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Group Divisible Designs with Two Associate Classes: $n=2$ or $m=2$

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In this paper we find necessary and sufficient conditions for the existence of a group divisible design GDD(n, m) of index (λ_1, λ_2) in which $n=2$ or $m=2$, thereby completing the solution of the existence problem for all n, m, λ_1, and λ_2. In the process, necessary and sufficient conditions are found for the existence of an x-regular partial triple system whose complement in λK_n has a 1-factorization.

1. INTRODUCTION

A *group divisible design* GDD$(n, m; k; \lambda_1, \lambda_2)$ is an ordered triple (V, G, B) where V is a set of *varieties* or *symbols*, G is a partition of V into m sets of size n, each set being called a *group*, and B is a collection of subsets of V, called *blocks*, each of size k, such that

(1) each pair of symbols that occur together in the same group occur together in exactly λ_1 blocks, and

(2) each pair of symbols that occur together in no group occur together in exactly λ_2 blocks.

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Elements occurring together in the same group are called *first associates*, and elements occurring in different groups are called *second associates*. We say that the GDD is *defined on the set V*.

GDDs with $\lambda_1 = 0$ have been of great use in constructing other designs. But just as interesting to the statisticians are GDDs in which $\lambda_1 > 0$. According to Raghavarao [12] partially balanced designs with two association classes were classified in 1952 by Bose and Shimamoto into five types: group divisible designs, simple, triangular, latin square type and cyclic. We will concentrate here on group divisible designs. For a wealth of information on GDDs see Raghavarao [12]. Clatworthy [3] gives tables for all the classes of GDDs.

In this paper, we consider the existence of GDDs in the case where $k = 3$. To simplify the notation, let a GDD$(n, m; 3; \lambda_1, \lambda_2)$ be denoted by a GDD(n, m) of index (λ_1, λ_2), and let a block of size 3 be called a triple. Together with Dinesh Sarvate, we have already completely solved this existence problem in the case where $n, m \geqslant 3$, proving the following result.

THEOREM 1.1 [7]. *Let* $n, m \geqslant 3$ *and* $\lambda_1, \lambda_2 \geqslant 1$. *There exists a* GDD$(n, m)$ *of index* (λ_1, λ_2) *if and only if*

(1) 2 *divides* $\lambda_1(n-1) + \lambda_2(m-1)\, n$, *and*

(2) 3 *divides* $\lambda_1 mn(n-1) + \lambda_2 m(m-1)\, n^2$.

GDDs with $m = 1$ (so λ_2 is irrelevant) and $k = 3$ are the well known triple systems (TS), so we denote a GDD$(n, 1; 3; \lambda_1, \lambda_2)$ by a TS(n) of index λ_1. Since a TS(n) has only one group, $V = G$ so it can be simply represented by (V, B). We will use the following famous result.

THEOREM 1.2 ([8], or see [10]). *Let* $n \geqslant 3$. *There exists a* TS(n) *of index* λ *iff*

(a) 2 *divides* $\lambda(n-1)$, *and*

(b) 3 *divides* $\lambda n(n-1)$.

So the existence of a GDD(n, m) of order (λ_1, λ_2) has been completely settled if $m = 1$ or $n = 1$ (and so also if $\lambda_2 = 0$). It has also been settled if $\lambda_1 = 0$ with the following result.

THEOREM 1.3 [9]. *There exists a* GDD(n, m) *of index* $(0, \lambda_2)$ *iff*

(a) 2 *divides* $\lambda_2(m-1)\, n$,

(b) 3 *divides* $\lambda_2 m(m-1)\, n^2$, *and*

(c) $m \geqslant 3$.

In this paper, the cases where $m=2$ and where $n=2$ are solved (see Theorem 5.1), thus completing the solution of the existence problem for a GDD(n, m) of index (λ_1, λ_2) (see Theorem 5.2). At first sight, this would seem to be quite simple to handle compared to the myriad of cases that have to be considered to prove Theorem 1.1. However, it turns out to be a very interesting case, requiring different solution techniques and another necessary condition. In particular, one technique developed here requires knowing when there exists a multigraph on n vertices whose edges can be partitioned into triples, and whose complement in λK_n has a 1-factorization (see Theorem 2.9). This result is of interest in its own right (see [15], for example).

Graph theoretically, a GDD(n, m) of index (λ_1, λ_2) is a partition of the edges of a graph H into copies of K_3 (each K_3 is also called a *triple*), where H is the multigraph with vertex set $V = V_0 \cup V_1 \cup \cdots \cup V_{m-1}$, $|V_i| = n$ for each $i \in \mathbf{Z}_m$, in which two vertices are joined by λ_1 edges if they both occur in V_i for some i, and otherwise are joined by λ_2 edges. Edges joining vertices in the same or different groups are called *pure* or *cross* edges respectively. This description of a GDD will often be used in this paper.

It is worth remarking that many papers have been written on GDDs; for example considering the case where $k=4$ [2], and the case where not all groups have the same size [4]. See [5] for many references.

2. PRELIMINARY RESULTS: $m=2$

In this section we obtain several building blocks. In Section 3, these will be put together in various ways to show that the following necessary conditions are sufficient for the existence of a GDD$(n, 2)$ of index (λ_1, λ_2) (see Theorem 3.7).

LEMMA 2.1. *If there exists a* GDD$(n, 2)$ *of index* (λ_1, λ_2) *then*

(1) 2 *divides* $\lambda_1(n-1) + \lambda_2 n$,

(2) 3 *divides* $\lambda_1 n(n-1) + \lambda_2 n^2$, *and*

(3) $\lambda_1 \geqslant \lambda_2 n/2(n-1)$.

Proof. Conditions (1) and (2) follow because each vertex must have even degree, and the number of edges must be divisible by 3. (3) follows since any cross edge must be contained in a triple that contains another cross edge and a pure edge, so the number of pure edges must be at least half the number of cross edges. ▮

We now proceed to produce some building blocks and other useful results.

LEMMA 2.2. *Let $n \geqslant 3$. There exists a* GDD$(n, 2)$ *of index* $(n, 2n-2)$.

Proof. Define

$$B = \{\{(a, 0), (b, 0), (c, 1)\}, \{(a, 1), (b, 1), (c, 0)\} \mid 0 \leqslant a < b \leqslant n-1, c \in \mathbf{Z}_n\}.$$

Then $(\mathbf{Z}_n \times \mathbf{Z}_2, \{\mathbf{Z}_n \times \{i\} \mid i \in \mathbf{Z}_2\}, B)$ is a GDD$(n, 2)$ of index $(n, 2n-2)$. ▮

The following is a result of Petersen.

THEOREM 2.3 [11]. *Let H be a regular multigraph of even degree. Then there exists a 2-factorization of H.*

Lemma 2.4 is a special case of a result of Rodger and Stubbs.

LEMMA 2.4 [16]. *Let $\lambda, n \geqslant 1$. Suppose that $0 \leqslant x \leqslant \lambda(n-1)$, x is even, and 3 divides xn. Then there exists an x-regular mutligraph of multiplicity at most λ with n vertices whose edges can be partitioned into triples.*

These two results can be combined to obtain Corollary 2.5. Let $E(H)$ be the set of edges in H.

COROLLARY 2.5. *Suppose that $\lambda, n \geqslant 1$, $0 \leqslant x \leqslant \lambda(n-1)$, 3 divides xn, and $\lambda(n-1)$ and x are even. Then there exists an x-regular multigraph H of multiplicity at most λ with n vertices whose edges can be partitioned into triples, such that $\lambda K_n - E(H)$ has a 2-factorization.*

Proof. Choose H using Lemma 2.4, then apply Theorem 2.3 to $\lambda K_n - E(H)$. ▮

We will need a companion result to Corollary 2.5 to cope with the situation where $\lambda(n-1)$ is odd. Obtaining this result will require the following results, the first by Stern and Lenz, the second by Rees, and the third by Simpson. For any $D \subseteq \mathbf{Z}_{\lfloor n/2 \rfloor} \backslash \{0\}$, let $H[D]$ be the graph with vertex set $\mathbf{Z}_n$ and edge set $\{\{j, j+d\} \mid d \in D, j \in \mathbf{Z}_n\}$, reducing the sum modulo n.

LEMMA 2.6 [18]. *There exists a 1-factorization of $H[D]$ if and only if there exists a $d \in D$ such that $n/gcd(n, d)$ is even.*

Note that if $d = n/2 \in D$ then since $n/gcd(n, d)$ is even, $H[D]$ has a 1-factorization.

THEOREM 2.7 [15]. *For all $n \equiv 0 \pmod 6$ and for all even x with $0 \leqslant x < n$ except for $(n, x) \in \{(12, 10), (6, 4)\}$, there exists an x-regular simple graph H on n vertices whose edges can be resolvably partitioned into triples, such that $K_n - E(H)$ has a 1-factorization.*

THEOREM 2.8 [17]. *For any* $y \geqslant 1$ *and for some* $s \in \{3y, 3y+1\}$, *the integers in* $\{y+1, y+2, ..., 3y+1\} \backslash \{s\}$ *can be partitioned into pairs* (a_i, b_i) *with* $b_i > a_i$ *such that* $\{b_i - a_i \mid 1 \leqslant i \leqslant y\} = \{1, 2, ..., y\}$.

We can now present the companion to Corollary 2.5. It is a result that is of interest in its own right.

THEOREM 2.9. *Suppose that* $\lambda \geqslant 1$ *and* $n \geqslant 3$. *Then*

(i) *there exists an x-regular graph H on n vertices and of multiplicity at most* λ *whose edges can be partitioned into triples, such that*

(ii) $\lambda K_n - E(H)$ *has a 1-factorization.*

if and only if $0 \leqslant x \leqslant \lambda(n-1)$, *if* $x > 0$ *then* 3 *divides* xn, *if* $x < \lambda(n-1)$ *then* 2 *divides n, and* 2 *divides x.*

Proof. It is obvious that if (i) and (ii) are true then $0 \leqslant x \leqslant \lambda(n-1)$, if $x > 0$ then 3 divides xn, if $x < \lambda(n-1)$ then 2 divides n, and 2 divides x; therefore we will now prove the opposite statement.

For each $\lambda \geqslant 1$ and each even $n \geqslant 4$, let $S(n, \lambda)$ be the set of integers x for which (i) and (ii) are true. Let $\ell = 2$ if $n \equiv 0$ or 4 (mod 6) and let $\ell = 6$ if $n \equiv 2 \pmod 6$.

Since there exists a 1-factorization of K_n, $0 \in S(n, \lambda)$, and for $x < \lambda(n-1)$ if $x \in S(n, \lambda)$ then $x \in S(n, \lambda')$ for all $\lambda' \geqslant \lambda$. Also, by Theorem 1.2 there exists a TS(n) of index ℓ; so if $x = y\ell(n-1) + x'$ with $0 \leqslant x' < \lambda(n-1)$ and $\lambda \leqslant \ell$, and if $x' \in S(n, \lambda)$, then $x \in S(n, \lambda + y\ell)$. Therefore we need only consider the cases where $0 < x < \ell(n-1)$.

Suppose that $n \equiv 0 \pmod 6$. We need only consider the cases where $x < 2(n-1)$. If $x < n$ then the result follows from Theorem 2.7 unless $(n, x) \in \{(12, 10), (6, 4)\}$. Fortunately, since we do not require the set of triples to be resolvable, we can obtain solutions in these cases too: for each $m \in \{3, 6\}$ the complement of the edges in the triples of a GDD$(2, m)$ of index $(0, 1)$ is a 1-factor. If $n \leqslant x \leqslant 2n-4$ then we can simply combine a solution where $x' = n-2$ and $\lambda' = 1$ with a solution where $x'' = x-(n-2)$ and $\lambda'' = 1$.

If $n \equiv 2$ or 4 (mod 6) then since x is even and 3 divides xn, we have that $x \equiv 0 \pmod 6$, so let $x = 6y$. If $x = n-2$ then $n \equiv 2 \pmod 6$; since there exists a GDD$(2, 3y+1)$ of index $(0, 1)$ we have that $n-2 \in S(n, 1)$. If $x < n-2$ then define s, a_i and b_i as in Theorem 2.8, and let $T = \{\{j, a_i + j, b_i + j\} \mid j \in \mathbf{Z}_n\}$, reducing sums modulo n. Then T is a set of triples that partition $H = H[D']$ where $D' = \{1, 2, ..., 3y+1\} \backslash \{s\}$, and $K_n - E(H) = H[D]$ where $D = \{1, 2, ..., n/2\} \backslash D'$. Since $x < n-2$, $n/2 \in D$, so $K_n - E(H)$ has a 1-factorization by Lemma 2.6. So it remains to consider $x \geqslant n$.

If $n \equiv 4 \pmod 6$ then $\ell = 2$ so we can assume that $x < 2(n-1)$; so $n+2 \leqslant x \leqslant 2n-8$ (since $x \equiv 0 \pmod 6$). We can combine a solution where $x' = n-4$ and $\lambda' = 1$ with a solution where $x'' = x-(n-4) \leqslant n-4$ and $\lambda'' = 1$.

If $n \equiv 2 \pmod 6$ then $\ell = 6$, so we can assume that $x < 6(n-1)$; so $n+4 \leqslant x \leqslant 6n-12$ (since $x \equiv 0 \pmod 6$). Let ℓ' be such that $\ell'(n-2) < x \leqslant (\ell'+1)(n-2)$. Combine ℓ' solutions where $x' = n-2$ and $\lambda' = 1$ with a solution where $x'' = x - \ell'(n-2) \leqslant n-2$ and $\lambda'' = 1$. ▮

It will be useful to let $[x, y, z]$ denote the graph with vertex set $\mathbf{Z}_n \times \mathbf{Z}_2$ in which two vertices (u, i) and (v, j) are joined by x edges if $i = j = 0$, by y edges if $i \neq j$, and by z edges if $i = j = 1$.

The next four results are crucial building blocks in the construction of the GDD's in Section 3.

Lemma 2.10. *For each $i \in \mathbf{Z}_2$, let T_i be an xn-regular multigraph on the vertex set $\mathbf{Z}_n \times \{i\}$ that has a 1-factorization. Then there exists a set of triples whose edges partition the edges of $[0, x, 0] + T_i$.*

Proof. Partition the xn 1-factors in a 1-factorization of T_i into n sets $S_0, S_1, ..., S_{n-1}$, each of size x. For each $a \in \mathbf{Z}_n$ and for each edge $\{(u, i), (v, i)\}$ in a 1-factor in S_a, let B contain the triple $\{(u, i), (v, i), (a, i+1)\}$, reducing the sum modulo 2. ▮

A *quasigroup* $(\mathbb{Z}_n, \circ)$ of *order* n is an $n \times n$ array in which each cell contains exactly one symbol, and each symbol in $\mathbb{Z}_n$ occurs exactly once in each row and exactly once in each column; if cell (a, b) contains c then we write $a \circ b = c$. A quasigroup $(\mathbb{Z}_n, \circ)$ is *symmetric* if $a \circ b = b \circ a$ for all $a, b \in \mathbb{Z}_n$, and is *idempotent* if $a \circ a = a$ for all $a \in \mathbb{Z}_n$. It is well known (and easy to see!) that there exists a symmetric idempotent quasigroup of order n for all odd $n \geqslant 1$.

Lemma 2.11. *Let n be odd, and let F be any 1-factor of $[0, 1, 0]$. Then there exists an edge-disjoint decomposition of $[1, 1, 0] - F$ and of $[0, 1, 1] - F$ into copies of K_3.*

Proof. Let $(\mathbf{Z}_n, \circ)$ be a symmetric idempotent quasigroup of order n. Let $i \in \mathbf{Z}_n$ and let $F' = \{\{(a, 0), (a, 1)\} \mid a \in \mathbf{Z}_n\}$. Let $B'_i = \{\{(a, i), (b, i), (a \circ b, i+1)\} \mid 0 \leqslant a < b \leqslant n-1\}$, reducing $i+1$ modulo 2. Then clearly the triples in B'_i partition the edges in $[1, 1, 0] - F'$ or $[0, 1, 1] - F'$ if $i = 0$ or 1 respectively. The first coordinate of the symbols in the triples in B'_i whose second coordinate is $i+1$ can easily be renamed to produce a set of triples B_i that partition the edges of $[1, 1, 0] - F$ or $[0, 1, 1] - F$ as required. ▮

LEMMA 2.12. *Let $i \in \mathbf{Z}_2$, and let H_i be a 2x-regular graph on the vertex set $\mathbf{Z}_n \times \{i\}$. Then there exists a 2x-regular multigraph T consisting of 2x 1-factors, each being in $[0, 1, 0]$, such that there exists an edge-disjoint decomposition of $H_i + T$ into copies of K_3.*

Proof. By Theorem 2.3, H_i has a 2-factorization into x 2-factors $T_0, T_1, ..., T_{x-1}$. For each $j \in \mathbf{Z}_x$, T_j consists of vertex disjoint cycles which we can arbitrarily orient to form directed cycles; call the resulting directed graph T'_j. Let H'_i be the corresponding directed graph. For each directed edge (a, b) in T'_j, let $\{(a, i), (b, i+1)\} \in F_{2j}$ and $\{(a, i), (a, i+1)\} \in F_{2j+1}$. Let T be the $2x$-regular multigraph formed by the sum of $F_0, ..., F_{2x-1}$. Then $B = \{\{(a, i), (b, i), (b, i+1)\} \mid (a, b) \in E(H'_i)\}$ is a set of triples whose edges partition the edges of $H_i + T$. ∎

LEMMA 2.13. *Let $n \geqslant 4$ be even. Let $\varepsilon = 0$ if $n \equiv 0 \pmod 4$, $\varepsilon = 1$ if $n \equiv 6 \pmod{12}$, and $\varepsilon = 3$ if $n \equiv 2$ or $10 \pmod{12}$. For each $i \in \mathbf{Z}_2$ there exists a simple graph H_i on the vertex set $\mathbf{Z}_n \times \{i\}$* such that:

(i) *H_0 is $(n/2 + \varepsilon)$-regular and H_1 is $(n/2 - \varepsilon)$-regular,*

(ii) *the edges of $[0, 1, 0] + H_0 + H_1$, can be partitioned into triples, and*

(iii) *there exists a 1-factorization of $K_n - E(H_i)$, $i \in \mathbf{Z}_2$.*

Proof. Let $D = \{2k - 1 \mid 1 \leqslant k \leqslant n/4\}$. Define

$$D_0 = \begin{cases} D & \text{if } \varepsilon = 0, \\ D \cup \{2\} & \text{if } \varepsilon = 1, \\ D \cup \{2, 4\} & \text{if } \varepsilon = 3, \end{cases}$$

and define

$$D_1 = \begin{cases} D & \text{if } \varepsilon = 0, \\ (D \cup \{2\}) \backslash \{n/2 - 2\} & \text{if } \varepsilon = 1, \\ D \cup \{n/2 - 4\} & \text{if } \varepsilon = 3. \end{cases}$$

In any case, define $H_i = H[D_i]$ on the vertex set $\mathbf{Z}_n \times \{i\}$, for each $i \in \mathbf{Z}_2$. Then clearly H_i satisfies (i), and since $n/2 \notin D_i$ it follows from Lemma 2 that (iii) is satisfied.

If $\varepsilon = 0$ then let $B = \{\{(j, 0), (j + 2k - 1, 0), (j + k + n/4, 1)\}, \{(j, 1), (j + 2k - 1, 1), (j + k + n/4 - 1, 0)\} \mid j \in \mathbf{Z}_n, 1 \leqslant k \leqslant n/4\}$.

If $\varepsilon = 1$ then let $B = \{\{(j, 0), (j + 2k - 1, 0), (j + k + (n+2)/4, 1)\}, \{(j, 0), (j + 2, 0), (j + 1, 1)\} \mid j \in \mathbf{Z}_n, 1 \leqslant k \leqslant (n-2)/4\} \cup \{\{(j, 1), (j + 2k - 1, 1), (j + k + (n+2)/4, 0)\}, \{(j, 1), (j + 2, 1), (j + 2, 0)\} \mid j \in \mathbf{Z}_n, 1 \leqslant k \leqslant (n-6)/4\}$.

If $\varepsilon = 3$ then let $B = \{\{(j, 0), (j+2k-1, 0), (j+k+(n+6)/4, 1)\} \mid j \in \mathbf{Z}_n, 1 \leqslant k \leqslant (n-2)/4\} \cup \{\{(j, 0), (j+2, 0), (j+1, 1)\}, \{(j, 0), (j+4, 0), (j+2, 1)\}, \{(j, 1), (j+(n-4)/2, 1), (j+(n-4)/2, 0) \mid j \in \mathbf{Z}_n\} \cup \{\{(j, 1), (j+2k-1, 1), (j+k+(n-2)/4, 0)\} \mid j \in \mathbf{Z}_n, 1 \leqslant k \leqslant (n-10)/4\}$.

Then in each case, B is a set of triples which partition the edges of $[0, 1, 0] + H_0 + H_1$. ▮

The following structure will be needed in Section 3.

Let n be even, and let $\mathscr{H}$ be a partition of $\mathbf{Z}_n$ into sets of size 2. A *symmetric quasigroup* $(\mathbf{Z}_n, \circ)$ with *holes* $\mathscr{H}$ and of *order* n is an $n \times n$ array in which: cell (a, b) contains exactly one symbol in $\mathbf{Z}_n$ if $\{a, b\} \notin \mathscr{H}$ and no symbols if $\{a, b\} \in \mathscr{H}$; for each $a \in \mathbf{Z}_n$ row and column a contain each symbol in $\mathbf{Z}_n$ exactly once except for symbols a and b, where $\{a, b\} \in \mathscr{H}$; and cells (a, b) and (b, a) either contain the same symbol or are both empty, for $0 \leqslant a < b \leqslant n-1$; if cell (a, b) contains c then we write $a \circ b = c$. The following is well known (see [10], for example).

LEMMA 2.14. *For all even $n \geqslant 6$, there exists a symmetric quasigroup with holes $\mathscr{H}$ and of order n, where $\mathscr{H}$ is a partition of $\mathbf{Z}_n$ into sets of size* 2.

Since maximum packings and minimum coverings of triple systems have been completely determined, we have the following result.

LEMMA 2.15 [6, 9]. *Let $n \equiv 2 \pmod 6$, $n \geqslant 8$ and let L be a set of* 2 *independent edges in K_n. Then there exists an edge-disjoint decomposition of $(6y+2) K_n + 2L$ and of $(6y+4) K_n - 2L$ into copies of K_3, for all $y \geqslant 0$.*

Finally, it will probably help enormously to list the values of n that satisfy conditions (1) and (2) of Lemma 2.1 for all values of λ_1 and λ_2. This is done in Table I.

TABLE I

The Values of n (mod 6) for Each Value of λ_1 (mod 6) and λ_2 (mod 6) that Satisfy Conditions (1) and (2) of Lemma 2.1

λ_2: λ_1	0	1	2	3	4	5
0	any	0	0, 3	even	0, 3	0
1	1, 3	—	3	—	3, 5	—
2	0, 1, 3, 4	0	0, 2, 3, 5	0, 4	0, 3	0, 2
3	odd	—	3	—	3	—
4	0, 1, 3, 4	0, 2	0, 3	0, 4	0, 2, 3, 5	0
5	1, 3	—	3, 5	—	3	—

3. EXISTENCE WHEN $m=2$

We begin with a result that helps us deal with condition (3) of Lemma 2.1. It allows us to focus on large values of n, so then this lower bound on λ_1 will no longer be a moving target (that is, a function of n).

PROPOSITION 3.1. *If conditions* (1)–(3) *of Lemma* 2.1 *are sufficient for the existence of a* GDD$(n, 2)$ *of index* (λ_1, λ_2) *whenever* $\lambda_2 \leqslant 2(n-1)$, *then they are sufficient for all* $\lambda_2 \geqslant 1$.

Proof. Suppose that n, λ_1 and λ_2 satisfy Conditions (1)–(3) of Lemma 2.1, that $2x(n-1) < \lambda_2 \leqslant (2x+2)(n-1)$, and that $x \geqslant 1$. Then by (3), $\lambda_1 \geqslant \lambda_2/2 + x + \varepsilon$ where

$$\varepsilon = \begin{cases} \frac{1}{2} & \text{if } \lambda_2 \text{ is odd and } \lambda_2 < (2x+1)(n-1), \\ 1 & \text{if } \lambda_2 \text{ is even}, \\ \frac{3}{2} & \text{if } \lambda_2 \text{ is odd and } \lambda_2 \geqslant (2x+1)(n-1). \end{cases}$$

Let $\lambda_1' = \lambda_1 - xn$ and $\lambda_2' = \lambda_2 - 2x(n-1)$. Then $\lambda_2' \leqslant 2(n-1)$, and since $\lambda_1' = \lambda_1 - xn \geqslant \lambda_2/2 + x + \varepsilon - xn = (\lambda_2 - 2x(n-1))/2 + \varepsilon = \lambda_2'/2 + \varepsilon$, so $\lambda_1' \geqslant \lambda_2' n/2(n-1)$, so (3) is satisfied by n, λ_1' and λ_2'. (1) and (2) are easily seen to be satisfied too. Therefore, by our assumption there exists a GDD$(n, 2)$ of index $(\lambda_1 - xn, \lambda_2 - 2x(n-1))$. Also, by Lemma 2.2 there exists a GDD$(n, 2)$ of index $(xn, x(2n-2))$ for any $x \geqslant 1$. So together these two GDDs form a GDD$(n, 2)$ of index (λ_1, λ_2). ∎

Therefore, it remains to consider the case where $\lambda_2 \leqslant 2(n-1)$; or $n \geqslant \lambda_2/2 + 1$. Under this condition, (3) simply becomes $\lambda_1 \geqslant (\lambda_2 + 1)/2$. So throughout the rest of this section we will assume that n and λ_1 satisfy these lower bounds imposed by λ_2.

PROPOSITION 3.2. *Suppose that n is odd, $\lambda_1 \geqslant \lambda_2/2 + 1$ and $n \geqslant \lambda_2/2 + 1$. Let n, λ_1 and λ_2 satisfy conditions* (1) *and* (2) *of Lemma* 2.1. *Then there exists a* GDD$(n, 2)$ *of index* (λ_1, λ_2).

Proof. Since n is odd, λ_2 is even (see Table I). Let $\lambda = \lambda_1 - \lambda_2/2$. So $\lambda \geqslant 1$. The result will follow if we can find an integer t that satisfies the following conditions:

(i) $0 \leqslant 2t \leqslant \lambda(n-1)$ and 3 divides $(\lambda(n-1) - 2t)\, n$, and

(ii) $\lambda_2 - \lambda(n-1) \leqslant 2t \leqslant \lambda_2$, and 3 divides $(\lambda(n-1) - \lambda_2 + 2t)\, n$.

For, once these conditions are met, we can proceed as follows. Condition (i) ensures that the conditions of Corollary 2.5 are met when $x = \lambda(n-1) - 2t$, so there exists a $(\lambda(n-1) - 2t)$-regular graph H_0 on the vertex set $\mathbf{Z}_n \times \{0\}$ such that there exists a set B_0 of triples which partition the edges of H_0; so $\lambda K_n - E(H_0)$ is a $2t$-regular graph. Similarly, condition (ii) ensures that the conditions of Corollary 2.5 are met with $x = \lambda(n-1) - \lambda_2 + 2t$, so there exists a $(\lambda(n-1) - \lambda_2 + 2t)$-regular graph H_1 on the vertex set $\mathbf{Z}_n \times \{1\}$ such that there exists a set B_1 of triples which partition the edges of H_1; so $\lambda K_n - E(H_1)$ is a $(\lambda_2 - 2t)$-regular graph. Since λ_2 is even, by Lemma 2.12 there exists a set F_0 of $2t$ 1-factors and a set F_1 of $\lambda_2 - 2t$ 1-factors, each 1-factor being in $[0, 1, 0]$, such that for each $i \in \mathbf{Z}_2$ there exists a collection B'_i of triples which partition the edges of $\lambda K_n - E(H_i)$ and the edges in the 1-factors in F_i. Finally, if F is the λ_2-regular multigraph consisting of all the edges in F_0 and F_1, then by Lemma 2.11 there exists a collection B of triples that partition the edges of $[\lambda_2/2, \lambda_2, \lambda_2/2] - E(F)$. Then each edge $\{(u, i), (v, i)\}$ with $i \in \mathbf{Z}_2$ is contained in λ triples in B_i and B'_i, and is in $\lambda_2/2$ triples in B, and clearly each edge $\{(u, 0), (v, 1)\}$ is in λ_2 triples, so the result will follow. So it remains to find an appropriate integer t. Recall that $\lambda \geqslant 1$.

If $\lambda_2 = 6x + 2$ and $n \equiv 3 \pmod 6$ then $\lambda_1 \geqslant 3x + 2$ (since $\lambda_1 \geqslant \lambda_2/2 + 1$) and $n \geqslant 3x + 3$ (since $n \geqslant \lambda_2/2 + 1$). Choose $t = \lceil (3x+1)/2 \rceil$. Then $2t \leqslant n - 1$, 3 divides n, and $\lambda_2 - (n-1) \leqslant 2t$.

If $\lambda_2 = 6x + 2$ and $n \equiv 5 \pmod 6$ then $\lambda_1 \equiv 2 \pmod 3$ (see Table I), so $\lambda \equiv 1 \pmod 3$. If x is odd then $n \geqslant 3x + 2$, so choose $t = (3x+1)/2$. If x is even then $n \geqslant 3x + 5$ (since $n \equiv 5 \pmod 6$), so choose $t = (3x+4)/2$.

If $\lambda_2 = 6x + 4$ and $n \equiv 3 \pmod 6$ then $n \geqslant 3x + 3$, so choose $t = \lceil (3x+1)/2 \rceil$.

If $\lambda_2 = 6x + 4$ and $n \equiv 5 \pmod 6$ then $\lambda_1 \equiv 1 \pmod 3$ (see Table I). If x is even then $n \geqslant 3x + 5$, so choose $t = (3x+2)/2$. If x is odd then $n \geqslant 3x + 8$, so choose $t = (3x+5)/2$.

If $\lambda_2 = 6x$ and $n \equiv 1 \pmod 6$ then: if x is odd then $n \geqslant 3x + 4$, so choose $t = (3x+3)/2$; if x is even then $n \geqslant 3x + 1$, so choose $t = 3x/2$.

If $\lambda_2 = 6x$ and $n \equiv 3 \pmod 6$ then $n \geqslant 3x + 3$, so choose $t = \lceil 3x/2 \rceil$.

If $\lambda_2 = 6x$ and $n \equiv 5 \pmod 6$ then $\lambda_1 \equiv 0 \pmod 3$ (see Table I) and so $\lambda \geqslant 3$, and $n \geqslant 3x + 2$. If x is even then choose $t = 3x/2$, and if x is odd then choose $t = (3x+3)/2$. ∎

It turns out that if λ_2 is odd then we need to consider the smallest value of λ_1 by itself.

PROPOSITION 3.3. *Suppose that λ_2 is odd and $\lambda_1 = (\lambda_2 + 1)/2$. Let n, λ_1 and λ_2 satisfy conditions* (1)–(3) *of Lemma* 2.1. *Then there exists a* GDD$(n, 2)$ *of index* (λ_1, λ_2).

Proof. By (3) of Lemma 2.1, $n \geqslant \lambda_2 + 1$. Since λ_2 is odd, n and λ_1 are even (see Table I), so we can write $\lambda_1 = 6x + 2y$, $\lambda_2 = 12x + 4y - 1$, and $n \geqslant 12x + 4y$, where $y \in \mathbf{Z}_3$. So Table I shows that λ_1, λ_2 and n are restricted even more: if $\lambda_1 \equiv 0 \pmod{6}$ then $\lambda_2 \equiv 5 \pmod{6}$ so $n \equiv 0 \pmod{6}$; if $\lambda_1 \equiv 2 \pmod{6}$ then $\lambda_2 \equiv 3 \pmod{6}$ so $n \equiv 0$ or $4 \pmod{6}$; and if $\lambda_1 \equiv 4 \pmod{6}$ then $\lambda_2 \equiv 1 \pmod{6}$ so $n \equiv 0$ or $2 \pmod{6}$. Notice that in every case

(a) either $n \equiv 0 \pmod{6}$ or $n/2 - \lambda_1 \equiv 0 \pmod{3}$.

It will also be useful later to notice that if $n \equiv 2$ or $10 \pmod{12}$ then $\lambda_1 \equiv 4$ or $2 \pmod{6}$ respectively, and so since $n/2 \geqslant (\lambda_2 + 1)/2 = \lambda_1$ we have:

(b) if $n \equiv 2$ or $10 \pmod{12}$ then $n/2 \geqslant \lambda_1 + 3$;

and if $n \equiv 6 \pmod{12}$ then $n/2$ is odd, so we have:

(c) if $n \equiv 6 \pmod{12}$ then $n/2 \geqslant \lambda_1 + 1$.

Let ε be defined as in Lemma 2.13. By Lemma 2.13, for each $i \in \mathbf{Z}_2$, there exists a simple graph H_i on the vertex set $\mathbf{Z}_n \times \{i\}$ satisfying (i)–(iii). Let B_0 be a set of triples that partitions the edges of $[0, 1, 0] + H_0 + H_1$ (see (ii)). By (iii), $K_n - E(H_i)$ can be partitioned into $n - 1 - (n/2 + (-1)^i \varepsilon) = n/2 - 1 - (-1)^i \varepsilon$ 1-factors.

We want to apply Theorem 2.9 with $x = n/2 - \lambda_1 - (-1)^i \varepsilon$ and $\lambda = 1$, so we have some things to check. If $n \equiv 2$ or $4 \pmod{6}$ then $\varepsilon \in \{0, 3\}$, so by (a) we have that 3 divides xn. In each case $n/2 - (-1)^i \varepsilon$ is even, so x is even because λ_1 is even. Clearly $x \leqslant n - 1$, and by (b) and (c) we have that $x \geqslant 0$.

Therefore, by Theorem 2.9, for each $i \in \mathbf{Z}_2$ there exists a set of triples B'_i and there exists an $(n/2 - \lambda_1 - (-1)^i \varepsilon)$—regular graph H'_i with vertex set $\mathbf{Z}_n \times \{i\}$ whose edges are partitioned by the triples in B'_i such that $K_n - E(H'_i)$ has a 1-factorization into $n - 1 - (n/2 - \lambda_1 - (-1)^i \varepsilon) = n/2 + \lambda_1 - 1 + (-1)^i \varepsilon$ 1-factors.

Finally, for each $i \in \mathbf{Z}_2$, since $\lambda_1 \geqslant 2$ we can take the $(\lambda_1 - 2)(n - 1)$ 1-factors in a 1-factorization of $(\lambda_1 - 2) K_n$ on the vertex set $\mathbf{Z}_n \times \{i\}$. So for each $i \in \mathbf{Z}_2$, altogether on the vertex set $\mathbf{Z}_n \times \{i\}$ we have defined $(n/2 - 1 - (-1)^i \varepsilon) + (n/2 + \lambda_1 - 1 + (-1)^i \varepsilon) + (\lambda_1 - 2)(n - 1) = n(\lambda_1 - 1) = n(\lambda_2 - 1)/2$ 1-factors. By Lemma 2.10, there exists a set B_1 of triples that partition the edges in these 1-factors together with the edges in $[0, \lambda_2 - 1, 0]$.

Then clearly the triples in B_0, B_1, B'_0 and B'_1 form a GDD$(n, 2)$ of index (λ_1, λ_2). ∎

Before presenting our last proposition, we need to deal with two exceptional cases.

LEMMA 3.4. *Let* $n \equiv 2$ *or* 4 (mod 6), $\lambda_1 = 6y+6$, $\lambda_2 = 12y+9$ *and* $n \geqslant 6y+6$. *Then there exists a* GDD$(n, 2)$ *of index* (λ_1, λ_2).

Proof. If $n \equiv 2$ (mod 6) then there exists a TS$(2n)$ of index 2, and by Proposition 3.3 there exists a GDD$(n, 2)$ of index $(6y+4, 12y+7)$, which together produce a GDD$(n, 2)$ of index $(6y+6, 12y+9)$.

If $n \equiv 4$ (mod 6) then define ε as in Lemma 2.13. By Lemma 2.13, for each $i \in \mathbf{Z}_2$ there exists a simple graph H_i on the vertex set $\mathbf{Z}_n \times \{i\}$ that is $(n/2 + (-1)^i \varepsilon)$-regular, such that there exists a set B of triples that partition the edges of $[0, 1, 0] + H_0 + H_1$, and such that $K_n - E(H_i)$ has a 1-factorization into a set $F_1(i)$ of $n/2 - 1 - (-1)^i \varepsilon$ 1-factors. Since 6 divides $x = 3n/2 - 6y - 6 - (-1)^i \varepsilon$ and $0 \leqslant x \leqslant n-1$, by Theorem 2.9, for each $i \in \mathbf{Z}_2$ there exists a set B_i of triples and an x-regular graph H_i in $(6y+5) K_n$ defined on the vertex set $\mathbf{Z}_n \times \{i\}$ whose edges are partitioned by the triples in B_i, such that $(6y+5) K_n - E(H)$ has a 1-factorization into a set $F_2(i)$ of $(6y+5)(n-1) - x$ 1-factors. In $F_1(i)$ and $F_2(i)$, $i \in \mathbf{Z}_2$ there are a total of $(6y+4) n$ 1-factors, which altogether with the edges in $[0, 12y+8, 0]$ can be partitioned into a set B' of triples (by Lemma 2.10).

Clearly the triples in B, B', B_0 and B_1 together form a GDD$(n, 2)$ of index $(6y+6, 12y+9)$. ∎

LEMMA 3.5. *Let* $\lambda_1 \equiv 4$ (mod 6), $\lambda_2 = 1$ *and* $n \equiv 2$ (mod 6). *Let* n, λ_1 *and* λ_2 *satisfy conditions* (1)–(3) *of Lemma* 2.1. *Then there exists a* GDD$(n, 2)$ *of index* (λ_1, λ_2).

Proof. Let $\lambda_1 = 6y+4$. Let $F = \{\{2a, 2a+1\} \mid a \in \mathbf{Z}_{n/2}\}$ and $F_0 = \{\{(a, 0), (b, 0)\} \mid \{a, b\} \in F\}$. Let $L = \{\{0, 1\}, \{2, 3\}\}$, and for each $i \in \mathbf{Z}_2$ let $L_i = \{\{(a, i), (b, i)\} \mid \{a, b\} \in L\}$.

Let $(\mathbf{Z}_n, \circ)$ be a symmetric quasigroup with holes F and of order n (see Lemma 2.14). Define

$$\begin{aligned} B = {} & \{\{(a, 0), (b, 0), (a \circ b, 1)\} \mid 0 \leqslant a < b \leqslant n-1, \{a, b\} \notin F\} \\ & \cup \{\{(2a, 0), (2a+1, 0), (2a, 1)\}, \{(2a, 0), \\ & \quad (2a+1, 0), (2a+1, 1)\} \mid 2 \leqslant a < n/2\} \\ & \cup \{(2a, 0), (2a, 1), (2a+1, 1)\}, \{(2a+1, 0), \\ & \quad (2a, 1), (2a+1, 1)\} \mid 0 \leqslant a \leqslant 1\}. \end{aligned}$$

Then the triples in B contain: each edge $\{(a, 0), (b, 0)\}$ exactly once if $\{a, b\} \notin F$, exactly twice if $\{a, b\} \in F \backslash L$, and not at all if $\{a, b\} \in L$; each edge $\{(a, 0), (b, 1)\}$ exactly once; and each edge $\{(a, 1), (b, 1)\}$ exactly twice if $\{a, b\} \in L$, and otherwise not at all.

Using Lemma 2.15, let B_0 be a collection of triples that partition the edges of $(6y+2)\,K_n+2L_0$ on the vertex set $\mathbf{Z}_n\times\{0\}$, and let B_1 be a collection of triples that partition the edges of $(6y+4)\,K_n-2L_1$ on the vertex set $\mathbf{Z}_n\times\{1\}$.

Finally, let $(\mathbf{Z}_n\times\{0\}, F_0, B')$ be a GDD$(n/2, 2)$ of index $(0, 1)$.

Then the triples in B, B', B_0 and B_1 together form a GDD$(n, 2)$ of index $(6y+4, 1)$. ∎

PROPOSITION 3.6. *Suppose that n is even, $\lambda_1 \geqslant \lambda_2/2+1$ and $n \geqslant \lambda_2/2+1$. Let n, λ_1 and λ_2 satisfy conditions* (1) *and* (2) *of Lemma* 2.1. *Then there exists a* GDD$(n, 2)$ *of index* (λ_1, λ_2).

Proof. The result will follow if we can find an integer t that satisfies the following conditions:

(i) $0 \leqslant t$, $nt \leqslant \lambda_1(n-1)$, and 3 divides $(\lambda_1(n-1)-tn)\,n$, and

(ii) $t \leqslant \lambda_2$, $(\lambda_2-t)\,n \leqslant \lambda_1(n-1)$, and 3 divides $(\lambda_1(n-1)-(\lambda_2-t)\,n)\,n$.

For, once these conditions are met, we proceed as follows.

Since n is even λ_1 is even, so $(\lambda_1(n-1)-tn)$ is even. Therefore, by Theorem 2.9 and using (i), there exists a $(\lambda_1(n-1)-tn)$-regular graph H_0 on the vertex set $\mathbf{Z}_n\times\{0\}$ of multiplicity at most λ_1 and there exists a set B_0 of triples such that: these triples partition the edges of H_0; and $T_0 = \lambda_1 K_n - E(H_0)$ has a 1-factorization into tn 1-factors. Similarly, by Theorem 2.9 and (ii), there exists a $(\lambda_1(n-1)-(\lambda_2-t)\,n)$-regular graph H_1 on the vertex set $\mathbf{Z}_n\times\{1\}$ and there exists a set B_1 of triples such that: these triples partition the edges of H_1; and $T_1 = \lambda K_n - E(H_1)$ has a 1-factorization into $(\lambda_2-t)\,n$ 1-factors. Finally, by Lemma 2.10, there exists a set B of triples which partition the edges of $[0, \lambda_2, 0] + T_0 + T_1$. Then clearly the triples in B_0, B_1 and B together form a GDD$(n, 2)$ of index (λ_1, λ_2). So it remains to find a suitable value of t in each case.

In the following, to check that $tn \leqslant \lambda_1(n-1)$ it is easier to check that $t \leqslant (\lambda_1-t)(n-1)$. Also, we will choose t so that $t \geqslant \lambda_2/2$, in which case $tn \leqslant \lambda_1(n-1)$ implies that $(\lambda_2-t)\,n \leqslant \lambda_1(n-1)$.

If $\lambda_2 = 6x$ then $\lambda_1 \geqslant 3x+1$ and $n \geqslant 3x+1$. Choose $t = 3x$. From Table I, 3 divides λ_1, n or $n-1$, and since 3 divides t, the divisibility by 3 conditions in (i)–(ii) are met.

If $\lambda_2 = 6x+1$ and $n \equiv 0 \pmod 6$ then $\lambda_1 \geqslant 3x+2$ and $n \geqslant 3x+2$. Choose $t = 3x+1$.

If $\lambda_2 = 6x+1$ and $n \equiv 2 \pmod 6$, then $\lambda_1 \equiv 4 \pmod 6$ (see Table I), so $\lambda_1 \geqslant 3x+4$ and $n \geqslant 3x+2$. Choose $t = 3x+2$. Then all conditions in (i)–(ii) are met except that if $x=0$ then $\lambda_2 < t$; but then we seek a GDD$(n, 2)$ of index $(6y+4, 1)$ which was constructed in Lemma 3.5.

If $\lambda_2 = 6x+2$ then $\lambda_1 \geqslant 3x+2$ and $n \geqslant 3x+2$. Choose $t=3x+1$.

If $\lambda_2 = 6x+3$ and $n \equiv 0 \pmod 6$ then $\lambda_1 \geqslant 3x+3$ and $n \geqslant 3x+3$. Choose $t=3x+2$.

If $\lambda_2 = 6x+3$ and $n \equiv 2 \pmod 6$ then $\lambda_1 \equiv 0 \pmod 6$ (see Table I), so $\lambda_1 \geqslant 3x+3$ and $n \geqslant 3x+5$. Choose $t=3x+3$. Then all conditions in (i)–(ii) are met except that if $\lambda_1 = 3x+3$ then $nt > \lambda_1(n-1)$. However, if $\lambda_1 = 3x+3$ then we can write $\lambda_1 = 6y+6$, $\lambda_2 = 12y+9$ and $n \equiv 2 \pmod 6$, so we can use Lemma 3.4.

If $\lambda_2 = 6x+3$ and $n \equiv 4 \pmod 6$ then $\lambda_1 \geqslant 3x+3$ and $n \geqslant 3x+4$. Choose $t=3x+3$. Then all conditions in (i)–(ii) are satisfied unless $\lambda_1 = 3x+3$, for then $nt > \lambda_1(n-1)$. If $\lambda_1 = 3x+3$ then again the GDD can be obtained from Lemma 3.4.

If $\lambda_2 = 6x+4$ then $\lambda_1 \geqslant 3x+3$ and $n \geqslant 3x+3$. Choose $t=3x+2$.

If $\lambda_2 = 6x+5$ and $n \equiv 0 \pmod 6$ then $\lambda_1 \geqslant 3x+4$ and $n \geqslant 3x+6$. Choose $t=3x+3$.

If $\lambda_2 = 6x+5$ and $n \equiv 2 \pmod 6$ then $\lambda_1 \equiv 2 \pmod 6$ (see Table I), so $\lambda_1 \geqslant 3x+5$ and $n \geqslant 3x+5$. Choose $t=3x+4$. ∎

Finally, we can present the main result.

THEOREM 3.7. *Let* $n \geqslant 3$ *and* $\lambda_1, \lambda_2 \geqslant 1$. *There exists a* GDD$(n, 2)$ *of index* (λ_1, λ_2) *if and only if*

(1) 2 *divides* $\lambda_1(n-1) + \lambda_2 n$,

(2) 3 *divides* $\lambda_1 n(n-1) + \lambda_2 n^2$, *and*

(3) $\lambda_1 \geqslant \lambda_2 n / 2(n-1)$.

Proof. By Proposition 3.1, it suffices to consider the case where $\lambda_2 \leqslant 2(n-1)$, so $n \geqslant \lambda_2/2+1$ and therefore by (3) $\lambda_1 \geqslant (\lambda_2+1)/2$. If n is odd (so λ_2 is even) the result follows from Proposition 3.2. If $\lambda_1 = (\lambda_2+1)/2$ then the result follows from Proposition 3.3. If n is even and $\lambda_1 \geqslant \lambda_2/2+1$ then the result follows from Proposition 3.6. ∎

4. EXISTENCE WHEN $n=2$

In this section we prove that the following necessary conditions for the existence of a GDD$(2, m)$ of index (λ_1, λ_2) are sufficient (see Theorem 4.10).

LEMMA 4.1. *If there exists a* GDD$(2, m)$ *of index* (λ_1, λ_2) *then*

(1) 2 *divides* $\lambda_1 + 2\lambda_2(m-1)$,

(2) 3 *divides* $\lambda_1 m + 2\lambda_2 m(m-1)$, *and*

(3) $\lambda_1 \leqslant (m-1)\lambda_2$.

Remark. Condition (1) implies that λ_1 is even.

Proof. Both (1) and (2) follow since each vertex must have even degree and the number of edges must be divisible by 3. (3) follows since each pure edge is contained in a triple containing two cross edges, so the number of pure edges is at most half the number of cross edges. ∎

We can easily handle the case where $n = m = 2$ now.

LEMMA 4.2. *There exists a* GDD$(2, 2)$ *of index* (λ_1, λ_2) *if conditions* (1)–(3) *of Lemma* 4.1 *and condition* (3) *of Lemma* 2.1 *hold.*

Proof. The conditions (3) of Lemma 2.1 and 4.1 imply that $\lambda_1 = \lambda_2$, so the GDD$(2, 2)$ of index (λ_1, λ_2) must be a TS(4) of index $\lambda = \lambda_1 = \lambda_2$. By Theorem 1.2 conditions (1) and (2) of Lemma 4.1 ensure that a TS(4) of index λ exists. ∎

In view of this result, throughout the rest of this section we can assume that $m \geqslant 3$.

Our proof that conditions (1)–(3) of Lemma 4.1 are sufficient for the existence of a GDD$(2, m)$ of index (λ_1, λ_2) relies heavily on the following lemma.

LEMMA 4.3. *If there exists an edge-disjoint decomposition of* $\lambda_2 K_m$ *into a collection* B *of copies of* K_3 *and a spanning subgraph* H *such that the edges of* H *can be directed to form* H^+ *so that in* H^+ *each vertex has out-degree* $\lambda_1/2$, *then there exists a* GDD$(2, m)$ *of index* (λ_1, λ_2).

Proof. Suppose $\lambda_2 K_m$ on the vertex set $\mathbf{Z}_m$ has been decomposed into a directed graph H^+ and a collection B of K_3's as described. Let

$$B_1 = \{\{(0, a), (1, a), (0, b)\}, \{(0, a), (1, a), (1, b)\} \mid (a, b) \text{ is a directed edge in } H^+\}$$

and

$$B_2 = \{\{(0, a), (0, b), (0, c)\}, \{(1, a), (1, b), (0, c)\}, \{(1, a), (0, b), (1, c)\}, \{(0, a), (1, b), (1, c)\} \mid \{a, b, c\} \text{ is a triple in } B'\}.$$

Then since each vertex $a \in \mathbf{Z}_m$ has out-degree $\lambda_1/2$ in H^+, the edge $\{(0, a), (1, a)\}$ is in λ_1 triples defined in B_1. Also, for each $a \neq b$, the edge $\{a, b\}$ occurs x times in H and $\lambda_2 - x$ times in copies of K_3 in B, so the edges $\{(i, a), (j, b)\}$, $i, j \in \mathbf{Z}_2$ occur in x triples in B_1 and $\lambda_2 - x$ triples in B_2. Therefore $(\mathbf{Z}_2 \times \mathbf{Z}_m, \{\mathbf{Z}_2 \times \{i\} \mid i \in \mathbf{Z}_m\}, B_1 \cup B_2)$ is a GDD$(2, m)$ of index (λ_1, λ_2). ∎

PROPOSITION 4.4. *Suppose that $\lambda_2(m-1)$ is even and $m \geqslant 3$. Then conditions* (1)–(3) *of Lemma* 4.1 *are sufficient for the existence of a* GDD$(2, m)$ *of index* (λ_1, λ_2).

Proof. Recall that (1) of Lemma 4.1 implies that λ_1 is even. Condition (2) implies that 3 divides $\lambda_2 m(m-1) - \lambda_1 m$. Condition (3) implies that $\lambda_2(m-1) - \lambda_1 \geqslant 0$. Therefore we can apply Corollary 2.5 with $x = \lambda_2(m-1) - \lambda_1$ and $\lambda = \lambda_2$ (and replacing n with m) to produce an x-regular multigraph G of multiplicity at most λ_2 whose edges can be partitioned into triples, so that $H = \lambda_2 K_m - E(G)$ has a 2-factorization into λ_1 2-factors. Each 2-factor consists of edge-disjoint cycles that can be oriented to form directed cycles. The resulting directed graph H^+ has out-degree $\lambda_1/2$ at each vertex, so the proposition follows from Lemma 4.3. ∎

It remains to consider the case where $\lambda_2(m-1)$ is odd, so we know that λ_2 is odd, and λ_1 and m are even. We begin by showing that it essentially suffices to consider the case where $\lambda_2 = 1$. It may help to consult Table II which lists the values of m (mod 6) that satisfy conditions (1)–(2) of Lemma 4.1.

A Kirkman triple system KTS(n) is a TS(n) (V, B) of index 1 and order n in which B can be partitioned into sets of size $n/3$ so that each such set is a partition of V. We will use the following theorem in the proof of Proposition 4.7.

THEOREM 4.5 [13]. *For all $n \equiv 3 \pmod 6$ there exists a* KTS(n).

TABLE II

The Values of n (mod 6) for Each Value of λ_1 (mod 6) and λ_2 (mod 6) that Satisfy Conditions 1–2 of Lemma 4.1 when $\lambda_2(m-1)$ Is Odd

λ_2: λ_1	1	3	5
0	0, 4	0, 2, 4	0, 4
2	0	0	0, 2
4	0, 2	0	0

Similarly a GDD(n, m) (V, G, B) of index $(0, \lambda_2)$ is *resolvable* if B can be partitioned into sets of size $|V|/3$ so that each such set is a partition of V. We will use the following special case of a result of Assaf, Hartman, Rees and Stinson.

THEOREM 4.6 [1, 14]. *For all* $m \geqslant 4$, *there exists a resolvable* GDD$(6, m)$ *of index* $(0, 1)$.

PROPOSITION 4.7. *Suppose* $\lambda_2(m-1)$ *is odd and* $m \geqslant 4$. *If conditions* (1)–(3) *of Lemma* 4.1 *are sufficient for the existence of a* GDD$(2, m)$ *of index* (λ_1, λ_2) *when* $\lambda_2 = 1$, *then they are sufficient for all* $\lambda_2 \geqslant 1$, *except possibly for the case where* $\lambda_2 = 5 \pmod 6$, $m \equiv 2 \pmod 6$ *and* $\lambda_1 = 2$.

Proof. Suppose $\lambda_2(m-1)$ is odd, and suppose conditions (1)–(3) of Lemma 4.1 are sufficient for the existence of a GDD$(2, m^*)$ of index $(\lambda_1^*, 1)$ for all $m^* \geqslant 3$ and $\lambda_1^* \geqslant 1$. Suppose that: $m \geqslant 4$; $\lambda_1, \lambda_2 \geqslant 1$; if $\lambda_2 \equiv 5 \pmod 6$ and $m \equiv 2 \pmod 6$ then $\lambda_1 > 2$, and that m, λ_1 and λ_2 satisfy the conditions (1)–(3) of Lemma 4.1. Let $\lambda_2' = \lambda_2 - 1$ and $\lambda_2'' = 1$. We consider the case $m \equiv 0$, 2 and 4 (mod 6) in turn.

Case 1: $m \equiv 0 \pmod 6$. Let $\lambda_1' = \min\{\lambda_2'(m-1), \lambda_1\}$ and $\lambda_1'' = \lambda_1 - \lambda_1'$. Then since λ_1' and λ_2' are even, and since 3 divides m, we have that m, λ_1' and λ_2' satisfy conditions (1)–(3) of Lemma 4.1. So by Proposition 4.4, there exists a GDD$(2, m)$ of index (λ_1', λ_2'). Also, λ_1'' is even, and since $\lambda_1 \leqslant (m-1)\lambda_2$, we have that $\lambda_1'' \leqslant m-1$, so m, λ_1'' and $\lambda_2'' = 1$ satisfy conditions (1)–(3) of Lemma 4.1. So by our assumption there exists a GDD$(2, m)$ of index $(\lambda_1'', \lambda_2'')$. Together these two GDDs form a GDD$(2, m)$ of index (λ_1, λ_2) as required.

Case 2: $m \equiv 2 \pmod 6$. Let $m = 6x + 2$ where $x \geqslant 1$, and let $\lambda_2 = 6y + \varepsilon$ where $\varepsilon \in \{1, 3, 5\}$ and $y \geqslant 0$. Then from Table II, $\lambda_1 = 6z + \varepsilon - 3$ where $z \geqslant 1$ (recall that this proposition does not consider the case where $\lambda_2 \equiv 5 \pmod 6$, $m \equiv 2 \pmod 6$ and $\lambda_1 = 2$, so $z \neq 0$). Then by (3), $\lambda_1 \leqslant (m-1)\lambda_2 = (6x+1)(6y+\varepsilon) = 6(6xy + \varepsilon x + y) + \varepsilon$, so since $\lambda_1 = 6z + \varepsilon - 3$ it must be that in this case

$$\lambda_1 \leqslant (m-1)\lambda_2 - 3.$$

Define λ_1'' to be the largest integer congruent to 4 (mod 6) such that $\lambda_1'' \leqslant \min\{m-4, \lambda_1\}$, and write $\lambda_1'' = 6z'' + 4$. Notice that $z'' \geqslant 0$ since $x \geqslant 1$ and $z \geqslant 1$ (so $\lambda_1 \geqslant 4$). Define $\lambda_1' = \lambda_1 - \lambda_1''$, so $\lambda_1' = 6z + \varepsilon - 3 - 6z'' - 4 = 6(z - z'' - 1) + \varepsilon - 1$.

Since $\lambda_1'' \leqslant m-4$, $\lambda_1'' \equiv 4 \pmod 6$ and $\lambda_2'' = 1$, we have that m, λ_1'' and λ_2'' satisfy conditions (1)–(3) of Lemma 4.1, so by assumption there exists a GDD$(2, m)$ of index $(\lambda_1'', \lambda_2'')$.

If $\lambda_1 \geqslant m-4$ then $\lambda_1'' = m-4$ so $\lambda_1' = \lambda_1 - (m-4) \leqslant (m-1)\,\lambda_2 - 3 - (m-4) = (m-1)\,\lambda_2'$. If $\lambda_1 < (m-4)$ then $\lambda_1' \leqslant \lambda_1 < m-4 < (m-1)\,\lambda_2'$. So in any case m, λ_1' and λ_2' satisfy (3) of Lemma 4.1.

If $\lambda_1 \geqslant m-4$ then $\lambda_1' = \lambda_1 - (m-4) \equiv \varepsilon - 1 \pmod 6$, and $\lambda_2' = \lambda_2 - 1 \equiv \varepsilon - 1 \pmod 6$, so $\lambda_1' \equiv \lambda_2' \pmod 6$. Also, if $\lambda_1 < m-4$ then $\lambda_1 = 0$, 2 or 4 (mod 6) if $\lambda_2 \equiv 1$, 3 or 5 (mod 6) respectively (see Table II); so $\lambda_2' = \lambda_2 - 1 \equiv \lambda_1' \pmod 6$. Therefore, in any case, we have m, λ_1' and λ_2' satisfy (2) of Lemma 4.1, and clearly (1) is satisfied. So, since λ_2' is even, by Proposition 4.4 there exists a GDD$(2, m)$ of index (λ_1', λ_2').

Together, the GDD$(2, m)$ of index (λ_1', λ_2') and that of index $(\lambda_2'', \lambda_2'')$ form a GDD$(2, m)$ of index (λ_1, λ_2) as required.

Case 3: $m \equiv 4 \pmod 6$. Let $m = 6x+4$, where $x \geqslant 0$, and let $\lambda_2 = 6y + \varepsilon$ where $\varepsilon \in \{1, 3, 5\}$ and $y \geqslant 0$. From Table II, we can write $\lambda_1 = 6z$, where $z \geqslant 1$. Then by (3), $\lambda_1 \leqslant (m-1)\,\lambda_2 = (6x+3)(6y+\varepsilon) = 6(6xy + 3y + \varepsilon x) + 3\varepsilon$, so since ε is odd and $\lambda_1 = 6z$ it must that

$$\lambda_1 \leqslant (m-1)\,\lambda_2 - 3.$$

Define λ_1'' to be the largest integer congruent to 0 (mod 6) such that $\lambda_1 \leqslant \min\{m-4, \lambda_1\}$, and write $\lambda_1'' = 6z''$ where $z'' \geqslant 0$. Define $\lambda_1' = \lambda_1 - \lambda_1'' = 6(z - z'')$.

Since $\lambda_1'' \leqslant m-4$, $\lambda_1'' \equiv 0 \pmod 6$ and $\lambda_2 = 1$, we have that m, λ_1'' and λ_2'' satisfy conditions (1)–(3) of Lemma 4.1, so by assumption there exists a GDD(n, m) of index $(\lambda_1'', \lambda_2'')$.

If $\lambda_1'' \geqslant m-4$ then $\lambda_1'' = m-4$, so $\lambda_1' = \lambda_1 - (m-4) \leqslant (m-1)\,\lambda_2 - 3 - (m-4) = (m-1)\,\lambda_2'$. If $\lambda_1 < (m-4)$ then clearly $\lambda_1' \leqslant (m-1)\,\lambda_2'$. So m, λ_1' and λ_2' satisfy (3) of Lemma 4.1. Clearly 3 divides λ_1' and $(m-1)$, so (2) is satisfied, and 2 divides λ_1' and λ_2' so (1) is satisfied. Therefore by Proposition 4.4 there exists a GDD$(2, m)$ of index (λ_1', λ_2') which together with the GDD$(2, m)$ of index $(\lambda_1'', \lambda_2'')$ forms a GDD(n, m) of index (λ_1, λ_2). ∎

Proposition 4.8. *Suppose that* $m \geqslant 4$ *is even. Conditions* (1)–(3) *of Lemma* 4.1 *are sufficient for the existence of a* GDD$(2, m)$ *of index* $(\lambda_1, 1)$.

Proof. To prove this result we use Lemma 4.3, so we need to direct some edges in K_m to form the spanning subgraph H^+ in which each vertex has outdegree $\lambda_1/2$ so that the remaining undirected edges can be partitioned into copies of K_3. We will consider the cases $m = 0$, 2 and 4 (mod 6) in turn (see Table II). By (3) of Lemma 4.1 we have that $\lambda_1 \leqslant m-1$, and by Theorem 1.3 we can assume that $\lambda_1 > 0$. We begin with $m = 6x+4$ since it is the simplest case.

Case 1: $m = 6x+4$. Since $m = 6x+4$ and $\lambda_2 = 1$, we have $\lambda_1 = 6y$ (see Table II). Since $1 \leqslant \lambda_1 \leqslant m-1$, we have that $1 \leqslant y \leqslant x$. We define K_m on the vertex set $\{\infty\} \cup \mathbf{Z}_{6x+3}$. Let $(\mathbf{Z}_{6x+3}, T)$ be a KTS$(6x+3)$ with parallel classes $\pi_0, ..., \pi_{3x}$ (see Theorem 4.5). Clearly π_i has $2x+1$ triples. Partition the triples in π_0 into 2 sets T_0 and R_0 so that $|T_0| = y$. Direct some of the edges of K_m as follows to form H^+.

(i) Join ∞ with an edge directed to each vertex in each triple in T_0.

(ii) Join each vertex in each triple in R_0 with an edge directed to ∞.

(iii) Direct the edges in each triple in T_0 to form directed 3-cycles.

(iv) Direct the edges in each triple in $\bigcup_{i=1}^{3y-1} \pi_i$ to form directed 3-cycles (π_{3y-1} exists since $3y-1 < 3x$).

Then ∞ is incident with $3y = \lambda_1/2$ edges directed out in (i). Each $v \in \mathbf{Z}_{6x+3}$ is incident with 1 edge directed out in (ii)–(iii) and $3y-1$ edges directed out in (iv), so also has outdegree $\lambda_1/2$ in H^+. The edges in K_m remaining undirected are partitioned by the triples in $R_0 \cup (\bigcup_{i=3y}^{3x} \pi_i)$.

Case 2: $m = 6x+2$. Since $m = 6x+2$ and $\lambda_2 = 1$, we have that $\lambda_1 = 6y+4$. Since $\lambda_1 \leqslant m-1$, we have $0 \leqslant y < x$ and $x \geqslant 1$.

If $m = 8$ then $\lambda_1 = 4$. Define K_8 on the vertex set $\mathbf{Z}_4 \times \mathbf{Z}_2$. Let H^+ contain the directed edges in $\{((i, 0), (i, 1)) \mid i \in \mathbf{Z}_4\} \cup \{((i, 0), (i+2, 0)), ((i+2, 0), (i+1, 1)), ((i+3, 1), (i, 0)), ((i+1, 1), (i+2, 1)), ((i+3, 1), (i, 1)), ((i, 1), (i+3, 1)) \mid i \in \mathbf{Z}_2\}$ reducing sums modulo 4. Then each vertex has out-degree $2 = \lambda_1/2$, and the edges remaining undirected are partitioned by the triples in $\{\{(i, 0), (i+1, 0), (i+2, 1)\} \mid i \in \mathbf{Z}_4\}$.

So we can now assume that $x \geqslant 2$. We define K_m on the vertex set $\{\infty_i \mid i \in \mathbf{Z}_5\} \cup \mathbf{Z}_{6x-3}$. Let $(\mathbf{Z}_{6x-3}, T)$ be a KTS$(6x-3)$ with parallel classes $\pi_0, ..., \pi_{3x-3}$. Partition the triples in: π_0 into 3 sets T_0, T_1 and R_0; π_1 into 3 sets T_2, T_3 and R_1; π_2 into 2 sets T_4 and R_2; so that $|T_i| = y$ for $i \in \mathbf{Z}_5$ (π_2 exists since $x \geqslant 2$). Then $|R_0| = |R_1| = 2x-1-2y > 0$, and $|R_2| > 0$. Direct some of the edges of K_m as follows (to form H^+).

(i) H^+ contains the directed edges $\{(\infty_i, \infty_{i+1}), (\infty_i, \infty_{i+2}) \mid i \in \mathbf{Z}_5\}$, reducing the sum in the subscript modulo 5.

(ii) For each $i \in \mathbf{Z}_5$ direct the edge from ∞_i to each vertex in a triple in T_i.

(iii) For each vertex v in a triple in $T_0 \cup R_0$, $T_1 \cup R_0$, $T_2 \cup R_1$, $T_3 \cup R_1$, and R_2 direct the edge from v to ∞_1, ∞_0, ∞_3, ∞_2 and ∞_4 respectively.

(iv) Direct the edges in each triple in $\bigcup_{i \in \mathbf{Z}_5} T_i$ to form directed 3-cycles.

(v) Direct the edges in each triple in $\bigcup_{i=3}^{3y-1} \pi_i$ to form directed 3-cycles (π_{3y-1} exists since $3y-1 < 3x-3$).

The edges directed in (i)–(v) form H^+. For each $i \in \mathbf{Z}_5$, ∞_i is incident with 2 edges directed out in (i) and $3y$ directed out in (ii) so has outdegree $3y+2=\lambda_1/2$ in H^+. Each vertex $v \in \mathbf{Z}_{6x-3}$ is incident with 5 edges direct out in (iii)–(iv), and $3y-3$ directed out in (v), so also has degree $\lambda_1/2$. The edges in K_m remaining undirected are partitioned by the triples in $(\bigcup_{i \in \mathbf{Z}_3} R_i) \cup (\bigcup_{i=3y}^{3x-3} \pi_i)$.

Case 3: $m=6x$. In this case we have to consider 3 further cases, since $\lambda_1 \equiv 0$, 2 or 4 (mod 6) (see Table II), so we consider the cases $\lambda_1 \equiv 0$ or 2 (mod 6) and $\lambda_1 \equiv 4$ (mod 6) in turn.

Suppose that $\lambda_1 = 6y+2\varepsilon$, where $\varepsilon \in \{0, 1\}$, and suppose $\lambda_1 \neq 2$. Since $2 < \lambda_1 \leqslant m-1$, we have that $1 \leqslant y < x$, and so $x \geqslant 2$. Let K_m be defined on the vertex set $\{\infty_0, \infty_1, \infty_2\} \cup \mathbf{Z}_{6x-3}$, and let $(\mathbf{Z}_{6x-3}, T)$ be a KTS$(6x-3)$ with parallel classes $\pi_0, \pi_1, ..., \pi_{3x-3}$. Of course, π_i $(i \in \mathbf{Z}_{3x-3})$ contains $2x-1$ triples. Partition the triples in π_0 into 3 sets T_0, T_1 and R_0 so that $|T_0| = |T_1| = y$ (so $|R_0| = 2x-1-2y \geqslant 1$), and partition π_1 into 2 sets T_2 and R_1 so that $|T_2| = y$ (π_1 exists since $x \geqslant 2$). Direct some of the edges of K_m as follows (to form H^+).

(i) For each $i \in \mathbf{Z}_3$, direct the edge from ∞_i to each vertex in a triple in T_i.

(ii) For each vertex v in a triple in $T_0 \cup R_0$, $T_1 \cup R_0$ and R_1 direct the edge from v to ∞_1, ∞_0 and ∞_2 respectively.

(iii) Direct the edges in each triple in $T_0 \cup T_1 \cup T_2$ to form directed 3-cycles.

(iv) Direct the edges in each triple in $\bigcup_{i=2}^{3y-2+\varepsilon} \pi_i$ to form directed 3-cycles ($\pi_{3y-2+\varepsilon}$ exists since $3x-3 > 3y-2+\varepsilon$).

(v) If $\varepsilon = 1$ then direct the edges ∞_0 to ∞_1, ∞_1 to ∞_2, and ∞_2 to ∞_0.

The edges directed in (i)–(v) form H^+. For each $i \in \mathbf{Z}_3$, ∞_i has out degree $3|T_i| + \varepsilon = 3y+\varepsilon = \lambda_1/2$ (from (i) and (v)). For each $v \in \mathbf{Z}_{6x-3}$, v has 3 edges directed out defined in (ii) and (iii), and has $3y-3+\varepsilon$ edges directed out defined in (iv), so has out degree $\lambda_1/2$. The edges in K_m remaining undirected are partitioned by the triples in $R_0 \cup R_1 \cup \{\{\infty_0, \infty_1, \infty_2\}\} \cup (\bigcup_{i=3y-1}^{3x-3} \pi_i)$ if $\varepsilon = 0$, and by $R_0 \cup R_1 \cup (\bigcup_{i=3y}^{3x-3} \pi_i)$ if $\varepsilon = 1$.

Suppose that $\lambda_1 = 6y+4$ or $\lambda_1 = 2$, and suppose that $m \notin \{12, 18\}$. Since $1 \leqslant \lambda_1 \leqslant m-1$, we have that $0 \leqslant y < x$ and $x = 1$ or $x \geqslant 4$. We define K_m on the vertex set $\mathbf{Z}_x \times \mathbf{Z}_6$. Let $(\mathbf{Z}_x \times \mathbf{Z}_6, \{\{i\} \times \mathbf{Z}_6 \mid i \in \mathbf{Z}_x\}, T)$ be a resolvable GDD$(6, x)$ with parallel classes $\pi_0, \pi_1, ..., \pi_{3x-2}$ (see Theorem 4.6). Direct some of the edges of K_m as follows (to form H^+).

(i) If $\lambda_1 \neq 2$ then for each $i \in \mathbf{Z}_x$, let H^+ contain the directed edges $((i, 1), (i, 4))$, $((i, 1), (i, 5))$, $((i, 2), (i, 5))$, $((i, 2), (i, 0))$, $((i, 3), (i, 4))$,

$((i, 3), (i, 0))$, $((i, 0), (i, 1))$, $((i, 5), (i, 3))$, $((i, 4), (i, 2))$, $((i, 4), (i, 5))$, $((i, 5), (i, 0))$, and $((i, 0), (i, 4))$.

(ii) If $\lambda_1 = 2$ then let H^+ contain the directed edges $((i, 1), (i, 4))$, $((i, 2), (i, 4))$, $((i, 3), (i, 0))$, $((i, 4), (i, 0))$, $((i, 5), (i, 2))$, and $((i, 0), (i, 2))$.

(iii) Direct the edges in each triple in $\bigcup_{i \in \mathbf{Z}_{3y}} \pi_i$ to form directed 3-cycles (π_{3y-1} exists since $3y - 1 < 3x - 2$), and let H^+ contain these directed edges.

For each vertex $v \in \mathbf{Z}_x \times \mathbf{Z}_6$ there are 2 edges directed out of v defined in (i) if $\lambda_1 = 6y + 4$, there is 1 edge directed out of v in (ii) when $\lambda_1 = 2$, and in either case there are $3y$ edges directed out of v in (iii), so v has outdegree $\lambda_1/2$ in H^+. The edges of K_m that have not been directed are partitioned by the triples in $(\bigcup_{i=3y}^{3x-2} \pi_i) \cup \{\{(i, 1), (i, 2), (i, 3)\} \mid i \in \mathbf{Z}_x\}$ if $\lambda_1 = 6y + 4$, and $(\bigcup_{i=3y}^{3x-2} \pi_i) \cup \{\{(i, 1), (i, 2), (i, 3)\}, \{(i, 3), (1, 4), (i, 5)\}, \{(i, 5), (i, 0), (i, 1)\} \mid i \in \mathbf{Z}_x\}$ if $\lambda_1 = 2$.

Suppose that $\lambda_1 \in \{2, 4, 10\}$ and $m = 12$. If $\lambda_1 = 10$ then define K_{12} on the vertex set $\mathbf{Z}_2 \times \mathbf{Z}_6$, let H^+ contain the directed edges defined in (i) above for each $i \in \mathbf{Z}_2$, and add the directed edges in $\{((0, i), (1, i + j)), ((1, i), (0, i + j + 1)) \mid i \in \mathbf{Z}_6, j \in \mathbf{Z}_3\}$. Then each vertex has outdegree $5 = \lambda_1/2$, and the edges remaining undirected are partitioned by the triples $\{\{(i, 1), (i, 2), (i, 3)\} \mid i \in \mathbf{Z}_2\}$. If $\lambda_1 \in \{2, 4\}$ then define K_{12} on the vertex set $\{\infty_i \mid i \in \mathbf{Z}_3\} \cup (\mathbf{Z}_3 \times \mathbf{Z}_3)$. Let $(\mathbf{Z}_3 \times \mathbf{Z}_3, T)$ be a KTS(9) (see Theorem 4.5) with parallel classes π_i, $i \in \mathbf{Z}_4$ such that $\pi_4 = \{\{(i, 0), (i, 1), (i, 2)\} \mid i \in \mathbf{Z}_3\}$ and $\{(0, 0), (1, 1), (2, 2)\} \in \pi_3$ (clearly this is possible by renaming the symbols). Let $T_i = \{\{(i, j), (i, j + 1), \infty_{j+2}\} \mid i \in \mathbf{Z}_3, j \in \mathbf{Z}_3\}$, reducing sums modulo 3. For $\lambda_1 = 2$ let H^+ contain the directed edges in $\{((i, j), \infty_j), (\infty_j, \infty_{j+1}) \mid i, j \in \mathbf{Z}_3\}$, then the edges remaining undirected in K_{12} are partitioned by the triples in $\bigcup_{i \in \mathbf{Z}_3} (\pi_i \cup T_i)$. For $\lambda_1 = 4$ let H^+ contain the directed edges in $\{((i, j), \infty_j) \mid i, j \in \mathbf{Z}_3, i \neq j\} \cup \{(\infty_j, \infty_{j+1}), (\infty_j, (j, j)) \mid j \in \mathbf{Z}_3\}$ together with the directed edges in the directed 3-cycles formed from the triples in $\pi_2 \cup \{\{(0, 0), (1, 1), (2, 2)\}\}$; then the edges remaining undirected in K_{12} are partitioned by the triples in $(\pi_1 \cup \pi_3) \backslash \{\{(0, 0), (1, 1), (2, 2)\}\}$.

Suppose that $\lambda_1 \in \{2, 4, 10, 16\}$ and $m = 18$, and define K_{18} on the vertex set $\mathbf{Z}_3 \times \mathbf{Z}_6$. If $\lambda_1 = 16$ then for each $i \in \mathbf{Z}_3$ let H^+ contain the directed edges defined in (i) above together with the directed edges in $\{((i, j), (i + 1, k)) \mid j, k \in \mathbf{Z}_6, i \in \mathbf{Z}_3\}$; the edges remaining undirected are partitioned by the triples in $\{\{(i, 1), (i, 2), (i, 3)\} \mid i \in \mathbf{Z}_3\}$. If $\lambda_1 = 4$ or 2 then let H^+ contain the directed edges defined in (i) or (ii) above respectively; the edges remaining undirected are partitioned by the triples in a GDD(6, 3) of index (0, 1) (see Theorem 1.3) together with the triples in $\{\{(i, 1), (i, 2), (i, 3)\} \mid i \in \mathbf{Z}_3\}$ if $\lambda_1 = 4$ and in $\{\{(i, 1), (i, 2), (i, 3)\}, \{(i, 3), (i, 4), (i, 5)\}, \{(i, 5), (i, 0), (i, 1)\} \mid i \in \mathbf{Z}_x\}$ if $\lambda_1 = 2$. If $\lambda_1 = 10$ then let H^+ contain the directed edges in $\{((i, 2j), (i, 2j+1)) \mid i, j \in \mathbf{Z}_3\} \cup \{((i, j), (i+1, k)) \mid j, k \in \mathbf{Z}_6, j \neq k, i \in \mathbf{Z}_3\} \backslash \{((i, 2j), (i+1, 2j+2)) \mid i, j \in \mathbf{Z}_3\}$; then the edges remaining undirected are partitioned by the triples in $\{\{(0, j), (1, j), (2, j)\}, \{(0, 2k), (1, 2k+2), (2, 2k+4)\} \mid j \in \mathbf{Z}_6, k \in \mathbf{Z}_3\} \cup (\bigcup_{z \in \mathbf{Z}_3} B_z)$, where $(\{z\} \times \mathbf{Z}_6, \{\{(z, 2j), (z, 2j+1)\} \mid j \in \mathbf{Z}_3\}, B_z\}$ is a GDD(2, 3) of index (0, 1) (see Theorem 1.3). ▮

Proposition 4.9. *Let* $m \equiv 2 \pmod 6$, $m \geqslant 3$, $\lambda_1 = 2$ *and* $\lambda_2 \equiv 5 \pmod 6$. *There exists a* GDD$(2, m)$ *of index* (λ_1, λ_2).

Proof. Let $m = 6x+2$. Clearly it suffices to consider the case where $\lambda_2 = 5$ since by Theorem 1.3 there exists a GDD$(2, m)$ of index (0, 6). Also by Theorem 1.3, there exists a GDD$(2, 3x+1)$ of index (0, 1), so by taking 5 copies of this GDD, possibly with different groups, it is possible to define a collection B of triples that partition all the edges of $5K_n$, except for 5 1-factors (corresponding to the 5 sets of groups). Therefore it remains to find a set of 5 1-factors of $5K_{6x+2}$ whose edges are partitioned by a set B_1 of triples and a set of edges forming a spanning subgraph H that can be directed so that each vertex in the resulting directed graph H^+ has out-degree $\lambda_1/2 = 1$.

We define K_{6x+2} on the vertex set $\mathbf{Z}_{3x+1} \times \mathbf{Z}_2$. Let $B_1 = \{\{(i, 0), (i+1, 0), (i+2, 1)\} \mid i \in \mathbf{Z}_{3x+1}\}$ and let H^+ contain the directed edges in $\{((i, 0), (i, 1)), ((i, 1), (i+1, 1)) \mid i \in \mathbf{Z}_{3x+1}\}$, reducing the sum modulo $3x+1$. Then clearly each vertex in H^+ has outdegree 1 as required. $F_j = \{\{(i+j, 0), (i+2, 1)\} \mid i \in \mathbf{Z}_{3x+1}\}$ forms a 1-factor for each $j \in \mathbf{Z}_2$, and it is easy to see that the edges in $\{\{(i, 0), (i+1, 0)\}, \{(i, 0), (i, 1)\}, \{(i, 1), (i+1, 1)\} \mid i \in \mathbf{Z}_{3x+1}\}$ can be partitioned into 3 1-factors, so the result follows. ▮

We have now settled the case $n = 2$ as the following Theorem shows.

Theorem 4.10. *Let* $m \geqslant 2$ *and* $\lambda_1, \lambda_2 \geqslant 1$. *There exists a* GDD$(2, m)$ *of index* (λ_1, λ_2) *if and only if*

(1) 2 *divides* $\lambda_1 + 2\lambda_2(m-1)$,

(2) 3 *divides* $\lambda_1 m + 2\lambda_2 m(m-1)$, *and*

(3) $\lambda_1 \leqslant (m-1)\lambda_2$.

Proof. The necessity follows from Lemma 4.1. The sufficiency follows from Proposition 4.4 if $\lambda_2(m-1)$ is even, from Proposition 4.7 if $\lambda_2(m-1)$ is odd and $\lambda_2 = 1$, from Proposition 4.9 if $m \equiv 2 \pmod 6$, $\lambda_1 = 2$ and $\lambda_2 \equiv 5 \pmod 6$, and therefore from Proposition 4.7 in all other cases where $\lambda_2 \geqslant 2$. ▮

5. FINAL COMMENTS

We can now summarise the results on this paper with the following result (see Theorems 3.7 and 4.10).

THEOREM 5.1. *Let* $n=2$ *or* $m=2$, *and* $\lambda_1, \lambda_2 \geqslant 1$. *There exists a* GDD$(n, m)$ *of index* (λ_1, λ_2) *if and only if*

(1) 2 *divides* $\lambda_1(n-1)+\lambda_2(m-1)\,n$,

(2) 3 *divides* $\lambda_1 mn(n-1)+\lambda_2 m(m-1)\,n^2$,

(3) *if* $m=2$ *then* $\lambda_1 \geqslant \lambda_2 n/2(n-1)$, *and*

(4) *if* $n=2$ *then* $\lambda_1 \leqslant (m-1)\,\lambda_2$.

This can now be incorporated with Theorems 1.2, 1.3, and 1.1 to prove the following encompassing result.

THEOREM 5.2. *Let* n, m, $\lambda_2 \geqslant 1$ *and* $\lambda_1 \geqslant 0$. *There exists a* GDD(n, m) *of index* (λ_1, λ_2) *if and only if*

(1) 2 *divides* $\lambda_1(n-1)+\lambda_2(m-1)\,n$,

(2) 3 *divides* $\lambda_1 mn(n-1)+\lambda_2 m(m-1)\,n^2$,

(3) *if* $m=2$ *then* $\lambda_1 \geqslant \lambda_2 n/2(n-1)$, *and*

(4) *if* $n=2$ *then* $\lambda_1 \leqslant (m-1)\,\lambda_2$.

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