslant 0$ if $|\pi_r| > L_r$, and*

(3) *$\alpha_r \geqslant 0$ if $|\pi_r| < U_r$.*

Proof. Let $\Pi \equiv \Pi^{(L,U)}$. We recall from Eq. (2.4) that $A^\sigma = AI^\sigma$ for each $\sigma \in \Pi$.

(a) $\Rightarrow$ (b): Suppose $A^\pi = AI^\pi$ is a vertex of $P_A^{(L,U)}$. Then A^π is the unique maximizer over $P_A^{(L,U)}$ of some linear functional, say one that is represented by the matrix $C \in \mathbb{R}^{k\times p}$. Now, suppose that $A^\pi = AX$ with $X \in P_I^{(L,U)}$ and we will show that $X = I^\pi$. As $X \in P_I^{(L,U)}$, there exist partitions $\pi^1,\ldots,\pi^q$ in Π and positive coefficients $\alpha_1,\ldots,\alpha_q$ which sum to 1 such that $X = \sum_{s=1}^q \alpha_s I^{\pi^s}$; in particular, $\langle C, A^\pi\rangle = \langle C, AX\rangle = \sum_{s=1}^q \alpha_s \langle C, AI^{\pi s}\rangle = \sum_{s=1}^q \alpha_s \langle C, A^{\pi^s}\rangle$. As A^π is the unique maximizer over $P_A^{(L,U)}$ of the linear function represented by C, as the α_s's are positive and sum to 1 and as the A^{π^s}'s are in $P_A^{(L,U)}$, it follows that for each $s = 1,\ldots,q$, $\langle C, A^{\pi^s}\rangle = \langle C, A^\pi\rangle$ and $A^{\pi^s} = A^\pi$. Thus Proposition 1.2 implies that all π^s's coincide with π and therefore $X = \sum_{s=1}^q \alpha_s I^{\pi^s} = I^\pi$.

(b) $\Rightarrow$ (c): Suppose $A^\pi = AI^\pi$ is a unique representation of A^π as $A^\pi = AX$ with $X \in P_I^{(L,U)}$. Let Y be a matrix in $P_I^{(L,U)}$ satisfying $AY = A^\pi$. It then follows that $Y = I^\pi$, implying that $\langle Y, I^\pi\rangle = \langle I^\pi, I^\pi\rangle = n$. Thus $P_I^{(L,U)} \cap \{Y \in \mathbb{R}^{n\times k}\colon AY = A^\pi$ and $\langle I^\pi, Y\rangle \leqslant n-1\} = \emptyset$.

(c) $\Rightarrow$ (a): Assume that A^π is not a vertex of $P_A^{(L,U)}$. From Proposition 1.1, each vertex of $P_A^{(L,U)}$ is in the set $\{A^\pi\colon \pi \in \Pi\}$ and a standard result implies that A^π has a representation as a convex combination of vertices of $P_A^{(L,U)}$. Hence, there exist partitions $\pi^1,\ldots,\pi^q$ in Π, all distinct from π, and positive coefficients $\alpha_1,\ldots,\alpha_q$ which sum to 1 such that $A^\pi = \sum_{s=1}^q \alpha_s A^{\pi^s}$. As $A^{\pi^s} = AI^{\pi^s}$ for

$s = 1, \ldots, q$, hence, $A^{\pi} = \sum_{s=1}^{q} \alpha_s A I^{\pi^s} = A(\sum_{s=1}^{q} \alpha_s I^{\pi^s})$. Also, the convexity of $P_I^{(L,U)}$ assures that $\sum_{s=1}^{q} \alpha_s I^{\pi^s} \in P_I^{(L,U)}$. Now, as $\langle I^{\pi}, I^{\sigma} \rangle \leqslant n - 1$ for each p-partition σ that is distinct from π, we have that $\langle I^{\pi}, \sum_{s=1}^{q} \alpha_s I^{\pi^s} \rangle = \sum_{s=1}^{q} \alpha_s \langle I^{\pi}, I^{\pi^s} \rangle \leqslant \sum_{s=1}^{q} \alpha_s (n-1) = n - 1$. So, $X \equiv \sum_{s=1}^{q} \alpha_s I^{\pi^s} \in P_I^{(L,U)}$ satisfies $AX = A^{\pi}$ and $\langle I^{\pi}, X \rangle \leqslant n - 1$, demonstrating that $\{X \in P_I^{(L,U)}\colon AX = A^{\pi}$ and $\langle I^{\pi}, X \rangle \leqslant n - 1\}$ is not empty.

(a) $\iff$ (d): This equivalence is established in Theorem 5 of BHR. □

The three necessary and sufficient conditions for being a vertex of a bounded-shape partition polytope given in Theorem 3.1 yield computational methods with polynomial complexity in n, p and k; further discussion of such methods is deferred till the end of the current section, at which point the restrictive assumption that A's columns are nonzero and distinct is relaxed.

Condition (a) of Theorem 3.1 does not imply condition (b) when A's columns include repeated vectors and/or zero vectors; for example, if all columns of A coincide, each single-shape partition polytope contains a single point which is a vertex of the polytope and this vertex has multiple representations as A^{π} when $p > 1$ (in fact, this example neither satisfies the conclusion of Proposition 1.2). In Theorem 3.5 we identify variants of condition (b) which characterize vertices of bounded-shape partition polytopes without the assumption that A's columns are nonzero and distinct. A modification of condition (d) which is necessary for (a) and applies for the general case is developed in Section 4. We are not aware of a corresponding modification of condition (c).

A few additional definitions are needed before we are ready to explore general bounded-shape partition polytopes. Let $\tilde{n}$ be the number of nonzero distinct columns of A. We will consider matrices with $\tilde{n} + 1$ rows or $\tilde{n} + 1$ columns where these rows/columns are indexed by $0, 1, \ldots, \tilde{n}$. Further, when a matrix has $\tilde{n} + 1$ rows indexed by $0, 1, \ldots \tilde{n}$, we use *underlining* to denote the submatrix obtained by truncating the 0-row, so, if $B \in \mathbb{R}^{(\tilde{n}+1)\times n}$, then $\underline{B} \in \mathbb{R}^{\tilde{n}\times n}$.

Given a $k \times n$ matrix A, wet let $\ddot{A}$ be the $k \times \tilde{n}$ submatrix of A obtained by deleting zero and multiple columns that appear in A, for uniqueness we assume that the first of any group of repeated columns of A is preserved while the others are deleted, and the order of $\ddot{A}$'s columns is induced from A. Also, let $\mathring{A}$ be the $k \times (1 + \tilde{n})$ matrix obtained from $\ddot{A}$ by adding the zero vector as the 0-column. Of course, never does $\ddot{A}$ have a zero column but $\mathring{A}$ always does. Finally, let $J \in \mathbb{R}^{(1+\tilde{n})\times n}$ be the $\{0, 1\}$-matrix with $J^t = e^s$ (the s-unit vector in $\mathbb{R}^{\tilde{n}}$) if $A^t = \ddot{A}^s (\neq 0)$ and $J^t = e^0$ (the 0-unit vector in $\mathbb{R}^{(\tilde{n}+1)}$) if $A^t = 0$. For example, if

$$A = \begin{pmatrix} 1 & 0 & 3 & 3 & 1 \\ 2 & 0 & 4 & 4 & 2 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

then

$$\ddot{A}=\begin{pmatrix}1&3\\2&4\\1&0\end{pmatrix},\qquad \mathring{A}=\begin{pmatrix}0&1&3\\0&2&4\\0&1&0\end{pmatrix},$$

$$J=\begin{pmatrix}0&1&0&0&0\\1&0&0&0&1\\0&0&1&1&0\end{pmatrix}\quad\text{and}\quad \underline{J}=\begin{pmatrix}1&0&0&0&1\\0&0&1&1&0\end{pmatrix}.$$

The use of underlining to denote truncation of the zero row of a matrix having $1+\tilde{n}$ rows (introduced in the above paragraph) and the construction of $\mathring{A}$ from $\ddot{A}$ by augmenting it with a zero vector imply that for each matrix Y with $1+\tilde{n}$ rows we have that

$$\mathring{A}Y=\ddot{A}\underline{Y}, \tag{3.1}$$

in particular,

$$\mathring{A}J=A=\ddot{A}\underline{J}. \tag{3.2}$$

When A has no zero columns, the forthcoming development can be carried out without the use of $\mathring{A}$, but solely with the use of $\ddot{A}$. In particular, when A's columns are nonzero and distinct, $\ddot{A}=A$ and $\underline{J}=I\in\mathbb{R}^{n\times n}$.

We will use $J,\underline{J}$ and $A\in\mathbb{R}^{k\times n}$ as data-matrices. In particular, for a p-partition π Eq. (2.4) implies that $JI^\pi=J^\pi, \underline{J}I^\pi=\underline{J}^\pi, AI^\pi=A^\pi$ and, by Eq. (3.2)

$$A^\pi=AI^\pi=(\mathring{A}J)I^\pi=\mathring{A}(JI^\pi)=\mathring{A}J^\pi \tag{3.3}$$

and

$$A^\pi=AI^\pi=(\ddot{A}\underline{J})I^\pi=\ddot{A}(\underline{J}I)^\pi=\ddot{A}\underline{J}^\pi. \tag{3.4}$$

The next lemma shows that for a set of partitions $\Pi, P_I^\Pi, P_J^\Pi, P_{\underline{J}}^\Pi$ and P_A^Π form a sequence of polytopes where each is a projection of its predecessors; further, the composite projection of P_I^Π onto P_A^Π is given by $X\to AX$. The decompositions we are about to establish are demonstrated in Fig. 1. We shall refer to P_I^Π as the *generalized transportation polytope corresponding to Π.*

$$\begin{array}{ccc} P_I^\Pi & & P_A^\Pi \\ \downarrow & & \uparrow \\ P_J^\Pi & \rightarrow & P_{\underline{J}}^\Pi \end{array}$$

Fig. 1.

Lemma 3.2. *Let $A \in \mathbb{R}^{k\times n}$ and let Π be a set of partitions. Then*:

(a) $P_J^\Pi = \{JX\colon\ X \in P_I^\Pi \subseteq \mathbb{R}^{n\times p}\}$,

(b) $P_{\underline{J}}^\Pi = \{\underline{Y}\colon\ Y \in P_J^\Pi \subseteq \mathbb{R}^{(1+\tilde{n})\times p}\}$,

(c) $P_A^\Pi = \{\ddot{A}Z\colon\ Z \in P_{\underline{J}}^\Pi \subseteq \mathbb{R}^{\tilde{n}\times p}\}$.

Further, the composite projections of P_I^Π onto $P_{\underline{J}}^\Pi$, of $P_I^\Pi y$ onto P_A^Π and of P_J^Π onto P_A^Π are given, respectively, by $X \to \underline{J}X, X \to \bar{A}X$ and $Y \to \mathring{A}Y$.

Proof. A standard argument about convex hulls shows that

$$\begin{aligned}\{JX\colon\ X \in P_I^\Pi\} &= \{JX\colon\ X \in \mathrm{conv}\{I^\pi\colon\ \pi \in \Pi\}\} = \mathrm{conv}\{JI^\pi\colon\ \pi \in \Pi\}\\ &= \mathrm{conv}\{J^\pi\colon\ \pi \in \Pi\} = P_J^\Pi,\end{aligned} \tag{3.5}$$

proving (a). The same argument combines with Eq. (3.4) to show that

$$\begin{aligned}\{\ddot{A}Z\colon\ Z \in P_{\underline{J}}^\Pi\} &= \{\ddot{A}Z\colon\ Z \in \mathrm{conv}\{\underline{J}^\pi\colon\ \pi \in \Pi\}\}\\ &= \mathrm{conv}\{\ddot{A}\underline{J}^\pi\colon\ \pi \in \Pi\} = \mathrm{conv}\{A^\pi\colon\ \pi \in \Pi\} = P_A^\Pi,\end{aligned} \tag{3.6}$$

proving (c). Next, to establish (b), observe that the projection mapping $Y \in \mathbb{R}^{(1+\tilde{n})\times p}$ into $\underline{Y} \in \mathbb{R}^{\tilde{n}\times p}$ by eliminating the 0 row is a linear operator; this operator is representable by a matrix, say $E \in \mathbb{R}^{\tilde{n}\times(1+\tilde{n})}$ with $EY = \underline{Y}$ for each $Y \in \mathbb{R}^{(1+\tilde{n})\times p}$. As in Eq. (3.6) we then get that

$$\begin{aligned}\{\underline{Y}\colon\ Y \in P_J^\Pi\} &= \{EY\colon\ Y \in P_J^\Pi\} = \{EY\colon\ Y \in \mathrm{conv}\{J^\pi\colon\ \pi \in \Pi\}\}\\ &= \mathrm{conv}\{EJ^\pi\colon\ \pi \in \Pi\} = \mathrm{conv}\{\underline{J}^\pi\colon\ \pi \in \Pi\} = P_{\underline{J}}^\Pi.\end{aligned} \tag{3.7}$$

Finally, the composite projections of P_I^Π onto $P_{\underline{J}}^\Pi$, of P_I^Π onto P_A^Π and of P_J^Π onto P_A^Π are given respectively, by $X \to \underline{JX} = \underline{J}X$, $X \to \ddot{A}(\underline{JX}) = \ddot{A}(\underline{J}X) = (\ddot{A}\underline{J})X = AX$ (here we use Eq. (3.2)), and $Y \to \ddot{A}\underline{Y} = \mathring{A}Y$ (here we use (3.1)). □

Corollary 3.3. *Let $A \in \mathbb{R}^{k\times n}$ and let Π be a set of p-partitions. Then the partition polytope P_A^Π is the image of the generalized transportation-polytope P_I^Π under the linear function mapping $X \in P_I^\Pi \subseteq \mathbb{R}^{n\times p}$ into $AX \in \mathbb{R}^{k\times p}$. In particular, for every p-partition $\pi, A^\pi = AI^\pi$.*

We next obtain an explicit representation, through a system of linear inequalities, for bounded-shape partition polytopes when the data matrix is J. The result resembles Lemma 2.1 which concerns the case where the data matrix is I.

Lemma 3.4. *Let L and U be positive integer p-vectors satisfying $L \leqslant U$ and $\sum_{j=1}^p L_j \leqslant n \leqslant \sum_{j=1}^p U_j$. Then $P_J^{(L,U)}$ is the solution set of the linear system*

$$Y_s^j \geqslant 0 \quad \textit{for } s = 0, 1, \ldots, \tilde{n} \textit{ and } j = 1, \ldots, p, \tag{3.8a}$$

$$\sum_{j=1}^{p} Y_s^j = (Je)_s \quad \textit{for } s = 0, 1, \ldots, \tilde{n}, \tag{3.8b}$$

$$L_j \leqslant \sum_{s=0}^{\tilde{n}} Y_s^j \leqslant U_j \quad \textit{for } j = 1, \ldots, p, \tag{3.8c}$$

with e as the vector $(1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$.

Proof. Trivially, each of the matrices J^π for $\pi \in \Pi^{(L,U)}$ satisfies (3.8), hence, the convex hull of these matrices, namely $P_J^{(L,U)}$, is contained in the solution set of (3.8) which we denote by K.

Adding variables Y_j^0 for $j = 1, \ldots, p$, replacing (3.8c) by the constraints $Y_j^0 = \sum_{s=0}^{\tilde{n}} Y_s^j$ and $L_j \leqslant Y_j^0 \leqslant U_j$ for $j = 1, \ldots, p$ and adding the constraint $\sum_{j=1}^{p} Y_j^0 = \sum_{s=0}^{\tilde{n}} (Je)_s = n$, the linear system (3.8) is expanded to a network flow problem with integer lower and upper bounds on arc-flows (see Fig. 2); in particular, standard results (that rely on the fact that the constraint matrix of the defining linear system is totally unimodular) assure that the polytope associated with the network flow problem has integral vertices (see Schrijver, 1986). The constructed expansion defines a one-to-one *linear* map from K onto this polytope, under which vertices are mapped onto vertices, and we conclude that all vertices of K are integer matrices.

Let Y^* be a vertex of K; it then follows from the above paragraph that Y^* is an integer matrix. For each $s = 0, 1, \ldots, \tilde{n}$, let $H_s \equiv \{t = 1, \ldots, n\colon A^t = \overset{\circ}{A}{}^s\}$; these sets partition N and for each s, $\sum_{j=1}^{p} (Y^*)_s^j = (Je)_s = |H_s|$. The latter together with the integrality of Y^* implies that for $s = 0, 1, \ldots, \tilde{n}$, there exists a (not necessarily unique) partition of H_s into sets $\sigma_{s_1}, \ldots, \sigma_{s_p}$ such that $|\sigma_{s_j}| = (Y^*)_s^j$. For $j = 1, \ldots, p$, let $\pi_j \equiv \bigcup_{s=0}^{\tilde{n}} \sigma_{s_j}$. It follows that $\pi_1, \ldots, \pi_p$

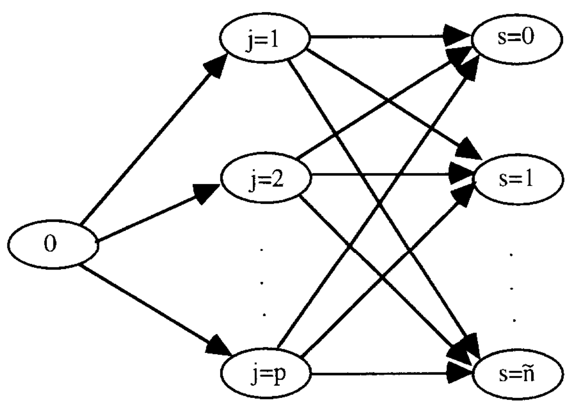

Fig. 2.

partition N; also, for $j = 1,\dots,p$, $|\pi_j| = \sum_{s=0}^{\tilde{n}} |\sigma_{s_j}| = \sum_{s=0}^{\tilde{n}} (Y^*)^j_s$ so (Eq. (3.8c)) implies that $L_j \leqslant |\pi_j| \leqslant U_j$. Thus, $\pi \in \Pi^{(L,U)}$. Next, for $s = 0,1,\dots,\tilde{n}$ and $j = 1,\dots,p$, $\sum_{t\in H_s}(I^\pi)^j_t = |\sigma_{s_j}| = (Y^*)^j_s$, hence, $(J^\pi)^j_s = (JI^\pi)^j_s = \sum_{t\in H_s}(I^\pi)^j_t = (Y^*)^j_s$, implying that $Y^* = J^\pi \in P_J^{(L,U)}$. So, each vertex of K is in $P_J^{(L,U)}$. By another standard result, K is the convex hull of its vertices, implying that K is contained in $P_J^{(L,U)}$. □

The explicit representation of $P_J^{(L,U)}$ in Lemma 3.4 does not generally extend to $P_J^{(L,U)}$; specifically, when zero vectors exist, they have to be accounted for the lower and upper bounds on the cardinality of the parts. But, part (b) of Lemma 3.2 shows that $P_J^{(L,U)}$ is a projection of $P_J^{(L,U)}$. When A has no zero column, $(Je)_0 = 0$ and all solutions of (3.8) have $Y^j_0 = 0$ for $j = 1,\dots,p$; by eliminating these variables, we get from (3.8) a characterization of $P_J^{(L,U)}$.

The next theorem provides three necessary and sufficient conditions for vectors corresponding to partitions to be vertices of bounded-shape partition polytopes, without the assumption that A's columns are nonzero and distinct. The three conditions look cumbersome and repetitive, but, they have distinct uses. Condition (b) extends the uniqueness of the representation of vertices when A's columns are nonzero and distinct (condition (b) of Theorem 3.1), while condition (d) explains the potential degrees of freedom in multiple representations of extreme points of the partition polytopes (see the discussions following the theorem and Corollary 3.6). Conditions (c) and (d) concern the polytopes $P_J^{(L,U)}$ and $P_I^{(L,U)}$ which have explicit linear inequalities representations (Lemma 2.1 and Corollary 3.3) and are therefore useful for computable tests; in fact, the representation of $P_J^{(L,U)}$ is used to establish the (most difficult) implication (a) ⇒ (c).

Theorem 3.5. *Let $A \in \mathbb{R}^{k\times n}$, let $L_1,\dots,L_p$, $U_1,\dots,U_p$ be positive integers satisfying $L \leqslant U$ and $\sum_{j=1}^p L_j \leqslant n \leqslant \sum_{j=1}^p U_j$, and let $\pi \in \Pi^{(L,U)}$. Then the following are equivalent:*

(a) *A^π is a vertex of $P_A^{(L,U)}$,*

(b) *$\{Z \in P_{\underline{J}}^{(L,U)}\colon \ddot{A}Z = A^\pi\} = \{\underline{J}^\pi\}$ and $\underline{J}^\pi$ is a vertex of $P_{\underline{J}}^{(L,U)}$, and*

(c) *$\{Y \in P_J^{(L,U)}\colon \mathring{A}Y = A^\pi\} = \{Y \in P_J^{(L,U)}\colon Y = \underline{J}^\pi\}$ and $\underline{J}^\pi$ is a vertex of $P_{\underline{J}}^{(L,U)}$, and*

(d) *$\{X \in P_I^{(L,U)}\colon AX = A^\pi\} = \{X \in P_I^{(L,U)}\colon \underline{J}X = \underline{J}^\pi\}$ and $\underline{J}^\pi$ is a vertex of $P_{\underline{J}}^{(L,U)}$.*

Proof. (a) ⇒ (c): Suppose A^π is a vertex of $P_A^{(L,U)}$. To see that $\{Y \in P_J^{(L,U)}\colon \underline{Y} = \underline{J}^\pi\} \subseteq \{Y \in P_J^{(L,U)}\colon \mathring{A}Y = A^\pi\}$, let $Y \in P_J^{(L,U)}$ satisfy $\underline{Y} = \underline{J}^\pi$. It then follows from Eq. (3.1) and Eq. (3.4) that $\mathring{A}Y = \ddot{A}\underline{Y} = \ddot{A}\underline{J}^\pi = A^\pi$. To see the reverse inclusion, let $Y \in P_J^{(L,U)}$ satisfy $\mathring{A}Y = A^\pi$. Consider the network flow (with integer lower and upper bounds on arc-flows) expansion of the linear system (3.8) as described in the proof of Lemma 3.4; see Fig. 2. Lemma 3.4 and its proof show that $P_J^{(L,U)}$ is in one-to-one correspondence with the solution-set of this network flow problem with vertices mapped onto vertices. Thus, we may and will identify vectors in $P_J^{(L,U)}$, that is, solutions of (3.8), with their expansion to solutions of the network flow problem; in particular, this is the case for Y and J^π. Also, a *circuit* z is a nonzero, normalized, minimal support solution of the (homogenous) system

$$\sum_{j=1}^{p} z_j^0 = 0, \tag{3.9a}$$

$$z_j^0 - \sum_{s=0}^{\tilde{n}} z_s^j = 0 \quad \text{for } j = 1, \ldots, p, \tag{3.9b}$$

$$\sum_{j=1}^{p} z_s^j = 0 \quad \text{for } s = 0, 1, \ldots, \tilde{n}, \tag{3.9c}$$

where *normalized* means that $\|z\|_\infty = 1$ (with $\| \ \|_\infty$ denoting the l_∞ norm) and *minimal support* means that the set of nonzero variables of no solution of (3.9) is strictly contained in that of z. Standard results show that the coordinates of a circuit z are all $-1, 0$ and 1 and that each node of the network presented in Fig. 2 appears in either 0 or 2 indices corresponding to nonzero coordinates of z. Also, each column of a circuit z has at most two nonzero elements which can take only the values -1 and 1; as $\mathring{A}^0 = 0$ and $\mathring{A}^1, \ldots, \mathring{A}^{\tilde{n}}$ are nonzero and distinct, it follows that

$$\text{circuit } z \text{ satisfies } \mathring{A}z = 0 \text{ if and only if } \underline{z} = 0. \tag{3.10}$$

As Eq. (3.9b) determines the z_j^0's of a circuit z from the remaining coordinates, we identify such a circuit with its projection $\underline{z} \in \mathbb{R}^{\tilde{n} \times p}$.

As A^π is a vertex of $P_A^{(L,U)}$, A^π is the unique maximizer over $P_A^{(L,U)}$ of some linear function; let such a linear function be represented by the matrix $C \in \mathbb{R}^{k \times p}$. Also, as Y and J^π are solutions of the network flow problem, a standard result about network flows (e.g., Denardo, 1982 p. 99) implies that $Y - J^\pi$ can be decomposed into a sum $\sum_{t=1}^{q} \beta_t z^t$ where for each $t = 1, \ldots, q$, z^t is a circuit of the network flow problem, β_t is a positive number and $J^\pi + \beta_t z^t$ is a feasible solution of the network flow problem, that is, $J^\pi + \beta_t z^t \in P_J^{(L,U)}$, and by Lemma 3.2, $\mathring{A}J^\pi + \beta_t \mathring{A}z^t = \mathring{A}(J^\pi + \beta_t z^t) \in P_A^{(L,U)}$. Now, Eq. (3.3) and the unique optimality of A^π over $P_J^{(L,U)}$ under the linear function represented by C implies that for $t = 1, \ldots, q$, $\langle C, A^\pi \rangle \geqslant \langle C, \mathring{A}J^\pi + \beta_t \mathring{A}z^t \rangle = \langle C, A^\pi \rangle + \beta_t \langle C, \mathring{A}z^t \rangle$ with equality holding if and only if $\mathring{A}J^\pi + \beta_t \mathring{A}z^t = A^\pi$, that is, $\langle C, \mathring{A}z^t \rangle \leqslant 0$ with equality holding if and only if $\mathring{A}z^t = 0$, and by Eq. (3.10) this is the case if and only if $z^t = 0$. As we are assuming that $\mathring{A}Y = A^\pi$ and $Y = J^\pi + \sum_{t=1}^{q} \beta_t z^t$, we have that $\langle C, A^\pi \rangle = \langle C, \mathring{A}Y \rangle = \langle C, \mathring{A}J^\pi + \sum_{t=1}^{q} \beta_t \mathring{A}z^t \rangle = \langle C, A^\pi \rangle + \sum_{t=1}^{q} \beta_t \langle C, \mathring{A}z^t \rangle \leqslant \langle C, A^\pi \rangle$;

it follows that the inequalities in the above string hold as equalities. Thus for $t = 1, \ldots, q, \langle C, \mathring{A}z^t \rangle = 0$ and consequently $\underline{z}^t = 0$. Thus $\underline{Y} = \underline{J}^\pi + \sum_{t=1}^q \beta_t \underline{z}^t = \underline{J}^\pi$.

It remains to show that $\underline{J}^\pi$ is a vertex of $P_J^{(L,U)}$. Indeed, assume that $\underline{J}^\pi$ has a representation $\underline{J}^\pi = \beta Z^1 + (1-\beta)Z^2$ for some $0 < \beta < 1$ and $Z^1, Z^2 \in P_J^{(L,U)}$, and we will show that $Z^1 = Z^2 = \underline{J}^\pi$. By Eq. (3.4), $A^\pi = \ddot{A}\underline{J}^\pi$, and by Lemma 3.2 $\ddot{A}Z^1$ and $\ddot{A}Z^2$ are in $P_A^{(L,U)}$. As A^π is assumed to be a vertex of $P_A^{(L,U)}$ and $A^\pi = \ddot{A}\underline{J}^\pi = \beta\ddot{A}Z^1 + (1-\beta)\ddot{A}Z^2$, we conclude that $A^\pi = \ddot{A}Z^1 = \ddot{A}Z^2$. By part (b) of Lemma 3.2 there exist matrices Y^1 and Y^2 in $P_J^{(L,U)}$ with $\underline{Y}^1 = Z^1$ and $\underline{Y}^2 = Z^2$, and Eq. (3.1) implies that $\mathring{A}Y^1 = \ddot{A}\underline{Y}^1 = \ddot{A}Z^1 = A^\pi$ and $\mathring{A}Y^2 = \ddot{A}\underline{Y}^2 = \ddot{A}Z^2 = A^\pi$. Hence, the established conclusion $\{Y \in P_J^{(L,U)}:\ \mathring{A}Y = A^\pi\} = \{Y \in P_J^{(L,U)}:\ \underline{Y} = \underline{J}^\pi\}$ implies that $Z^1 = \underline{Y}^1 = \underline{J}^\pi$ and $Z^2 = \underline{Y}^2 = \underline{J}^\pi$.

(c) $\Rightarrow$ (b): It suffices to show that if $\{Y \in P_J^{(L,U)}: \mathring{A}Y = A^\pi\} = \{Y \in P_J^{(L,U)}:\ \underline{Y} = \underline{J}^\pi\}$, then $\{Z \in P_J^{(L,U)}:\ \ddot{A}Z = A^\pi\} = \{\underline{J}^\pi\}$. So, assume that the first equality holds. By Eq. (3.4), $\ddot{A}\underline{J}^\pi = A^\pi$; as $J^\pi \in P_J^{(L,U)}$, it follows that $\underline{J}^\pi \in \{Z \in P_J^{(L,U)}:\ \ddot{A}Z = A^\pi\}$. To establish the reverse inclusion let $Z \in P_J^{(L,U)}$ satisfy $\ddot{A}Z = A^\pi$. It then follows from part (b) of Lemma 3.2 that $Z = \underline{Y}$ for some $Y \in P_J^{(L,U)}$; for such Y we have from Eq. (3.1) that $\mathring{A}Y = \ddot{A}\underline{Y} = \ddot{A}Z = A^\pi$ and, by assumption, this implies that $\underline{Y} = \underline{J}^\pi$, that is, $Z = \underline{Y} = \underline{J}^\pi$.

(b) $\Rightarrow$ (d): From Eqs. (3.2) and (3.4), if $X \in P_I^{(L,U)}$ and $\underline{J}X = \underline{J}^\pi$ then $AX = (\ddot{A}\underline{J})X = \ddot{A}(\underline{J}X) = \ddot{A}\underline{J}^\pi = A^\pi$. So, $\{X \in P_I^{(L,U)}:\ \underline{J}X = \underline{J}^\pi\} \subseteq \{X \in P_I^{(L,U)}:\ AX = A^\pi\}$. Thus, it suffices to show that if $\{X \in P_I^{(L,U)}:\ \underline{J}X = \underline{J}^\pi\} \subset \{X \in P_I^{(L,U)}:\ AX = A^\pi\}$ then there exists a matrix $Z \in P_J^{(L,U)}$ with $\ddot{A}Z = A^\pi$ and $Z \neq \underline{J}^\pi$. So, suppose that $X \in P_I^{(L,U)}$ satisfies $AX = A^\pi$ and $\underline{J}X \neq \underline{J}^\pi$, and let $Z \equiv \underline{J}X$. Then $Z = \underline{J}X \neq \underline{J}^\pi$ and by Lemma 3.2 and Eq. (3.2), respectively, $Z = \underline{J}X \in P_J^{(L,U)}$ and $\ddot{A}Z = \ddot{A}(\underline{J}X) = (\ddot{A}\underline{J})X = AX = A^\pi$.

(d) $\Rightarrow$ (a): Suppose condition (d) holds. We will assume that A^π is not a vertex of $P_A^{(L,U)}$ and establish a contradiction. From Proposition 1.1, each vertex of $P_A^{(L,U)}$ is in the set $\{A^\pi:\ \pi \in \Pi\}$ and a standard result assures that A^π has a representation as a convex combination of vertices of $P_A^{(L,U)}$. Hence, there exist partitions $\pi^1, \ldots, \pi^q$ in Π and positive coefficients $\alpha_1, \ldots, \alpha_q$ which sum to 1 such that $A^\pi = \sum_{t=1}^q \alpha_t A^{\pi^t}$ and each A^{π^t} is a vertex of $P_A^{(L,U)}$. As $A^{\pi^t} = AI^{\pi^t}$ for $t = 1, \ldots, q$, implying that $A^\pi = \sum_{t=1}^q \alpha_t A^{\pi^t} = \sum_{t=1}^q \alpha_t AI^{\pi^t} = A(\sum_{t=1}^q \alpha_t I^{\pi^t})$. As $P_I^{(L,U)}$ is convex, $X \equiv \sum_{t=1}^q \alpha_t I^{\pi^t} \in P_I^{(L,U)}$. So, $X \in P_I^{(L,U)}$ and $AX = A^\pi$; hence, condition (d) implies that $\underline{J}X = \underline{J}^\pi$, that is, $\underline{J}^\pi = \underline{J}X = \underline{J}(\sum_{t=1}^q \alpha_t I^{\pi^t}) = \sum_{t=1}^q \alpha_t \underline{J}I^{\pi^t} = \sum_{t=1}^q \alpha_t \underline{J}^{\pi^t}$. As $\underline{J}^\pi$ is assumed to be a vertex of $P_J^{(L,U)}$, as all the α_t's are positive and as all the $\underline{J}^{\pi^t}$'s are in $P_J^{(L,U)}$, we conclude that for $t = 1, \ldots, q, \underline{J}^{\pi^t} = \underline{J}^\pi$ and therefore $A^{\pi^t} = \ddot{A}\underline{J}^{\pi^t} = \ddot{A}\underline{J}^\pi = A^\pi$. As each A^{π^t} is assumed to be a vertex of $P_J^{(L,U)}$ whereas A^π is not, we reached a contradiction which proves the asserted implication. □

The equivalence of conditions (a) and (d) in Theorem 3.5 shows that for multiple representations of vertices of $P_A^{(L,U)}$ in the form AX where

$X \in P_I^{(L,U)}, \underline{J}X$ is unique. Restricting this condition to vectors associated with partitions we get the following corollary of Theorem 3.5 which extends Proposition 1.2 to situations where A's columns may include repeated vectors and/or zero vectors. The corollary shows that the freedom of selecting a partition corresponding to a particular vertex of $P_A^{(L,U)}$ reduces to the exchange indices associated with common vectors and to the shift indices associated with the zero vector.

Corollary 3.6. *Let $A \in \mathbb{R}^{k\times n}$, let $L_1,\ldots,L_p,U_1,\ldots,U_p$ be positive integers satisfying $L \leqslant U$ and $\sum_{j=1}^{p} L_j \leqslant n \leqslant \sum_{j=1}^{p} U_j$, and let V be a vertex of $P_A^{(L,U)}$. Then $\underline{J}^{\pi}$ coincide for partitions $\pi \in \Pi^{(L,U)}$ with $A^{\pi} = V$.*

Proof. The implication (a) ⇒ (b) in Theorem 3.5 implies that if $V = A^{\pi} = A^{\pi'}$ for $\pi, \pi' \in \Pi$, then $\underline{J}^{\pi} = \underline{J}^{\pi'}$. □

Condition (c) of Theorem 3.5 is next used to describe a test for vectors associated with partitions to be vertices of given bounded-shape partition polytopes. An alternative method is described in Hwang et al. (unpublished manuscript), and a test for vertices of partition polytopes determined by arbitrary shape-constraints (not necessarily through lower and upper bounds) are provided in Hwang et al. (submitted).

Testing if a vector A^{π} is a vertex of the bounded-shape partition polytope: Let A, L, U and π be as in Theorem 3.5. Our test for determining whether or not A^{π} is a vertex of $P_A^{(L,U)}$ has two parts.

The first part of the test determines whether or not $\underline{J}^{\pi}$ is a vertex of $P_{\underline{J}}^{(L,U)}$. We observe that $\underline{J}^{\pi}$ is a vertex of $P_{\underline{J}}^{(L,U)}$ if and only if for every representation of $\underline{J}^{\pi}$ as $\frac{1}{2}(Y' + Y'')$ with Y' and Y'' in $P_{\underline{J}}^{(L,U)}$ we have that $Y' = Y''$. This condition holds if and only if for each $s = 0, 1, \ldots, \tilde{n}$ and $j = 1, \ldots, p$ the maximum of $(Y' - Y'')_s^j$ over Y' and Y'' satisfying $Y' \in P_{\underline{J}}^{(L,U)}, Y'' \in P_{\underline{J}}^{(L,U)}$ and $Y = \frac{1}{2}(Y' + Y'')$ is zero, which is verifiable by solving $\tilde{n}p$ linear programs where each has $\tilde{n}p$ variables and $2(\tilde{n} + p) + \tilde{n}p$ constraints. When A has no zero columns, $J = \underline{J}$ and a test for being a vertex of $P_{\underline{J}}^{(L,U)} = P_J^{(L,U)}$ can be developed from the explicit representation of $P_{\bar{J}}^{(L,U)}$ available from the Lemma 3.4.

The second part of the test determines whether or not $\underline{Y} = \underline{J}^{\pi}$ for each matrix Y satisfying $Y \in P_J^{(L,U)}$ and $\mathring{A}Y = A^{\pi}$. In view of Lemma 3.4 (with the expansion discussed in the proof of the lemma), the assertion that $Y \in P_J^{(L,U)}$ is characterized by a linear system having $(\tilde{n} + 1)(p + 1)$ variables, $(\tilde{n} + 1) + p$ constraints, nonnegativity constraints and lower and upper bounds on p variable; also, the requirement $\mathring{A}Y = A^{\pi}$ reduces to another kp constraints. Testing whether or not each solution Y to the joint system satisfies $\underline{Y} = \underline{J}^{\pi}$ can be accomplished by maximizing and minimizing Y_s^j for $j = 1, \ldots, p$ and $s = 1, \ldots, \tilde{n}$ over the joint system, that is, by solving $2\tilde{n}p$ corresponding linear programs.

The above method for testing whether or not a vector A^{π} is a vertex of a bounded-shape partition polytope depends on condition (c) of Theorem 3.5 which reduces to condition (b) of Theorem 3.1 when A's columns are nonzero and distinct. We observe that conditions (c) and (d) of Theorem 3.1 yield alternative computational methods under the restricted assumption of that theorem. Indeed, condition (c) concerns solvability of a (sparse) linear system with np $\{0,1\}$-variables and $(n+1)(k+1)$ equality and weak inequality constraints, and condition (d) concerns solvability of a linear system having $(k+1)p$ variables and $(p-1)n$ strict inequality constraints. Each of these tests is obviously polynomial in k, n and p.

4. Vertex characterization

In this section we tighten the necessary condition of Proposition 1.3 for being a vertex of a constrained-shape partition polytope to obtain a condition which is both necessary and sufficient. The result extends Theorem 5 of BHR by relaxing the assumption that the columns of A are nonzero and distinct; see Theorem 3.1. Our analysis is carried out in two steps. First, we tighten the necessary condition of Proposition 1.3 to obtain a stronger necessary condition, then, we modify this tighter condition to obtain a condition which is both necessary and sufficient.

We recall that $\operatorname{argmax}_{x\in A} f(x)$ refers to the set of maximizers of the function $f(.)$ over A.

Theorem 4.1. *Let $L_1,\dots,L_p,U_1,\dots,U_p$ be positive integers satisfying $L \leqslant U$ and $\sum_{j=1}^{p} L_j \leqslant n \leqslant \sum_{j=1}^{p} U_j$, let Π be the set of $\Gamma^{(L,U)}$-shape partitions, let $\pi\in\Pi$ where A^{π} is a vertex of $P_A^{(L,U)}$ and for $r,s\in\{1,\dots,p\}$ let $\Lambda^{rs}\equiv\{A^u\colon u\in\pi_r\}\cap\{A^u\colon u\in\pi_s\}$. Then for some matrix $C\in\mathbb{R}^{k\times p}$ and vector $\alpha\in\mathbb{R}^p$,*

(a) $\alpha_r\leqslant 0$ *for $r=1,\dots,p$ satisfying $L_r<|\pi_r|$,*

(b) $\alpha_r\geqslant 0$ *for $r=1,\dots,p$ satisfying $U_r>|\pi_r|$,*

(c) $(C^s-C^r)^{\mathrm{T}}A^u\leqslant\alpha_r-\alpha_s$ *for distinct indices $r,s\in\{1,\dots,p\}$ and $u\in\pi_r$, and*

(d) *if $(C^s-C^r)^{\mathrm{T}}A^u=\alpha_r-\alpha_s$ for distinct indices $r, s\in\{1,\dots,p\}$ and $u\in\pi_r$, then :*

 (i) $A^u\in\Lambda^{rs}$ *if $\Lambda^{rs}\neq\emptyset$, and*

 (ii) $A^u=0$, $|\pi_r|>L_r$ *and* $|\pi_s|<U_s$ *if* $\Lambda^{rs}=\emptyset$.

Proof. Our proof modifies the arguments proving the necessity of the condition characterizing a vertex of bounded-shape partition polytopes as given in Theorem 5 of BHR under the assumption that the columns of A are nonzero and distinct.

As A^π is a vertex of P_A^Π it is the unique maximizer over P_A^Π of some linear function, say one that is determined by the matrix $C \in \mathbb{R}^{k \times p}$; so

$$\langle C, X\rangle < \langle C, A^\pi\rangle \quad \text{for each } X \in P_A^\Pi \setminus \{A^\pi\}. \tag{4.1}$$

As in BHR, consider the *linear assignment problem* with indices $0, 1, \ldots, p$ and cost-coefficients

$$d_{rs} = \begin{cases} \max_{u \in \pi_r}\{(C^s - C^r)^{\mathrm{T}} A^u & \text{if } r, s \geqslant 1, \\ -\infty & \text{if } r = 0, s \geqslant 1 \text{ and } |\pi_s| = L_s, \\ -\infty & \text{if } s = 0, r \geqslant 1 \text{ and } |\pi_r| = U_r, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Circuits (with respect to this problem) are then nonzero, normalized, minimal support solution of

$$\sum_{r=0}^{p} z_{rs} = 0 \quad \text{for } s = 0, 1, \ldots, p \tag{4.3}$$

and

$$\sum_{s=0}^{p} z_{rs} = 0 \quad \text{for } r = 0, 1, \ldots, p, \tag{4.4}$$

where *normalized* means that $\max\{|z_{rs}|\colon r, s = 0, 1, \ldots, p\} = 1$ and *minimal support* means that if $z' \neq 0$ satisfies Eqs. (4.3) and (4.4), then the set of nonzero coordinates of z' is not strictly included in that of z.

With I as the $(p+1) \times (p+1)$ identity, it is shown in BHR that under the assumption that the columns of A are nonzero and distinct, if z is a circuit with $I + z \geqslant 0$, then

$$\sum_{r=0}^{p}\sum_{s=0}^{p} d_{rs} z_{rs} \leqslant 0, \tag{4.5}$$

and when (4.5) holds as equality, $z_{rs} = 0$ for all indices $r, s = 1, \ldots, p$. While the arguments of BHR establishing the inequalities of (4.5) are applicable when the assumption that the columns of A are nonzero and distinct is relaxed, the conclusions from equality in (4.5) need not hold and such situations are examined in the next paragraph. Still, as in BHR, the (weak) inequalities of (4.5) suffice to show that the identity is optimal for our linear assignment problem, and further, Linear Programming Duality and the Weak and Strong Complementarity Theorems imply the existence of a vector $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_p)$ satisfying (a), (b) and

$$d_{rs} \leqslant \alpha_r - \alpha_s \quad \text{for all distinct indices } r, s \in \{1, \ldots, p\}, \tag{4.6}$$

with strict inequality holding in (4.6) if and only if $z_{rs} = 0$ for all circuits z satisfying (4.5) with equality.

We next examine circuits z for which $I + z \geqslant 0$ and (4.5) holds as equality. Let z be such a circuit. As in BHR, it follows that there exist distinct indices $r_1, r_2, \ldots, r_q$ where $q \geqslant 2$ such that, with $r_{q+1} \equiv r_1$,

$$Z_{rs} = \begin{cases} 1 & \text{for } (r,s) = (r_t, r_{t+1}),\ t = 1,\ldots,q \\ -1 & \text{for } (r,s) = (r_t, r_t), \quad t = 1,\ldots,q, \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Without loss of generality assume that $r_1 = \min\{r_t\colon\ t = 1,\ldots,q\}$ and for $t = 1,\ldots,q$ with $r_t \geqslant 1$ let u_t be a maximizer or $(C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^u$ over $u \in \pi_{r_t}$. Now, if (4.5) holds as equality and $r_1 \geqslant 1$, the arguments of BHR yield a partition π' obtained by cyclic shifts of the vectors $A^{u_1}, A^{u_2}, \ldots, A^{u_q}$ with $\langle C, A^{\pi'}\rangle = \langle C, A^{\pi}\rangle$. It then follows from (4.1) that $A^{\pi} = A^{\pi'}$, implying that, with $u_{q+1} \equiv u_1, A^{u_t} = A^{u_{t+1}}$, for $t = 1,\ldots,q$; in particular, in this case the common vector $A^{u_1} = A^{u_2} = \cdots = A^{u_q}$ is in $\bigcap_{t=1}^{q} A^{r_t, r_{t+1}}$. Also, if (4.5) holds as equality and $r_1 = 0$, the arguments of BHR show that $L_{r_2} < |\pi_{r_2}|,\ |\pi_{r_q}| < U_{r_q}$ and the existence of a partition π' obtained by shifts of the vectors $A^{u_2}, A^{u_3}, \ldots, A^{u_{q-1}}$, with $\langle C, A^{\pi'}\rangle = \langle C, A^{\pi}\rangle$. From (4.1) we then have that $A^{\pi} = A^{\pi'}$, implying that $A^{u_2} = 0$ and $A^{u_t} = A^{u_{t+1}}$ for $t = 2, 3, \ldots, q-2$. So, we have that $L_{r_2} < |\pi_{r_2}|, |\pi_{r_q}| < U_{r_q}$ and $0 = A^{u_2} = A^{u_3} = \cdots = A^{u_{q-1}}$; in particular, $0 \in \bigcap_{t=2}^{q-2} A^{r_t, r_{t+1}}$. Of course, some of these conclusion are vacuous if $q = 2$ or if $q = 3$.

Let $r, s \in \{1,\ldots,p\}$ with $r \neq s$ and let $u \in \pi_r$. The definition of the d_{rs}'s in Eq. (4.2) and (4.6) then imply that $(C^s - C^r)A^u \leqslant d_{rs} \leqslant \alpha_r - \alpha_s$, establishing (c); further, if $(C^s - C^r)A^u = \alpha_r - \alpha_s$, then $u \in \operatorname{argmax}_{x\in\pi_r}(C^s - C^r)A^x$ and $d_{rs} = \alpha_r - \alpha_s$. We establish (d) by further analysis of the case where $(C^s - C^r)A^u = \alpha_r - \alpha_s$ and considering two cases:

Case I: $A^{rs} \neq \emptyset$. In this case there exist $w \in \pi_r$ and $v \in \pi_s$ with $A^w = A^v$. As we already demonstrated that $u \in \operatorname{argmax}_{x\in\pi_r}(C^s - C^r)A^x$, we have that $(C^s - C^r)A^u \geqslant (C^s - C^r)A^w = (C^s - C^r)A^v$. We will prove by contradiction that $A^u = A^v$. Suppose $A^u \neq A^v$. Let σ be the partition obtained from π by switching v from π_s to π_r and u from π_r to π_s. Then $\sigma \in \Pi$ and $A^{\sigma} = A^{\pi} + (A^u - A^v)(e^s - e^r)^{\mathrm{T}} \neq A^{\pi}$. So, $A^{\sigma} \in P_A^{\Pi} \setminus \{A^{\pi}\}$, and (4.1) implies that $\langle C, A^{\pi}\rangle > \langle C, A^{\sigma}\rangle = \langle C, A^{\pi} + (A^u - A^v)(e^s - e^r)^{\mathrm{T}}\rangle$; hence,

$$\begin{aligned} 0 &> \langle C, (A^u - A^v)(e^s - e^r)^{\mathrm{T}}\rangle = \langle C(e^s - e^r), (A^u - A^v)\rangle \\ &= \langle (C^s - C^r), (A^u - A^v)\rangle = (C^s - C^r)^{\mathrm{T}} A^u - (C^s - C^r)^{\mathrm{T}} A^v, \end{aligned}$$

in contradiction to the assertion $(C^s - C^r)A^u \geqslant (C^s - C^r)A^v$. The contradiction proves that $A^u = A^v$, in particular, $A^u \in A^{rs}$.

Case II: $A^{rs} \neq \emptyset$. We have already concluded that $d_{rs} = \alpha_r - \alpha_s$ and therefore there exists a circuit z that satisfies $z + I \geqslant 0$, (4.5) with equality and $z_{rs} \neq 0$; it then follows that z has the representation (4.7), further, without loss of generality we assume that $r_1 = \min_{t=1,\ldots,q} r_t$. The assertion $A^{rs} \neq \emptyset$ combines with the analysis of equality in (4.5) to imply that necessarily $q = 3, r_1 = 0$,

$(r,s) = (r_2, r_3), |\pi_r| > L_r$ and $|\pi_s| < U_s$ and $A^{u_2} = 0$; further, as $u \in \mathrm{argmax}_{x\in\pi_r}(C^s - C^r)A^x, A^{u_2}$ can be selected as A^u in the construction of the partition π' from π and z, implying that $A^u = A^{u_2} = 0$. □

It is easy to verify that the necessary condition for being a vertex of a bounded-shape partition polytope asserted in Theorem 4.1 implies the necessary condition of Proposition 1.3 (which applies to the more general constrained-shape partition polytopes). The necessary condition of Theorem 4.1 is next modified to obtain a condition which is both necessary and sufficient (cf., Theorem 5 of BHR). The task is accomplished by tightening the conclusions from equalities $(C^r - C^s)A^v = \alpha_s - \alpha_r$ in (d). But, the new condition lacks a simple geometric/algebraic motivation of the (necessary) condition of Theorem 4.1; further, verification of the polynomial test for vertices described in Section 3 in simpler.

Theorem 4.2. *Let $L_1, \ldots, L_p, U_1, \ldots, U_p$ be positive integers satisfying $L \leqslant U$ and $\sum_{j=1}^p L_j \leqslant n \leqslant \sum_{j=1}^p U_j$, let Π be the set of $\Gamma^{(L,U)}$-shape partitions and let $\pi \in \Pi$. Then A^π is a vertex of P_A^Π if and only if for some matrix $C \in \mathbb{R}^{k\times p}$ and vector $\alpha \in \mathbb{R}^p$.*

(a) *$\alpha_r \leqslant 0$ for $r = 1, \ldots, p$ satisfying $L_r < |\pi_r|$, and*

(b) *$\alpha_r \geqslant 0$ for $r = 1, \ldots, p$ satisfying $U_r > |\pi_r|$;*

(c) *$(C^s - C^r)^{\mathrm{T}} A^u \leqslant \alpha_r - \alpha_s$ for distinct indices $r, s \in \{1, \ldots, p\}$ and $u \in \pi_r$;*

(d_1) *if $q \geqslant 2, r_1, \ldots, r_q$ are distinct indices in $\{1, \ldots, p\}, u_1, \ldots, u_q$ are indices in $\{1, \ldots, n\}$ with $u_t \in \pi_t$ and (with $r_{q+1} = 1$) $(C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^{u_t} = \alpha_{r_t} - \alpha_{r_{t-1}}$ for $t = 1, \ldots, q$, then $A^{u_1} = A^{u_2} = \cdots = A^{u_q}$;*

(d_2) *if $q \geqslant 2, r_1, \ldots, r_q$ are distinct indices in $\{1, \ldots, p\}$ with$L_{r_1} > |\pi_{r_1}|$ and $U_{r_q} < |\pi_{r_q}|$, and $u_1, \ldots, U_{q-1}$ are indices in $\{1, \ldots, n\}$ with $u_t \in \pi_t$ and $(C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^{u_t} = \alpha_{r_t} - \alpha_{r_{t+1}}$* for *$t = 1, \ldots, q-1$, then $A^{u_1} = A^{u_2} = \cdots = A^{u_{q-1}} = 0$.*

Proof. *Necessity*: Assume that A^π is a vertex of P_A^Π and C and α are constructed as in the proof of Theorem 4.1. In particular, (a)–(c) are satisfied, the coefficients d_{rs} for $r, s = 1, \ldots, p$ given by Eq. (4.2) satisfy (4.6), and (from the arguments of the proof of Theorem 4.1) if z is a circuit (for the corresponding assignment problem) with $I + z \geqslant 0$, with Eq. (4.7) in force and with equality holding in (4.5), then:

(i) if $r_1 > 0$ then $\mathrm{argmax}_{x\in\pi_{r_t}}(C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^x$ is invariant of $t = 1, \ldots, q$ and the common set consists of a single element, and

(ii) if $r_1 = 0$ then $\mathrm{argmax}_{x\in\pi_{r_t}}(C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^x = \{0\}$ for $t = 1, \ldots, q-1$.

Suppose $q \geqslant 2, r_1, \ldots, r_q$ are distinct indices in $\{1, \ldots, p\}$ and $u_1, \ldots, u_q$ are indices in $\{1, \ldots, n\}$ with $u_t \in \pi_t$ and (with $r_{q+1} = 1)(C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^{u_t} = \alpha_{r_t} - \alpha_{r_{t+1}}$ for $t = 1, \ldots, q$. For each t, (c) and (4.6) show that $(C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^{u_t} \leqslant \max_{x\in\pi_{r_t}} (C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^x = d_{r_t, r_{t+1}} \leqslant \alpha_{r_t} - \alpha_{r_{t+1}} (C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^{u_t}$ and therefore $A^{u_t} \in \mathrm{argmax}_{x\in\pi_{r_t}}(C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^x$ and $d_{r_t, r_{t+1}} = \alpha_{r_t} - \alpha_{r_{t+1}}$. It follows that the vector z defined from $r_1, \ldots, r_q$ by the right-hand side of (Eq. (4.7)) is a circuit satisfying

$$\sum_{r=0}^{p}\sum_{s=0}^{p} d_{rs}z_{rs} = \sum_{t=1}^{q} d_{r_t,r_{t+1}} = \sum_{t=1}^{q} \alpha_{r_t} - \alpha_{r_{t+1}} = 0,$$

As z is a circuit satisfying (4.5) with equality, assertion (i) of the above paragraph shows that $\operatorname{argmax}_{x\in\pi_{r_t}}(C^{r_{t+1}} - C^{r_t})^{\mathrm{T}}A^x$ is invariant of $t = 1, \ldots, q$ and the common set consists of a single element, hence, the fact that $A^{u_t} \in \operatorname{argmax}_{x\in\pi_{r_t}}(C^{r_{t+1}} - C^{r_t})^{\mathrm{T}}A^x$ for $t = 1, \ldots, q$ implies that $A^{u_1} = A^{u_2} = \cdots = A^{u_q}$, and the proof of $(\mathrm{d})_1$ is complete.

A similar line of argument applies for the case considered under d_2 except that in order to obtain a circuit of the assignment problem it is necessary to augment $r_1, \ldots, r_q$ with $r_{q+1} = 0$.

Sufficiency: Suppose C and α satisfy conditions (a)–(d). We show that A^π is a vertex of P_A^Π by showing that $\langle C, X\rangle \leqslant \langle C, A^\pi\rangle$ for each $X \in P_A^\Pi$ and equality holds only if $X = A^\pi$.

By Lemma 2.1, P_I^Π is the solution set of (2.5). Adding variables Y_j^0 for $j = 1, \ldots, p$, replacing (2.5c) by the constraints $Y_j^0 = \sum_{s=0}^{\tilde{n}} Y_s^j$ and $L_j \leqslant Y_j^0 \leqslant U_j$ for $j = 1, \ldots, p$ and adding the constraint $\sum_{j=1}^{p} Y_j^0 = \sum_{s=0}^{\tilde{n}} (Je)_s = n$, the linear system (2.5) is expanded to a network flow problem with integer lower and upper bounds on arc-flows (see the more detailed construction in Lemma 3.4 and the corresponding Fig. 1 which applies to P_J^Π rather than P_J^Π); as the augmented variables are uniquely determined by the original ones, we identify feasible flows of the network and elements in P_I^Π (see the proof of Theorem 3.5 for details that apply to P_J^Π). We consider *circuits* of the network which are nonzero, normalized, minimal support solution of (2.5) (see the proof of Lemma 3.4 and Theorem 4.1 for more detailed definitions of circuits for other network flows). A circuit z for which $I^\pi + z \in P_I^\Pi$ may have one of two representation. It is either identified with sequences $r_1, \ldots, r_q$ of distinct indices in $\{1, \ldots, p\}$ and $u_1, \ldots, u_q$ of distinct indices in $\{1, \ldots, n\}$ such that $q \geqslant 2$, $u_t \in \pi_t$ for $t = 1, \ldots, q$ and, with $r_{q+1} = 1$,

$$z_{ur} = \begin{cases} 1 & \text{for } (u,r) = (u_t, r_{t+1}),\ t = 1, \ldots, q, \\ -1 & \text{for } (u,r) = (u_t, r_t), \quad t = 1, \ldots, q, \\ 0 & \text{otherwise,} \end{cases} \tag{4.8}$$

or with sequences $r_1, \ldots, r_q$ of distinct indices in $\{1, \ldots, p\}$ and $u_1, \ldots, u_{q-1}$ of distinct indices in $\{1, \ldots, n\}$ such that $q \geqslant 2$, $u_t \in \pi_t$ for $t = 1, \ldots, q-1$, $|\pi_{r_1}| > L_{r_1}$, $|\pi_{r_q}| < U_{r_q}$ and

$$z_{ur} = \begin{cases} 1 & \text{for } (u,r) = (u_t, r_{t+1}),\ t = 2, \ldots, q, \\ -1 & \text{for } (u,r) = (u_t, r_t), \quad t = 1, \ldots, q-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

In particular, each such a circuit corresponds to a unique partition $\sigma(z)$ with $I^{\sigma(z)} = I^{\pi} + z$. A circuit with representation (4.8) satisfies $Az = \sum_{t=1}^{q} A^{u_t}(e^{r_{t+1}} - e^{r_t})^{\mathrm{T}}$ and

$$\langle C, Az\rangle = \sum_{t=1}^{q}\langle C, A^{u_t}(e^{r_{t+1}} - e^{r_t})^{\mathrm{T}}\rangle = \sum_{t=1}^{q}\langle C(e^{r_{t+1}} - e^{r_t}), A^{u_t}\rangle$$
$$= \sum_{t=1}^{q}\langle (C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^{u_t} \leqslant \sum_{t=1}^{q}(\alpha_{r_t} - \alpha_{r_{t-1}}) = 0, \tag{4.10}$$

and similarly a circuit with representation (4.9) satisfies $Az = \sum_{t=1}^{q-1} A^{u_t}(e^{r_{t-1}} - e^{r_t})^{\mathrm{T}}$ and

$$\langle C, Az\rangle = \sum_{t=1}^{q-1}(C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^{u_t} \leqslant \sum_{t=1}^{q-1}(\alpha_{r_t} - \alpha_{r_{t-1}}) = \alpha_{r_1} - \alpha_{r_q} \leqslant 0, \tag{4.11}$$

where the last inequality follows as (a) with $|\pi_{r_1}| > L_{r_1}$ imply that $\alpha_{r_1} \leqslant 0$ and (b) with $|\pi_{r_q}| < U_{r_q}$ imply that $\alpha_{r_q} \geqslant 0$. We conclude that if $\langle C, Az\rangle = 0$, then $\langle (C^{r_{t+1}} - C^{r_t})^{\mathrm{T}} A^{u_t} = \alpha_{r_t} - \alpha_{r_{t+1}}$ for all relevant indices t, and therefore (d) implies that $Az = 0$. So, $Az = 0$ for each circuit z satisfying $\langle C, Az\rangle = 0$.

Let $Y \in P_A^{\Pi}$. By Corollary 3.3 $Y = AX$ for some $X \in P_I^{\Pi}$. Next, a standard result about network flows (used in the proofs of Lemma 3.4 and Theorem 4.1) assures that $X - I^{\pi}$ can be decomposed into a sum $\sum_{k=1}^{q}\beta_k z^k$ where for each $k = 1,\ldots,q$, z^k is a circuit of our network flow problem, β_k is a positive number and $I^{\pi} + \beta_k z^k \in P_I^{\Pi}$, the latter implying (again, by Corollary 3.3) that $A^{\pi} + \beta_k A z^k = A(I^{\pi} + \beta_k z^k) \in P_A^{\Pi}$. From Eqs. (4.10) and (4.11) we have that $\langle C, Az^k\rangle \leqslant 0$ for each k, implying that $\langle C, A^{\pi} + \beta_k A z^k\rangle \leqslant \langle C, A^{\pi}\rangle$; hence, $\langle C, Y\rangle - \langle C, A^{\pi}\rangle = \langle C, A(X - I^{\pi})\rangle = \sum_{k=1}^{q}\beta_k\langle C, Az^k\rangle \leqslant 0$ with equality holding only if $\langle C, Az^k\rangle = 0$ for each k. As the above paragraph shows that $Az^k = 0$ whenever $\langle C, Az^k\rangle = 0$, we conclude that if $\langle C, Y\rangle = \langle C, A^{\pi}\rangle$ then $Az^k = 0$ for each k, implying that $Y = AX = A(I^{\pi} + \sum_{k=1}^{q}\beta_k z^k) = A^{\pi} + \sum_{k=1}^{q}\beta_k A z^k = A^{\pi}$. So, indeed, we established that $\langle C, X\rangle \leqslant \langle C, A^{\pi}\rangle$ for each $X \in P_A^{\Pi}$ with equality holding only for $X = A^{\pi}$. □

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